These are lecture notes for the Oxford University MMathPhys course The Standard Model in Trinity Term 2017. None of the material presented in these notes is original and it has been collected by me from different sources as a guide for the students taking the course. These notes should not be used as a substitution of books and original articles. The notes have not been checked or proofread, so the students should be aware of possible typos and use them with care. A list of books and useful reviews, from which most of the material of this lecture notes has been taken is below.

References:


5. P. Nason, Introduction to QCD, lecture notes available online at 
   http://moby.mib.infn.it/ nason/misc/QCD-intro.ps.gz

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1 Lecture 1

1.1 Motivation of colour as a symmetry

The proliferation of hadrons discovered in the late 1950’s and 1960’s led to the introduction of quarks as fundamental particles all hadrons were made of. Assuming these quarks were close to identical, regular patterns in the classification of mesons and baryons were found based on exchanging the quark content of each of the hadrons. This is a flavour symmetry, which we will describe in more detail later. One of the consequences of the quark model was that some baryons are made of three identical quarks, for example

\[ \Delta^{++} \sim (u, uu) \quad \Omega \sim (s, s, s) \]

Since quarks are fermions, the total wave function of the quarks in a baryon must be anti-symmetric. Since it is impossible to construct an anti-symmetric state with three identical fermions, the existence of those baryons implied that there must be one additional quantum number associated to quarks. Indeed, if each quark carries a new conserved quantity, which may be viewed as additional index on each quark \( i = 1, 2, 3 \) one can easily construct such a state as

\[ B = \epsilon_{ijk} q_i^j q_i^k. \]

This new quantum number is called colour. Since quantum numbers correspond to conserved properties, this observation means that there must be some unitary transformation of the quarks with different colour index. We may introduce this transformation as a matrix transformation in colour space

\[ q_i' = U_{ij} q_j. \]

Another important feature of this quantum number is that it is not observed in hadrons. Neither mesons nor baryons show the appearance of a colour index. This implies that the baryon field described above must be invariant under a colour transformation. Examining how the baryon field rotates under colour we obtain.

\[ B' = \epsilon_{i'j'k'} q_{i'}^j q_{i'}^j q_{i'}^k = \epsilon_{i'j'k'} U_{i'j} U_{j'k} q_i^j q_k \]

\[ = \det \{U\} \epsilon_{ijk} q_i^j q_k = \det \{U\} B. \]

Demanding that the quark field is invariant imposes the constraint

\[ \det \{U\} = 1. \]

Therefore the symmetry transformation must be unitary and with unit determinant. The set of \( N \times N \) of special unitary matrices form a lie group called SU(N). This fixes the symmetry of which colour is a quantum number to be SU(3). Below we review a few basic facts of this group.
1.2 Basic facts about SU(N)

Given the importance of the SU(N) group, before discussing the QCD dynamics, we will describe a few of its properties to fix notation and conventions.

The elements of SU(N) may be represented by the exponential map

$$U = \exp \{ i \alpha^a t^a \},$$

where $\alpha^a$ are constants and $t^a$ are matrices known as generators of the algebra of the group. The index $a = 1..N^2 - 1$, since the number of independent special and unitary matrices is $N^2 - 1$. In the particular case of $N=3$, the number of generators of the group is $N^2 - 1 = 8$.

Defined as in eq. (7) the group generators are hermitian

$$(t^a)^\dagger = t^a.$$ (8)

The explicit expression of the group generators $t^a$ depend on the representation of the group. When the group acts on a $N$-dimensional complex vector space, i.e., when the group elements are special and unitary matrices, the group is said to be in the fundamental representation. These matrices can be chosen such that they satisfy an orthogonality condition given by

$$\operatorname{tr} \{ t^a, t^b \} = T_R \delta^{ab},$$

with $T_R$ a normalisation constant, which depends on the representation. For the fundamental representation, it is customary to choose $T_R = 1/2$. For the particular case of $N = 3$ a convenient choice of basis for the group generators, consistent with these conventions, is given by the Gell-Mann matrices,

$$t^a = \frac{1}{2} \lambda^a,$$ (10)

with

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ (11)

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$ (12)

$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$ (13)

Independently of the representation of the group, the generators form an algebra, which means that their commutator is also a element of the algebra. This means that

$$[t^a, t^b] = i f^{abc} t^c,$$ (14)
where $f^{abc}$ are a set of (real) numbers called the structure constants of the group. For the particular case of $N = 3$, the commutation relations of the Gell-Mann matrices leads to

$$f^{123} = 1,$$

$$f^{147} = f^{156} = f^{246} = f^{345} = -f^{367} = \frac{1}{2},$$

$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}.$$  

(15)  

