Cosmology: from theory to observables

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Last class we learned that the CMB encodes information about the Universe 380,000 yrs after the Big Bang. In particular, there are acoustic wave patterns encoded in the temperature fluctuations of the CMB which allow us to probe the composition of the Universe and the statistics of the primordial quantum fluctuations present during inflation. After recombination, the pull of gravity on baryons led to the formation of stars and galaxies out of the original potential fluctuations. Those galaxies, which we can map throughout the history of our Universe, thus encode important cosmological information. Today, we will learn how to extract that information.

We will focus on probing the composition of a Universe governed by General Relativity, but you should also bear in mind that the theoretical predictions for some of the observables we will be discussing change if the laws of physics depart from GR, as in the case for "modified gravity" theory.

1 Preliminaries

As in the previous class, we will work with a perturbed Minkowski metric of the following form

$$ds^2 = a^2(\eta)[(1+2\Psi)d\eta^2 - (1-2\Phi)\delta_{ij}dx^i dx^j] \tag{1}$$

where η is conformal time: $d\eta = dt/a(t)$.

1.1 Primordial perturbations from inflation

Throughout past lectures in this course, you learned that quantum fluctuations present during inflation generated a *spectrum* of perturbations of the gravitational potential present in the Universe. You phrased this in terms of perturbations in the so called "comoving curvature perturbation",

$$\mathcal{R} = -\Phi - \frac{\mathcal{H}(\Phi' + \mathcal{H}\Phi)}{4\pi G a^2(\bar{\rho} + \bar{P})}. \tag{2}$$

a conserved quantity on large scales. It is impossible to predict the value of $\mathcal{R}$ at any given point of space and time, due to its quantum origin. But you were able to make a prediction for its statistical properties. Namely, you found a prediction for the *power spectrum* of the curvature:

$$\Delta^2_{\mathcal{R}}(k) \equiv A_s \left(\frac{k}{k_*}\right)^{n_s - 1} \tag{3}$$

where A_s is an overall amplitude, k_* is a reference scale in Fourier space and n_s is the tilt of the power spectrum.

Furthermore, you learned that there are a set of equations that allow you to make predictions for how these perturbations in the curvature propagate into perturbations of the total matter density field and (last class) the temperature of the cosmic microwave background. Those equations were: the Einstein equations, the conservation of the stress-energy tensor and the Bolztmann equation. Solving those equations allowed you to relate the primordial curvature perturbation to perturbations in the density of any given component of our Universe via a *transfer function*, $T(k,z)$. For example, perturbations on the matter field, δ_m, at redshift z are related to the primordial perturbation $\mathcal{R}$ by

$$\delta_m(\mathbf{k}, z) = T(k,z)\mathcal{R}(\mathbf{k}) \tag{4}$$

and the power spectrum of δ_m is related to the power spectrum of the curvature by

$$P_m(k,z) \equiv |\delta_m(\mathbf{k},z)|^2 = T^2(k,z)|\mathcal{R}(\mathbf{k})|^2 \tag{5}$$

Eq. (5) is a powerful one. It means we can connect observables of the matter density field today to the initial conditions of inflation. It is hard, though, to measure the distribution of all matter in the Universe. We'll see there is a way. But to start with, we'll start with the matter that it is easier to see and count: galaxies.

1.2 Growth of matter pertubations

When density perturbations in the matter field are small, they grow linearly with time. In this case, it proves convenient to define a "growth function", $\mathcal{G}(z)$, which connects the value of $z = 0$ (today) to their value at some previous redshift in the following way:

$$\delta_m(\mathbf{x},z) \equiv \frac{\rho_m(\mathbf{x},z) - \bar{\rho}_m(z)}{\bar{\rho}_m(z)} \simeq \delta_m(\mathbf{x}, z=0)\frac{\mathcal{G}(z)}{\mathcal{G}(z=0)}. \tag{6}$$

The combination of Einstein equations and the Boltzmann equation give us a second order differential equation for the matter density perturbations,

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0. \tag{7}$$

When equations (6) and (7) are put together, one can find an expression for the growth function

$$\mathcal{G}(a) = \frac{5\Omega_m}{2}\frac{H(a)}{H_0}\int_0^a \frac{da'}{[a'H(a')/H_0]^3}. \tag{8}$$

1.3 Background expansion

The rate at which the Universe expands also depends on its composition. It is important to keep this in mind for several reasons. First of all, with some observables, it is possible to get a very direct measurement of the expansion rate. When we think about making maps of galaxies, we have to remember that we do not measure their distances along the line of sight directly. What we have access to are their redshifts and their positions in the sky.

