– The Standard Model –

Problem Set: 03
Lecturer: Dr. Fabian Ruehle
Submission deadline: Friday, June 8th, 11:00AM

3.1 Seesaw mechanism and neutrino oscillations *10 points*

We write a four component Dirac spinor Ψ_D in the chiral representation as a composition of two Weyl spinors. A *Majorana spinor* Ψ_M is a Dirac spinor Ψ_D that satisfies the constraint $\Psi^c_\mathrm{M} := C\overline{\Psi}^T_\mathrm{M} = \Psi_\mathrm{M}$, where $C = \mathrm{i}\gamma^2\gamma^0$ is the charge conjugation operator.

(a) The most general mass term for a Dirac spinor is the *Dirac–Majorana mass term*. Write down the mass term of Ψ in terms of the mass matrix

$$\mathcal{M} = \begin{pmatrix} m_\mathrm{L} & m_\mathrm{D} \\ m_\mathrm{D} & m_\mathrm{R} \end{pmatrix}$$

and the independent spinor components Ψ_L, Ψ_R. Why do we typically take $m_{\mathrm{M,left}} = 0$, $m_\mathrm{D} \sim 100$ GeV, $m_{\mathrm{M,\ right}} \sim 10^{16}$ GeV in the Standard Model extended by the right-handed neutrino? What are the mass eigenvalues and eigenstates for these parameters? Quantify the effect of mass mixing. (*4 points*)

(b) Since neutrinos are massive, we can have flavor mixing, i.e. the mass eigenstates are not necessarily flavor eigenstates. Let us assume that there are n orthonormal flavor (interaction) eigenstates $|\nu_\alpha\rangle$ that are transformed into n mass eigenstates ν_i via a unitary mixing matrix U, $|\nu_\alpha\rangle = U_{\alpha i}|\nu_i\rangle$. Assuming the mass eigenstates $|\nu_\mathrm{i}\rangle$ are stationary states and were emitted with momentum p by a source at $x = 0$ at $t = 0$, what is the form of $|\nu_i(x,t)\rangle$? What is the relativistic Hamiltonian to first non-vanishing order in m/p? (*1 point*)

(c) A neutrino detector is built at a distance L from a source that produced neutrinos in an eigenstate $|\nu_\alpha\rangle$. Compute the amplitude for detecting a neutrino in an eigenstate $|\nu_\beta\rangle$ in the ultra-relativistic limit $v = 1,\ p = E$. Obtain the transition probability P in terms of the differences of the mass squares $\Delta m^2_{ij} := m^2_i - m^2_j$. What is the probability of finding the original flavor? (*3 points*)

(d) Now assume that we have two flavors and one mixing angle θ. What is the form of U? Compute $P(\alpha \to \beta)$ and $P(\alpha \to \alpha)$ for this case. Under which condition can one flavor completely rotate into another one? (*2 points*)

3.2: Gauge anomalies *10 points*

Consider a theory with (Abelian or non-Abelian) gauge fields A^a_μ and N left–chiral Weyl fermions Ψ_i with gauge charges q^a_i. Consider the Feynman graph in figure 1. There, $j^a_\mu = \frac{\delta S}{\delta A^{a,\mu}}$ is the current coupling to the gauge field A^a_μ and a, b, c label various gauge

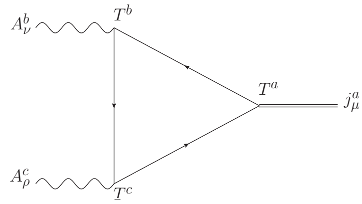

Figure 1: 4D triangle anomaly graph

symmetries that occur in theory. The T^a are either the Abelian charges q^a or the non–Abelian generators in the respective representation. The particles running in the loop are all N chiral fermions in the theory. A gauge theory is said to be anomalous if the symmetry is not preserved under quantization. This manifests in an anomalous variation of the path integral measure, which is proportional to the graph figure 1. The induced effective change of the action is

$$\delta S_{\text{anom}} \propto \int \mathrm{d}^4 x \lambda^a \epsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^c_{\rho\sigma}\,,$$

where λ^a is the gauge parameter.