(16)  

(17)

In general, the structure constants are anti-symmetric in their indices. In addition, given the commutator identity

$$[t^a, [t^b, t^c]] + [t^b, [t^c, t^a]] + [t^c, [t^a, t^b]] = 0,$$

the structure constants satisfy the Jacobi identity

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0.$$

(18)  

(19)

This relation shows that if we define the $N^2 - 1$ real $(N^2 - 1) \times (N^2 - 1)$ matrices

$$(T^a)_{bc} \equiv -if^{abc},$$

(20)

these also satisfy the algebra

$$[T^a, T^b] = if^{abc} T^c.$$

(21)

We conclude then that these matrices form another representation of the SU(N) group, acting on a real vector space of dimension $N^2 - 1$. This representation is called the fundamental representation. The fundamental and adjoint representations are the two representation we need to construct the QCD Lagrangian.

Another important property of the different representations, that we will use in these lectures, are the values of the Casimir operators. The Casimir operators are defined as

$$T^2 = T^a_R T^a_R,$$

(22)

where the subindex R denotes de representation. This operator is an invariant of the algebra and its value is constant for each irreducible representation. For the fundamental and adjoint representations these are

$$t^a_{ij} t^b_{jk} = C_F \delta_{ij},$$

(23)

$$T^d_{ab} T^d_{bc} = C_A \delta_{ac},$$

(24)

where the constants $C_F$ and $C_A$ depend on the rank on the group and are given by

$$C_F = \frac{N^2 - 1}{2N},$$

(25)

$$C_A = N.$$  

(26)

The properties of the algebra we have just described appear frequently in perturbative computations.
2 Lecture 2

2.1 Gauge Symmetry and the QCD Lagrangian

As we have seen in our motivation for the colour degrees of freedom, the symmetry transformation colour is associated with, affects only quarks inside hadrons and not the hadrons themselves, which are singlets under this type of transformation. As a consequence, we should be able to perform different symmetry transformations at different points in space. This means that the dynamics of the strong force must be invariant if we perform a space-time dependent SU(3) rotation, which we can parametrise as

\[ q'(x) = \exp \{i\alpha^a(x)t^a\} q(x) \equiv U(x)q(x) , \quad (27) \]

These local transformations are called gauge transformations. These also appear in QED, where the difference that there only one phase, as opposed to a matrix, becomes space-time dependent. As in QED, demanding that this local symmetry is a symmetry of the theory constraints the possible terms that appear in the Lagrangian. Since the derivatives of fields that appear in kinetic terms do not transform nicely under these gauge transformations, we must introduce a new (gauge) field to construct a covariant derivative. Following the same logic as in QED, we define

\[ (D_\mu)_{ij} = \partial_\mu\delta_{ij} +igt^a_{ij}A^a , \quad (28) \]

where the index \( i,j \) are colour indexes in the fundamental representation and, therefore, the covariant derivative is a matrix in colour space. As announced, we have introduced a new set of gauge field \( A^a \), which in QCD are called gluons, with a an adjoint index. The coupling of the quark fields and the gauge fields is controlled by the coupling constant \( g \), which is the strong force analogue of the electric charge \( e \) in QED.

Even though \( A^a \) is labeled by and adjoint index, \( A \) does not transform in the adjoint representation of the gauge group. To determine how \( A \) transform, we analyse how the covariant derivative must transform to make the Lagrangian invariant. Requiring that

\[ \mathcal{L}_q = \bar{q} \left( i\not\!D - m \right) q , \quad (29) \]

is invariant implies that

\[ (D_\mu q)' = U(x)D_\mu q , \quad (30) \]

where we have suppressed the colour indexes for simplicity of the notation. Performing the transformation

\[ D'_\mu q' = (\partial_\mu +igt^a A^a_\mu) U q , \quad (31) \]

\[ = (\partial_\mu U + U\partial_\mu +igt^a A^a_\mu U) q \quad (32) \]

\[ = U (\partial_\mu +igt^a A^a_\mu) q , \quad (33) \]

implies

\[ \partial_\mu U +igt^a A^a_\mu U = Uigt^a A^a_\mu \quad (34) \]
and therefore,
\[ t^a A^\mu_a = U t^a A^\mu_a U^{-1} + \frac{i}{g} (\partial_\mu U) U^{-1}. \] (35)

This expression shows how the gauge fields transform for large gauge transformations, in which the parameters \( \alpha \) are arbitrary. For infinitesimal transformations, i.e., when \( \alpha \)'s are arbitrary small, we find
\[ A'_\mu^a = A^\mu_a - \frac{1}{g} (D_\mu \alpha)^a \] (36)
where \( \alpha \) possesses an adjoint index, and therefore \( D \) in this expression is the covariant derivative in the adjoint representation, given by
\[ (D_\mu)_{ab} = \partial_\mu \delta_{ab} + ig T^c A^c_{ab} \] (37)
The detailed derivation of this result is a good exercise. The form Eq. (35) is analogous to the transformation of the gauge field in QED, replacing the derivative by the covariant derivative.