For now, let's remind ourselves of the relation between distances and redshifts. We know that in the standard ΛCDM model, the expansion rate is given by the Friedmann equation

$$H^2(z) = H_0^2\left[\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_k(1+z)^2 + \Omega_\Lambda\right] \tag{9}$$

At the beginning of this class, you learned about different definitions of distances:

- Distances to objects a long the line of sight are given by

$$D_C(z) = \frac{c}{H_0}\int_0^z dz'\frac{H_0}{H(z')}. \tag{10}$$

- Comoving angular diameter distances relate an object's comoving size l to the angular size, $\theta = l/D_A$. In a Universe with no curvature, $D_c = D_A$.

2 Observables

We now have all the tools to move on to modelling the observables. We will today focus on galaxies alone, but you might be interested in reading about other cosmological probes like supernovae or the Lyman-α forest. All the observables that we are going to discuss are built from large area maps of the sky, where we can identify the positions and shapes of galaxies, as in Figure **??**

2.1 Clustering of galaxies

Imagine that you map the redshifts and locations on the sky of a large number of galaxies. The three-dimensional position of each galaxy will be given by

$$\mathbf{x} = D_c(z)(\theta_x, \theta_y, 1) \tag{11}$$

We can then obtain the density of galaxies at a given redshift as

$$\delta_g(\mathbf{x}, z) \equiv \frac{n_g(\mathbf{x}, z) - \bar{n}_g(z)}{\bar{n}_g(z)} \tag{12}$$

How can we relate this observable to the matter field? Due to the gravitational pull of the dark matter, galaxies tend to form in overdense regions of the Universe. When the density perturbations are small, we can assume that the density field of galaxies is linearly related to the total density field

$$\delta_g(\mathbf{x}, z) = b\, \delta_m(\mathbf{x}, z) + O(\delta^2) \tag{13}$$

where b is the galaxy "bias".

Similarly to the case of the CMB temperature fluctuations, Eq. (5) also tells us that we cannot predict but the statistical properties of δ_g, not its value at any particular point in space and time. But what we can predict if the galaxy power spectrum,

$$P_g(\mathbf{k}, z) = b^2\, |\delta(\mathbf{k}, z)|^2 \tag{14}$$

The real space analog of the power spectrum is called the *correlation function*, defined as

$$\begin{aligned} \xi_g(\mathbf{x} - \mathbf{x}', z) &\equiv \langle \delta_g(\mathbf{x}, z) \delta_g(\mathbf{x}', z) \rangle, \\ &= \int \frac{d^3k}{(2\pi)^3} P_g(k) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \end{aligned} \tag{15}$$

which is what we can measure by counting pairs of galaxies separated by a given distance at a given redshift. We can thus connect the galaxy correlation function to the matter power spectrum

$$\xi_g(\mathbf{x} - \mathbf{x}', z) = b^2 \int \frac{d^3k}{(2\pi)^3} P_m(k) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \tag{16}$$

2.2 Baryon acoustic oscillations

Last class, we learned about the acoustic waves imprinted in the CMB. The structure of these acoustic waves remains encoded in the matter, and galaxy, density field as it evolves. Sound waves in the baryon-photon fluid could only be sustained before recombination. After recombination, we learned that phothons travel freely and hence the acoustic waves "freeze", leaving patterns in the density field.

Given the sound speed in the fluid, and the age of the Universe are recombination, there is a maximum distance that a sound wave could have travelled in that time, this is called the sound horizon and is given by

$$D_c^* = \int_0^{t_{\rm rec}} c_s(1+z)dt = \int_{z_{\rm rec}}^{\infty} \frac{c_s dz}{H(z)} \tag{17}$$

which is simply the comoving distance a perturbation travelling at c_s has displaced by recombination. The sound speed during this epoch was $c_s \simeq c/\sqrt{3}$.

