– The Standard Model –

Problem Set: 02
Lecturer: Dr. Fabian Ruehle
Submission deadline: Friday, May 25th, 11:00AM

2.1 The Standard Model Higgs effect *10 points*

The GSW theory is the part of the Standard Model that describes the electroweak interactions via a non-Abelian gauge theory with the gauge group $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. The corresponding Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &\overbrace{\overline{R}(\mathrm{i}\gamma^\mu D_\mu)R + \overline{L}(\mathrm{i}\gamma^\mu D_\mu)L}^{\text{kinetic energy of leptons and interactions with gauge bosons}} \quad \overbrace{- \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}}^{\text{kinetic energy of the gauge bosons and self-interactions}} \\ &\underbrace{+ (D_\mu\Phi)^\dagger(D^\mu\Phi) - \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2}_{\text{kinetic energy of Higgs and Higgs potential}} \quad \underbrace{-G_e\left(\overline{L}\Phi R + \overline{R}\Phi^\dagger L\right)}_{\text{Higgs Yukawa coupling}},\end{aligned} \tag{1}$$

with

$$D_\mu = \partial_\mu + \mathrm{i}g' T_Y B_\mu + \mathrm{i}g T^a W^a_\mu\,, \qquad T^a = \frac{1}{2}\sigma^a, \quad T_Y = Y\mathbb{1}_{2\times 2} \tag{2}$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu\,. \tag{3}$$

In this exercise, we study symmetry breaking to the electro-magnetic $\mathrm{U}(1)_{\mathrm{EM}}$ symmetry and how the vector bosons become massive.

(a) How does the covariant derivative (2) act on the left- and right-handed leptons doublets/singlet and on the Higgs-doublet? Show that the Lagrangian is gauge and Lorentz invariant. (*3 points*)

(b) In the Higgs potential we have $\mu^2 < 0$. Find the minimum and apply an $\mathrm{SU}(2)_L$ rotation to choose the vacuum expectation value (VEV) of the Higgs field to be of the form $\langle\Phi\rangle = \frac{1}{\sqrt{2}}(0, v)^T$. We can use this to write the Higgs fields as

$$\Phi(x) = \exp\left\{\frac{\mathrm{i}}{v}\xi^a(x)T^a\right\}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + \eta(x))\end{pmatrix},$$

where $\xi^a(x)$ and $\eta(x)$ are real fields and T^a are the generators of $\mathrm{SU}(2)$. Now we apply an $\mathrm{SU}(2)_L$ gauge transformation such that the angular excitations $\xi^a(x)$ vanish. This gauge transformation is called *unitary gauge*. Rewrite the Higgs potential in unitary gauge, find the mass of η, and compare the degrees of freedom in the Higgs sector to the situation before electroweak symmetry breaking. (*3 points*)

(c) Rewrite the kinetic energy of the Higgs field in (1) in unitary gauge. Define $W^{\pm\,\mu} := \frac{1}{\sqrt{2}}(W^{1\,\mu} \mp \mathrm{i}\,W^{2\,\mu})$ and find the SO(2) matrix

$$\mathcal{O} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}$$

that orthogonalizes the mass matrix of the generators W^3_μ and B_μ. Write $\mathcal{O}$ in terms of g and g' and read off the masses for $W^\pm_\mu$, Z_μ, and the photon A_μ. (*3 points*)

(d) Finally, rewrite the covariant derivative (2) in terms of the fields $W^\pm_\mu$, Z_μ and A_μ in the broken phase. Read off the generator of the electro-magnetic $\mathrm{U}(1)_{\mathrm{EM}}$ field and the electric charge e. (*1 point*)

2.2 V-A couplings *10 points*

In this exercise, we study how chirality and the V-A structure arises in the standard model after symmetry breaking.

(a) Rewrite the lepton kinetic terms in (1) in terms of Dirac spinors Ψ_D, the projection operators $P_{L/R} = \frac{1}{2}(1 \mp \gamma^5)$, and the physical fields $W^\pm_\mu$, Z_μ, A_μ after electroweak breaking. Which vector bosons couple chirally, which couple non-chirally? Determine the leptonic vector and axial-vector coupling constants c_V and c_A. (*4 points*)

(b) Now we introduce the quark fields. Repeat the analysis of (a) for the quarks using the quark kinetic terms

$$\mathcal{L}_{\text{Quark}} = \overline{Q}(\mathrm{i}\gamma^\mu D_\mu)Q + \overline{D}(\mathrm{i}\gamma^\mu D_\mu)D + \overline{U}(\mathrm{i}\gamma^\mu D_\mu)U \,.$$

Read off the couplings c_V, c_A and the electric charges. (*3 points*)

(c) Finally, derive the interaction vertices of the fields $W^\pm_\mu$, Z_μ and A_μ after electroweak symmetry breaking. (*3 points*)

2.3 Fermions and CKM Matrix *10 points*

In this exercise, we see how the Higgs mechanism gives a mass to fermions. In the process, we study the CKM matrix. Throughout the exercise, we assume N generations of quarks and leptons, and argue that $N \geq 3$ for CP violation. The Yukawa couplings are

$$\mathcal{L}_{\text{Yuk}} = \underbrace{C_L^{ij}\overline{R}_i\Phi L_j}_{\substack{\text{Lepton Yukawa}\\ \text{couplings}}} + \underbrace{C_U^{ij}\overline{U}_i(\varepsilon\Phi)Q_j}_{\substack{\text{Up-type Yukawa}\\ \text{couplings}}} + \underbrace{C_D^{ij}\overline{D}_i\Phi U_j}_{\substack{\text{Down-type Yukawa}\\ \text{couplings}}} + \text{h.c.} \tag{4}$$

with $C_L, C_U, C_D \in \mathbb{C}^{3\times 3}$.

(a) Show that the terms in (4) are gauge invariant and insert the Higgs VEV (in unitary gauge) to read off the mass matrices. (*4 points*)

(b) The Yukawa matrices $C_{L,U,D}$ can be diagonalized with a biunitary transformation

$$C_L = V_L D_L U_L^\dagger \,, \quad C_U = V_U D_U U_U^\dagger \,, \quad C_D = V_D D_D U_D^\dagger$$

Use this rotation to obtain the mass eigenstates and insert them into the interaction terms derived in exercise 2.2. There, the index structure is diagonal, i.e. the family indices are contracted with a Kronecker delta. Show that the interactions with A_μ

and Z_μ stay diagonal (no tree-level flavor-changing neutral currents). For the charged currents, show that with a proper transformation of the neutrinos also the $W_\mu^\pm$ terms stay diagonal for the leptons, but not for the quarks. Why is this transformation on the neutrinos allowed? This shows us that within the standard model there is no mixing between the leptons. *(3 points)*

(c) From (b), we find the CKM matrix $V_{\text{CKM}} = U_U^\dagger U_D$. Show that V_{CKM} has $(N-1)^2$ physical parameters. *(2 points)*

(d) For N generations, how many parameters are SO(N) rotations and how many are complex phases? Show that CP violation necessitates $N \geq 3$. *(1 point)*