We can also define the non-abelian field strength as
\[ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^a_\mu A^b_\nu. \] (38)
The first term two terms are identical to the field strength in electromagnetism. The last term is purely non-abelian and implies that the gauge fields can interact with themselves, as we will see when we discuss the Feynman rules. Since this term is proportional to \( g \), if the gauge coupling vanishes, the non-abelian gauge theory becomes identical to the abelian one, up to the number of degrees of freedom. Defined in this way, the non-abelian field strength is gauge co-variant
\[ F'_{\mu\nu} = \exp \{ i \alpha^m T^m \} F_{\mu\nu} \] (39)
This result is easy to check for infinitesimal transformation, which you can do as a exercise. Because of this property and the fact that the adjoint representation is real \( T^a = - (T^a)^\ast \), it is also easy to see that
\[ \mathcal{L}_{\text{Gauge}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \] (40)
is gauge invariant. Combining Eq. (40) and Eq. (29), the classical QCD Lagrangian is given by
\[ \mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_f \bar{q}_f (iD - m_f) q_f, \] (41)
where we have added the index \( f = u, d, s, c, b, t \) is the quark flavour and \( m_f \) is the corresponding quark mass.

2.2 Symmetries of QCD
Prior to start analysing different processes described by QCD, it is important to describe the symmetries of the gauge theory. As we have seen, QCD enjoys a non-abelian gauge
symmetry, with which we have constructed the Lagrangian. But in addition, the classical Lagrangian, Eq. (41) possesses several global symmetries and approximate symmetries. Some of the symmetries of the classical Lagrangian are not preserved at quantum level, due to anomalies. We will discuss those symmetries below.

2.2.1 Conformal Symmetry

Let us first consider the QCD Lagrangian for vanishing quark masses $m_f$. In this case, the Lagrangian does not possess any dimension-full parameter and it is therefore invariant under scaling transformations

$$x \rightarrow \lambda x,$$

$$q_f \rightarrow \frac{1}{\lambda^{3/2}} q_f,$$

$$A \rightarrow \frac{1}{\lambda} A.$$

where we have suppressed all the colour indexes. Nevertheless, this symmetry of the classical Lagrangian is not preserved at quantum level. As we will see in the next lecture, after renormalisation a new scale emerges dynamically in the theory and, therefore, this symmetry is anomalous. For QCD this dynamical scale is of order $\Lambda_{QCD} \sim 200 \text{ MeV}$.

The emergence of this scale allows us to better understand why the $m_f \rightarrow 0$ limit is relevant for (low energy) QCD. The masses of the $u$ and $d$ quarks are $m_u \sim 2 \text{ MeV}$ $m_d \sim 5 \text{ MeV}$, both much smaller than $\Lambda_{QCD}$. Therefore, symmetries of the classical Lagrangian with $m_f = 0$ are also approximate symmetries of QCD in this flavour sector (provided they are not anomalous). The $s$ quark is also light, $m_s \sim 100 \text{ MeV} < \Lambda_{QCD}$; nevertheless, symmetries of the massless Lagrangian are less accurate for this flavours. All other quarks are much heavier than $\Lambda_{QCD}$.

2.2.2 Flavour Symmetries

For each individual flavour we can perform a unitary transformation, which leaves the Lagrangian invariant, of the type

$$q_f = \exp \{i \alpha \} q_f,$$

where the transformation is a global phase, which does not affect the colour, flavour or Lorentz indexes of the quark fields. The conserved current of this symmetry is

$$J^f_{\mu} = \bar{q}_f \gamma_\mu q_f, \quad (\text{no sum over } f \text{ index}),$$

which is the quark flavour current. The conserved quantities associated to these symmetries are the $u$, $d$ ... quark number conservation. This symmetry holds for all flavours in QCD.