While in Fourier space acoustic waves appear as a series of peaks, in the correlation function, they appear as a single peak at **fixed** comoving separation D_c^*. This feature can be measured in the correlation function of galaxies at different redshifts, and it will always have the same value. As a consequence it acts as a "standard ruler" for measuring the expansion of the Universe via Eq. (10).

2.3 Redshift space distortions

Closely related to the density field by the continuity equation, the velocity field also contains cosmological information. When density perturbations are small, thus linear, the continuity equation establishes a relation between the growth of the density constrast and the velocity field of matter through

$$\dot{\delta} + ikv = 0. \tag{18}$$

In Section 1.2, we said we would work under the assumption that density perturbations grow linearly, which allows us to connect the density perturbation today to another redshift via Eq. (6). We can re-write the continuity equation under this assumption as

$$v(k,\eta) = i\frac{d}{d\eta}\left[\frac{\delta_m}{\mathcal{G}}\mathcal{G}\right] = \frac{i\delta_m(k,\eta)}{k\mathcal{G}}\frac{d\mathcal{G}}{d\eta} \tag{19}$$

where η is conformal time, as usual.

It is common to work in terms of the logarithmic growth rate, f, which is defined as follows

$$f \equiv \frac{a}{\mathcal{G}}\frac{d\mathcal{G}}{da}, \tag{20}$$

and since $d/d\eta = a^2 H d/da$, the expression for the velocity field of matter is

$$v(k,a) = \frac{ifaH\delta_m(k,a)}{k}. \tag{21}$$

We have worked with the absolute value of the velocity thus far. The generalisation to the vectorial field is

$$\mathbf{v}(k,a) = ifaH\delta_m(k,a)\frac{\mathbf{k}}{k^2}. \tag{22}$$

In other words, gravity not only makes density perturbations in the matter grow, it also increases the velocity field in a proportional way when δ_m is small. Much of the information we can extract about the evolution of the Universe from the velocity field is containted in the function f. The logarithmic growth is very sensitive to Ω_m and this is often approximated as $f \propto \Omega_m^{0.6}$.

2.3.1 Measuring velocities

How do we measure the velocity field? As mentioned above, the information that we can extract from a galaxy catalogue is the position on the sky and the redshift of galaxies. The key is to realise that the redshift is in part caused by the Hubble expansion, in part due to the Doppler effect of the intrinsic velocity of a galaxy, which is induced by the gravity of the surrounding matter distribution. The two components that contribute to the redshift of an object are

$$z = H_0 x + \mathbf{v} \cdot \hat{\mathbf{x}}. \tag{23}$$

Thus, the velocity of the matter field will actually have an effect on the observed galaxy density field, δ_g. Inferred distances to galaxies will be modified by the addition of the velocity component such that the observed position, $\mathbf{x}_s$, will be a sum

$$x_s = x + \frac{\mathbf{v} \cdot \hat{\mathbf{x}}}{H_0}. \tag{24}$$

This means that the true galaxy density contrast, δ_g, is not exactly what we observe. What we observe is the density contrast in "redshift space", δ_s. We will now find a way to connect them. The number of galaxies, measured either in the space of the true or observed coordinates should not change, hence we can establish the following relation between the number densities in real and redshift space

$$\begin{aligned} n_g(\mathbf{x}_s) d^3\mathbf{x}_s &= n_g(\mathbf{x}) d^3\mathbf{x}, \\ n_g(\mathbf{x}_s) &= n_g(\mathbf{x}) J, \\ (1+\delta_s) &= (1+\delta_g) J. \end{aligned} \tag{25}$$

where J is the Jacobian of the coordinate transformation

$$J \equiv \left| \frac{d^3x}{d^3x_s} \right| = \frac{dx}{dx_s} \frac{x^2}{x_s^2}, \tag{26}$$

and we have replaced $n_g(\mathbf{x}_s) = \bar{n_g}(1+\delta_s)$ and $n_g(\mathbf{x}) = \bar{n_g}(1+\delta_g)$.