(a) We first discuss the case where all gauge symmetries are Abelian. Show that including the charges q_i^a of the fermions in the graph, the cancellation of the anomaly leads to the condition

$$\sum_{i=1}^{N} q_i^a q_i^b q_i^c = 0\,. \tag{1}$$

Why is it sufficient to consider only massless fermions? *(2 points)*

(b) Show that equation (1) is indeed fulfilled for the Standard Model hypercharge, i.e. with $T^a = T^b = T^c = T_Y$, where T_Y is the hypercharge generator. *(2 points)*

(c) Next, consider one U(1) and one non-Abelian SU(N) symmetry with the particles transforming only in the trivial or in the fundamental and anti-fundamental representation. Show that including group theory factors in the Feynman graphs leads to the constraint

$$\sum_{i=1}^{N} C_{\mathbf{R}}\, q_i^a = 0\,,$$

where $C_{\mathbf{R}}$ is the quadratic Casimir of the respective representation, $C_{\mathbf{R}} = \mathrm{tr}_{\mathbf{R}}\, T^a T^a$. Why is there no constraint containing two Abelian charges? *(2 points)*

(d) Check by inserting the proper quantum numbers that the $\mathrm{U(1)_Y} - \mathrm{SU(2)} - \mathrm{SU(2)}$ and the $\mathrm{U(1)_Y} - \mathrm{SU(3)} - \mathrm{SU(3)}$ anomaly vanishes in the Standard Model. *(2 points)*

(e) Finally we replace two gauge fields A^d_ρ by universal graviton couplings. Argue that this leads to the constraint

$$\sum_{i=1}^{N} q_i^a = 0\,.$$

Check that this is also fulfilled in the Standard Model. *(2 points)*

3.3 't Hooft-Polyakov monopole *10 points*

The 't Hooft-Polyakov monopole is a smooth topological field configuration with finite energy. We search for time-independent solutions that are stabilized by their behavior when approaching spatial infinities. Here $\partial\,\mathbb{R}^3 \cong S^2$.

(a) Argue that one real scalar or complex field with potential $V(\phi) = \frac{\lambda}{4}\left(|\phi|^2 - a^2\right)^2$ cannot lead to topologically non-trivial solutions. *Hint: Study the maps $\partial\mathbb{R}^3 \to M$ where M is the set of stable extrema of $V(\phi)$.* (*2 points*)

(b) Show that there are no topologically non-trivial solutions with finite energy in a pure scalar field theory by checking the behavior of the Hamiltonian in spherical coordinates as $r \to \infty$? (*2 points*)

By introducing a gauge symmetry, the gauge field in the covariant derivative will cancel the divergent behavior. For the *'t Hooft-Polyakov model*, we introduce an $SO(3)$ gauge field and a triplet of real scalars with Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu,a}F_{\mu\nu,a} + \frac{1}{2}(D_\mu\phi)_a(D^\mu\phi)^a - \frac{\lambda}{4}\left(\phi_a\phi^a - a^2\right)^2 ,$$

with

$$\begin{aligned} F_{\mu\nu,a} &= \partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} - e\epsilon_{abc}A_{\mu,b}A_{\nu,c} \,, \\ \left(D_\mu\phi\right)_a &= \partial_\mu\phi_a - e\epsilon_{abc}A_{\mu,b}\phi_c \,. \end{aligned}$$

(c) Find the minima of the scalar potential, the form of the vacuum, and the unbroken gauge symmetry. (*1 point*)

(d) Argue that $\phi_a(r \to \infty) \to a\hat{r}_a$ is topologically stable. Describe this configuration geometrically. (*1 point*)

(e) Show that if the gauge field behaves like $A_{0,a} = 0$, $A_{i,a}(r \to \infty) \to \epsilon_{iab}\frac{r_b}{er^2} + \mathcal{O}(r^{-2})$, the dangerous terms in the energy density cancel. (*2 points*)

(f) Show that this field configuration corresponds to a magnetic monopole of charge $g = 4\pi/e$. *Hint: Identify the unbroken U(1) generator.* (*2 points*)