In addition, as we have argued, restricting ourselves to the light flavours there are additional global symmetries of the Lagrangian associated to flavour exchanges. If we neglect the
quark masses, the different flavours are indistinguishable in the Lagrangian. We therefore can rotate the flavour indexes of the quarks by performing unitary transformations

\[ q'^f = \Omega f' q_f, \]

where the indexes \( f, f' \) make explicit that this rotation affects the flavour indexes (but not the colour nor spin). Restricting ourselves to the two lightest flavours, \( f = u, d \), the group of unitary transformations

\[ U(2)_V \sim SU(2)_V \times U(1)_V \]

where we have added the subindex \( V \), which denotes that the conserved currents of this symmetry are vectors. These unitary transformation may be also represented as

\[ \Omega = \exp \left\{ i \sum_0^3 \sigma_i \alpha^i \right\} \]

where \( \sigma_0 = 1 \), the identity matrix, and the rest are the Pauli matrices, which are the generators of SU(2). The parameters \( \alpha^i \) are constants and, since this is a global symmetry, do not depend on the space-time point. Using the standard procedure, it is easy to see that the Noether current associated to this symmetry is given by

\[ J^i_{\mu} = \bar{q} \gamma_{\mu} \sigma^i q, \]

where \( J^0_{\mu} \) is known as baryon current and \( J^3_{\mu} \) are the isospin currents. The \( u \) and \( d \) currents are a subset of those currents, which can be obtained from combinations of \( J^0 \) and \( J^3 \).

If instead of performing rotations involving only two light flavours we incorporate the \( s \) quark, we obtain a larger global symmetry group \( U(1) \times SU(3) \). It is important to stress that this flavour \( SU(3) \) has nothing to do with the gauge group. The latter is a local symmetry of the internal degrees of freedom of quarks, the former is a (approximate) global symmetry of the Lagrangian arising by exchanging the \( u, d \), and \( s \) quarks among themselves. As we have stressed, the fact that the \( m_s \) is comparable to the \( \Lambda_{QCD} \) makes the enlarged \( U(3) \) symmetry much less approximate than the isospin symmetry.

### 2.2.3 Chiral symmetry

Continuing with the massless limit, we can discuss another set of flavour symmetries associated to chirality. For massless fermions, helicity states, which are the projection of the spin of the fermion on the direction of their momentum, are preserved. This is a consequence of the fact that in the \( m_f = 0 \) limit, \( \mathcal{L}_q \), Eq. (29) splits into two as

\[ \mathcal{L}_q = \bar{q}_L i \slashed{D} q_L + \bar{q}_R i \slashed{D} q_R \]

where we have suppressed flavour indexes and

\[ q_L = P_L q, \quad P_L = \frac{1}{2} (1 - \gamma_5), \]

\[ q_R = P_R q, \quad P_R = \frac{1}{2} (1 + \gamma_5), \]
Therefore, in the massless limit the flavour symmetries of the classical Lagrangian are enlarged by rotating the $L$ and $R$ fields independently in flavour space.

\[ q_L' = \exp \left\{ 4 \sum_0^4 \alpha_L \sigma_i \right\} q_L, \quad (55) \]

\[ q_R' = \exp \left\{ 4 \sum_0^4 \alpha_R \sigma_i \right\} q_R, \quad (56) \]

The vector rotations we discussed before are obtained when $\alpha_L = \alpha_R$. If we choose $\alpha_L = -\alpha_R$ we obtain \textit{axial} currents, given by

\[ J^i_{A\mu} = \bar{q} \gamma^\mu \gamma_5 \sigma^i q \quad (57) \]

The total group of flavour transformations may then be expressed as

\[ U(1)_V \times SU(2)_V \times U(1)_A \times SU(2)_A \quad (58) \]

Unlike $U(1)_V$, $U(1)_A$ is anomalous, which means that loop diagrams explicitly break this symmetry. $U(1)_A$ is, therefore, \textit{not a symmetry of the quantum theory}. On the contrary $SU(2)_A$ is not anomalous and it is therefore a symmetry of the theory even at quantum level. However, this symmetry is not observed in nature, since the masses of vector and pseudo-vector mesons are different. This symmetry is, in fact, spontaneously broken; in the QCD vacuum there is a non-zero expectation value of the quark condensate

\[ \langle \Omega_{QCD} | \bar{q}q | \Omega_{QCD} \rangle \sim (250 \text{ MeV})^3 \quad (59) \]

The process of symmetry breaking is non-perturbative, as it can be inferred from the value of the condensate, and we will not describe it in this course. In the second part of the course we will explore in detail another example of spontaneous symmetry breaking. Nevertheless, independently of the precise mechanism of breaking, Goldstone theorem implies that there must be massless degrees of freedom in the QCD spectrum, associated to the breaking of the symmetry. Since chiral symmetry is only an approximate symmetry, these degrees of freedom are not identically massless, but much lighter than all other excitations. These are the pions. Enlarging the symmetry group to include the $s$ quark, there are additional light degrees of freedom, which correspond to kaons.