We will know Taylor expand J to find an expression for the relation between δ_s and δ_g. Starting from Eq. (24), we can replace dx_s/dx and x_s^2/x^2 directly

$$J = \left[1 + \frac{\partial}{\partial x} \left(\frac{\mathbf{v} \cdot \hat{\mathbf{x}}}{H_0} \right) \right]^{-1} \left(1 + \frac{\mathbf{v} \cdot \hat{\mathbf{x}}}{H_0 x} \right)^{-2}. \tag{27}$$

We can in fact neglect the factor that goes as $1/x$. This will be suppressed compared to the derivative of the velocity field by a factor kx. The argument is as follows: kx is the product of the typical size of modes we measure in our map of galaxies, times the size of the map. If we have a map of size x, we cannot measure modes of the order of $k \sim x^{-1}$. We could only hope to measure modes of order $k >> x^{-1}$. Hence, $kx >> 1$ and we can neglect the term with x in the denominator. Thus, if we keep only first order terms,

$$J \simeq \left[1 - \frac{\partial}{\partial x} \left(\frac{\mathbf{v} \cdot \hat{\mathbf{x}}}{H_0} \right) \right]. \tag{28}$$

Going back to Eq. (25), we can establish a connection between the real and redshift space density perturbations:

$$1 + \delta_s = (1+\delta_g) \left[1 - \frac{\partial}{\partial x} \left(\frac{\mathbf{v} \cdot \hat{\mathbf{x}}}{H_0} \right) \right] \tag{29}$$

and again restricting to first order terms,

$$\delta_s = \delta_g - \frac{\partial}{\partial x} \left(\frac{\mathbf{v} \cdot \hat{\mathbf{x}}}{H_0} \right) \tag{30}$$

2.3.2 Connection to the power spectrum

We need to go one step further and understand how the velocities modify the galaxy correlation function, or power spectrum. To simplify things, we will assume galaxies are very far away from us, so we can replace the generic $\hat{\mathbf{x}}$ unit vector by the unit vector along the line of sight direction, $\hat{\mathbf{z}}$. Let's now take the Fourier transform of the observed density contrast,

$$\tilde{\delta}_s = \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} \left[\delta_g(\mathbf{x}) - \frac{\partial}{\partial x}\left(\frac{\mathbf{v}\cdot\hat{\mathbf{z}}}{H_0}\right)\right] \tag{31}$$

If we replace the expression for the Fourier transform of the velocities, Eq. (22),

$$\tilde{\delta}_s = \tilde{\delta}_g - if \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{\partial}{\partial x}\left[\int \frac{d^3\mathbf{k}'}{(2\pi)^3} e^{i\mathbf{k}'\cdot\mathbf{x}} \tilde{\delta}_m \frac{\mathbf{k}'}{k'^2}\cdot\hat{\mathbf{z}}\right] \tag{32}$$

The derivative acts on the exponential, lowering a factor of $i\mathbf{k}'\cdot\hat{\mathbf{x}}$, which we approximate again as $i\mathbf{k}'\cdot\hat{\mathbf{z}}$. We exchange the order of the integrals, writing the expression in the following way,

$$\tilde{\delta}_s = \tilde{\delta}_g + \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \tilde{\delta}_m [f(\mathbf{k}'\cdot\hat{\mathbf{z}})^2] \int d^3\mathbf{x} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}. \tag{33}$$

If we now define the angle between $\mathbf{k}$ and the line of sight as $\mu_k = \mathbf{k}\cdot\hat{\mathbf{z}}$, the observed density contrast in redshift space is given by

$$\tilde{\delta}_s = \tilde{\delta}_g + f\mu_k^2\tilde{\delta}_m \tag{34}$$

and remembering that at large scales there is a factor of a bias between the density field and the galaxy density contrast,

$$\tilde{\delta}_s = \tilde{\delta}_g\left(1 + \frac{f}{b}\mu_k^2\right) \tag{35}$$

This tells us that the observed power spectrum of galaxies will be modified from its original form of Eq. 14 to be modulated by a factor which depends on the logarithmic growth and the angle between the Fourier mode and the line of sight.

2.4 Weak gravitational lensing

We have thus far restricted to studying galaxy positions and galaxy velocities and we have made the connection between their statistics, the primordial curvature and the expansion rate of the Universe in the ΛCDM model. One of the drawbacks that we face is the fact that the Universe is not only comprised of galaxies. To avoid the uncertainties related to the modelling of galaxy bias, we would prefer to have more direct access to the statistics of the full matter field. This can be accomplished via a different observable, "gravitational lensing".