### 2.3 CP symmetry and the $\theta$ parameter

We finish the discussion of symmetries of the Lagrangian with discrete symmetries. As in QED, the classical Lagrangian Eq. (41) is invariant under charge conjugation (C), parity (P) and time reversal symmetry (T), all of which have been experimentally tested in strong force mediated processes. Nevertheless, these symmetries deserve some additional considerations.

The classical Lagrangian Eq. (41) is not, in fact, the most general gauge invariant Lagrangian we can construct without adding dimension-full parameters. An additional term

\[ \mathcal{L}_\theta = -\frac{\theta g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a,\mu\nu} \quad (60) \]
with
\[ \tilde{F}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{a\rho\sigma}, \] (61)
can be added to the Lagrangian. The presence of \( \epsilon_{\mu\nu\rho\sigma} \) indicates that this term breaks CP symmetry. A similar term could have also been added to the QED Lagrangian. However, in QED this term is disregarded because it only contributes as a boundary term since
\[ F^{a\mu}_{\nu\rho} \tilde{F}^{a,\mu\nu} = \partial_\mu K^\mu, \] (62)
with
\[ K^\mu = 2 \epsilon^{\mu\nu\rho\sigma} A^a_\nu \left( \partial_\rho A^a_\sigma - \frac{2}{3} g f^{abc} A^b_\rho A^c_\sigma \right). \] (63)

In perturbative theory the same is true for QCD. However, when non-perturbative effects are taking into account, this is no longer true.

The origin of the relevance of this new term in QCD is the existence of gauge field configurations in the QCD vacuum which are singular at a discrete number of space points but which have regular field strengths, and therefore, well behaved actions. For these configurations, the boundary of space is now the spheres surrounding these singular points. It is a property of SU(3) that the integral of the vector \( K \) on the surface of the sphere is an integer. This is because the group SU(3) also contains a sphere, and the integer counts the number of times the sphere within the group winds around the boundary. This is known as a topological effect. These configurations lead to a sensitivity on the value of the parameter \( \theta \) in the QCD vacuum and spectrum.

Even though a \( \theta \) term can be added to the Lagrangian, experimental constraints on the CP violation by strong process constraint the value of \( \theta < 10^{-10} \). The smallness of this value is not understood and it is known as the strong CP problem. A popular mechanism that could explain this small value is that \( \theta \) is in fact a new field, called axion, which today acquires a vacuum expectation value very close to zero. We will not discuss this mechanism in this course, since it goes beyond our current understanding of the Standard Model.

### 2.4 Feynman Rules

To perform perturbative computation in QCD we need to discuss its Feynman rules. To do so, we must also consider additional terms in the Lagrangian, arising after quantisation. In general, the Lagrangian contains three types of terms
\[ L = L_{\text{classical}} + L_{\text{fixing}} + L_{\text{ghost}}, \] (64)
where \( L_{\text{fixing}} \) is the gauge fixing term and \( L_{\text{ghost}} \) is the ghost lagragian, which depends of the particular gauge fixing conditions. Some of the Feynman rules depend on the gauge fixing condition, while others are independent of the gauge fixing choice. To be concrete, we will show the Feynman rules in covariant gauge, which implies
\[ L_{\text{fixing}} = -\frac{1}{2\lambda} \left( \partial^\mu A^a_\mu \right)^2, \] (65)
\[ L_{\text{ghost}} = \partial_\mu C^a \dagger D^\mu_{ab} B^b, \] (66)
The gauge choice independent Feynman rules are displayed in figure 1. The fermion propagator is identical to the QED one, up to the trivial colour factor. The coupling of fermions to gauge bosons is analogous as well, with the addition of the colour matrix $t^a$ connecting the colours of the incoming and outgoing quarks. Unlike QED, however, gluons interact among themselves with three and four gluon interactions. This is a consequence of the triple gauge field terms in the non-Abelian field strength.

The gauge dependent Feynman rules are displayed in figure 2 in the covariant gauge. The gluon propagator depends explicitly of the gauge-fixing parameter $\lambda$. This is common to QED in covariant gauge. In addition, there is a trivial colour factor. The other two vertices correspond to the ghost fields, as encoded in $\mathcal{L}_{\text{ghost}}$. 
Figure 2: Gauge fixing dependent Feynman rules. Figure taken from Ellis, Stirling and Webber.
# Lecture 3

## 3.1 The QCD $\beta$-function

As we have mentioned, although the classical QCD Lagrangian is scale invariant, after renormalisation an arbitrary scale is introduced into the Lagrangian. Choosing, for example, dimensional regularisation, the divergences that appear in loop computations may be absorbed in a number of renormalised parameter and renormalised fields. Renormalised perturbation theory therefore states that the Lagrangian of the field theory in terms of bare quantities is identical to the renormalised Lagrangian

$$L_{\text{QCD}}(q_0, A_0, g_0, m_0) = L_{\text{R}}(q_R, A_R, g_R, m_R \mu),$$

where the renormalised fields are

$$q_0 = Z_2^{1/2} q_R, \quad A_0 = Z_3^{1/2} A_R,$$
$$g_0 = Z_g \mu^\epsilon g_R, \quad m_0 = Z_m m_R,$$

with $\epsilon = (4 - n)/2$ the dimensional regularisation parameter and $\mu$ an arbitrary mass scale which is introduced in the Lagrangian since in space-time dimensions different from $n=4$ the gauge coupling becomes dimension-full. The different renormalisation constants $Z_i(\epsilon)$ are divergent and are introduced to cancel the divergences in loops. These are related to the counter terms $\delta Z_i$ as $Z_i = 1 - \delta Z_i$. The expression for $Z_i$ depends on the renormalisation scheme. In minimal subtractions-like schemes, in which only the poles in $\epsilon$ and constants are removed from the divergent diagrams, the $Z_i$ do not depend on masses

$$Z_i = Z_i(\epsilon, g_R).$$

Because after renormalisation we have explicitly introduced an arbitrary scale $\mu$ in the Lagrangian, all calculations we perform become explicitly dependent on this parameter. Similarly, all parameters in the Lagrangian depend on $\mu$ via their relation with the bare parameters. We may then view the renormalisation procedure as a set of Lagrangians that are equivalent to each other as long as they are connected to the same bare Lagrangian. The parameter $\mu$ selects a curve in the space of possible Lagrangians which yield the same physics properties. We can study how the coupling changes in this set of Lagrangian by determining

$$\beta(g_R(\mu)) = \lim_{\epsilon \to 0} g_0^2 \frac{d}{dg_R} \delta Z_g(g_R).$$

Using the perturbative expansion, the $\beta$-function can be related to the (divergent) $Z_g$ constant, noting that $g_0$ does not depend on the scale, as

$$\beta(g_R) = \lim_{\epsilon \to 0} g_0^2 \frac{d}{dg_R} \delta Z_g(g_R).$$
To determine $Z_g$ we need to perform a loop computation. The diagrams needed to determine the $\beta$-function are shown in the figure (3). After analysing those diagrams, we obtain

$$\beta = -\frac{g_R^2}{(4\pi)^2} b_0, \quad b_0 = \left[ \frac{11}{3} N - \frac{2}{3} n_f \right].$$  

(73)

Note that for $n_f = 3...6$ active flavours, the $\beta$-function is negative. This implies that as the scale factor increases the coupling constant becomes smaller and smaller. This property is called asymptotic freedom, which implies that QCD at large values of the scale behaves like an almost free theory. Note also that this is the opposite behaviour to QED, which possesses a positive $\beta$-function.
3.2 Running coupling constant

As we have seen, the renomalisation procedure relates different Lagrangians which contains the same physics and the $\beta$ function allows us to relate the coupling constants of these theories to one another. At first sight this only concerns the "space of theories" and it seems not clear what such relation can have to physical observables. In this section we will discuss how this renormalisation group flow can be used to determine properties of such observables.

For simplicity, let us consider some cross section which depends only on one kinematic variable $Q$. An example of this is the total cross section of electron-positron annihilation to hadrons, where $Q = \sqrt{s}$, which we will analyse in the next section. After computing a calculation with our renormalised Lagrangian, and using dimensional analysis, we can express the cross section as

$$\sigma(Q) = \frac{1}{Q^2} f \left( \frac{Q^2}{\mu^2}, g_R(\mu), \frac{m_R^2}{\mu^2} \right)$$

where we have introduced the dimensionless function $f = Q^2 \sigma(Q)$. We will take the $m_R = 0$ limit for the moment and come back to this point later. In this limit, in a classical calculation $f$ should be a constant, independent of the energy $Q$. However, quantum effects introduce a momentum dependence in $f$ via ratios of $Q/\mu$.

Since a cross section is an observable that can be measure, it cannot depend of the arbitrary renormalisation scale $\mu$ which we must introduce to performed the regularised computation. Therefore, the computed function $f$ must satisfy

$$\mu \frac{d}{d\mu} f = 0 = \mu \frac{\partial}{\partial \mu} f + \beta \frac{\partial}{\partial g_R} f.$$  

This equation allows us to determine the function $f$ for different equivalent theories (different $\mu$ and $g_R$) for a fixed kinematical value $Q$.