Gravitational lensing is the distortion of the path of photons from a straight path due to deviations from a flat space metric. Of course, the way that we observe galaxies is by counting photons coming from them with our detectors. In the presence of metric perturbations, the path of these photons will not be straight, and the original image of the galaxy, $I_{\rm true}$, will be distorted as a result into $I_{\rm obs}$ when we observe it from Earth

$$I_{\rm obs}(\theta) = I_{\rm true}(\theta_S) \tag{36}$$

where θ are the observed coordinates of the photon on the sky, and θ_s are the original, undistorted ones at the source plane.

To relate θ and θ_s, we have to solve the geodesic equation for photons in the perturbed Minkowski metric. This will allow us to find the following relation

$$\theta_S^i = \theta^i + 2\int_0^{D_c} dD_c'\Phi_{,i}\left(1 - \frac{D_c'}{D_c}\right) \tag{37}$$

This tells us that the net displacement of the photon is a cumulative effect of the gradient of the potential along the line of sight.

If deflections are small, the distortions in the image will come from the first order contribution to θ_S^i, in other words

$$\theta_S \simeq \theta + \frac{\partial \theta_S}{\partial \theta} \cdot (\theta_S - \theta) \tag{38}$$

The matrix

$$A_{ij} \equiv \frac{\partial \theta_S^i}{\partial \theta^j} \tag{39}$$

is often written in the following way as well

$$A_{ij} = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} \tag{40}$$

where κ is called the "convergence" and γ_1 and γ_2 are the components of a complex "shear" field $\gamma = \gamma_1 + i\gamma_2$. We can see from Eq. (37) that these fields will be associated with derivatives of the Newtonian potential, $\Phi_{,ij}$.

We will now see how to associate these deflection fields to the shape of a galaxy. We are going to focus on one component alone, γ_1 and it is left for you to think about how the other two relate to quantities we can measure from our observations. One of the ways for quantifying the shape of an object in two dimensions is through its quadrupole moments,

$$q_{ij} \equiv \int d^2\theta I_{\rm obs}(\theta)\theta_i\theta_j \tag{41}$$

Effectively, what this means is that we are modelling the object as an ellipsoid. From these moments we can define an ellipticity

$$\epsilon_1 \equiv \frac{q_{xx} - q_{yy}}{q_{xx} + q_{yy}} \tag{42}$$

which we can relate to the true image of the galaxy by

$$\epsilon_1 = \frac{\int d^2\theta I_{\rm true}(\theta_S)(\theta_x\theta_x - \theta_y\theta_y)}{\int d^2\theta I_{\rm true}(\theta_S)(\theta_x\theta_x + \theta_y\theta_y)} \tag{43}$$

If we now replace Eq. (37) in the previous expression and stay at first order, we will find the following relation between ϵ_1 and γ_1

$$\epsilon_1 \simeq 2\gamma_1. \tag{44}$$

Hence, the distortion of the galaxy image, ϵ_1 is produced by the cumulative effect of the deflection of photons along the line of sight, which is caused by the perturbations of the Newtonian potential. Correlations between galaxy shapes will thus be related to correlations of the potential, and through the Poisson equation, of the density field of *all* matter along the line of sight.

Problems (optional)

- Estimate D_c^* assuming $c_s \simeq c/\sqrt{3}$, $z_{\rm rec} = 1100$, $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. Neglect radiation and curvature.
- Given Eq. (35), what will be the relation between the power spectrum of $\tilde{\delta}_s$ and of $\tilde{\delta}$?
- Start from Eqs. (43) and (37). Perform a coordinate transformation of the integral to integrate over the source coordinates θ_S instead of the observed ones. This can be achieved by using Eq. (39). Expanding at first order, arrive at Eq. (44).
- Find the determinant of matrix A_{ij} when the perturbations are small. Relate this to the ratio of fluxes of the original and observed source, i.e., the integrals over the brightness distributions $I_{\rm obs}$ and $I_{\rm true}$.

Bibliography

- Section 1 is a recap of the lecture notes on "Cosmology" by Daniel Baumann, available at
  http://www.damtp.cam.ac.uk/user/db275/Cosmology/Lectures.pdf
- "Observational probes of cosmic acceleration" by D. Weinberg et al.
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