However, we can use the fact that $f$ is dimensionless by construction to relate changes of $\mu$ to changes of $Q$

$$\mu \frac{\partial}{\partial \mu} f = -Q \frac{\partial}{\partial Q} f,$$

This relation allow us to transform the equation Eq. (75) into a evolution equation for the cross section which determines the cross section at different $Q$ values for a fixed $\mu$. After changing the $\mu$ and $Q$ derivatives, the solution of Eq. (75) leads to

$$f \left( \frac{Q}{\mu}, g_R(\mu) \right) = F \left( \tilde{g} \left( \frac{Q}{\mu} \right) \right),$$

i. e. the cross section becomes a function only of a running coupling $\tilde{g}$, which satisfies

$$\frac{\partial}{\partial \log(Q/\mu)} \tilde{g} = \beta(\tilde{g}), \quad \tilde{g} \left( Q = \mu \right) = g_R(\mu)$$

or, in other words, $\tilde{g}$ is determined implicitly by solving

$$\log \left( \frac{Q}{\mu} \right) = \int_{g_R(\mu)}^{\tilde{g}} \frac{dx}{\beta(x)}.$$
Using the one loop $\beta$-function Eq. (73) and introducing $\alpha_s = g_R^2 / 4\pi$, the solution of Eq. (79) is given by

$$\bar{\alpha}_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{b_0}{4\pi} \alpha_s(\mu^2) \log \frac{Q^2}{\mu^2}}.$$  

(80)

As a consequence of asymptotic freedom, independently of the value of $g_R(\mu)$, as the energy scale $Q$ grows the running coupling constant becomes smaller and smaller and pertubative techniques become better justified. On the contrary, as $Q$ decreases the coupling grows and non-perturbative dynamics become important.

The running coupling constant allows us to determine the evolution of a cross section with $Q$ provided we can determine the function of the coupling $F(g)$ appearing in the right hand side of Eq. (77). To do so, we must determine the cross section via a calculation at our scale of choice $\mu$ for one value of the energy scale $Q_0$. Then, the function is given by

$$F(g_R(\mu)) = f \left( \frac{Q_0}{\mu}, g_R(\mu) \right),$$  

(81)

which, since we have fixed both $\mu$ and $Q_0$ is only a function of $g_R$. Let us assume now that we choose $\mu$ large enough such that $g_R(\mu) \ll 1$ and perturbation theory is applicable. In this case, we can expand the cross section as

$$F(g_R(\mu)) = F^{(0)} + \alpha_s(\mu) F^{(1)} + ...$$  

(82)

where the coefficients $F^{(i)}$ are just numbers. In principle, we can choose any value of $\mu$ to perform this computation, provided $g$ is small. However, the running coupling instruct us on how to best select the scale. To see this, let us expand the running coupling in powers of $\alpha_s(\mu)$

$$\bar{\alpha}_s(Q^2) = \alpha_s(\mu^2) \left[ 1 + \sum_{n=1}^{\infty} \left( -\frac{b_0}{4\pi} \alpha_s(\mu) \log \frac{Q^2}{\mu^2} \right)^n \right].$$  

(83)

Inspecting this expression reveals something important. Even if we choose $\alpha_s(\mu) \ll 1$, if the momentum scale $Q$ at which we perform the calculation is very different from $\mu$, large logs in the calculation of the cross section appear. Since these large logs multiply each subleading power of the coupling, the combination $\alpha_s \log Q^2/\mu^2$ can become large, spoiling the perturbative series. The running coupling resums these large logs into the $\bar{\alpha}$. But this also shows that, in other to avoid these large logs, to determine $F$ it is best to choose a renormalisation scale $\mu \sim Q_0$.

Choosing, for concreteness $\mu = Q_0$, we can compute the cross section at $Q_0$ and determine the coefficients $F^{(i)}$ in Eq. (81). At this one scale, the cross section is fully determined by the perturbative expansion and we can compare it with a measurement to determine the value of $\alpha(Q_0)$. As an example, we could determine the total cross section of electron-proton annihilation into hadrons at $Q = M_Z$ and extract from there the value of $\alpha(Q_0 = M_Z)$. Experimentally, it is determined that $\alpha(M_Z) = 0.1181 \pm 0.0013$. Using the solution to Eq. (71)
given by Eq. (80) we can relate the QCD coupling constant at different scales. By a simple reorganisation of Eq. (80) we can express the coupling constant as

$$\alpha_s(Q^2) = \frac{4\pi}{b_0 \log \frac{Q^2}{\Lambda_{QCD}^2}}$$  \hspace{1cm} (84)$$

where $\Lambda_{QCD}$ is a dynamically generated scale which emerges from the process of renormalisation. Its precise value depends on the renormalisation scheme and on the number of active flavours. For three active flavours $\Lambda \sim 200$ MeV. As we can see, after determining $\Lambda_{QCD}$ the running coupling constant $\bar{\alpha}_s(Q) = \alpha_s(\mu = Q)$. Finally, lets us examine the cross section to fixed order in $\alpha_s(Q_0)$ at arbitrary momentum. Following the above, we conclude

$$\sigma(Q) = \sigma_0 + \alpha(Q)\sigma_1 + ...$$ \hspace{1cm} (85)$$

$$= \frac{1}{Q^2} \left( F^{(0)} + \alpha(Q)F^{(1)} + ... \right).$$  \hspace{1cm} (86)$$

The all orders calculation must be independent of our choice for $\mu$. However, this is not the case for any truncated order. Consider, for example the result of the perturbative expansion Eq. (82) evaluated at different scales, $\mu = \zeta Q_0$ with $\zeta$ a number of order 1. Using the $\beta$-function we can compute the relation between the coupling constants in two different scales

$$\alpha_s(\mu) = \alpha(Q_0) \left( 1 + \frac{b_0}{4\pi} \alpha(Q_0) \log \frac{1}{\zeta^2} + O(\alpha^2) \right)$$  \hspace{1cm} (87)$$

which means that the coupling constant between these two scales coincide only at leading order. Therefore, by varying the renormalisation scale we obtain answers which differ at a higher order in $\alpha_s$ than the truncation order (second order in the case of Eq. (82)). Therefore, we can use the dependence of the computation on the renormalisation scale as a way to estimate the contribution of higher order in the expansion beyond our chosen truncation order. It is a routine in perturbative computation to use those variations as measure of the theoretical uncertainty.

### 3.3 Infrared safety

We now turn back to the issue of the mass of the quarks, which in the previous analysis we have set to zero. In general, the quark mass parameter will not be zero, but, via renomalisation it also inherits a scale dependence, via the mass renormalisation constant. Therefore, also for the quarks mass there is a renormalisation flow equation

$$\mu \frac{d}{d\mu} m_R(\mu) = -\gamma_m(g_R)m_R,$$  \hspace{1cm} (88)$$

which, following the same steps that led to the definition of the running coupling, leads to a running mass that satisfies

$$\frac{\bar{m}(Q)}{\mu} = \frac{m_R(\mu)}{Q} \exp \left\{ - \int_0^{\log(Q/\mu)} dt \gamma_m(g(t)) \right\}.$$ \hspace{1cm} (89)$$
Since in an asymptotically free theory, at high scale $g$ is small, $\gamma_m$ can be treated perturbatively and it is therefore small. This implies that at large momentum the effect of a finite quark mass is small and the cross section is better and better approximated by the massless limit\(^1\).

However, not all cross sections are well defined in the $m \to 0$. As we will see in the next lecture, in this limit collinear singularities appear if the observable is not carefully defined, which makes the massless limit ill-defined. The set of observables which are well defined in this limit are generically called infrared safe and are particularly important in QCD, since they do not show sensitivity to the non-perturbative small values of the light quark masses. In the next lecture we will discuss this issue in more detail.

\(^1\)Note that should the theory not be asymptotically free, then $\gamma_m$ could grow at large energies and it would be not guaranteed that the mass parameter decouples.
4 Lecture 4

4.1 QCD in $e^+e^-$ collisions

Let’s start our analysis of QCD process in a particularly clean example, the annihilation of electro-positron pairs into hadrons. If we perform a fully inclusive measurement and we do not specify any cuts, this cross section is solely determined by the centre of mass energy of the lepton pair. This is the type of observables we studied in the previous section, for which the scale $Q = \sqrt{s}$.

Following the discussion in the previous lecture, for a large value of $Q$ it is convenient to choose $\mu = Q$ and expand the cross section as

$$\sigma_{e^+ e^- \rightarrow \text{hadrons}} = \sigma_0(Q) + \alpha(Q)\sigma_1$$

(90)

Therefore, in spite of being a complicated observable, with many low energy and non-perturbative processes occurring on an event-by-event basis, as long as we do a fully inclusive measurement $\sigma_0$ is given by the lowest order diagram, which is independent of $\alpha_s$, shown in figure (4).

![Figure 4: Leading order Feynman graph for $e^+e^-$ annihilation into hadrons.](image)

As shown in figure (4), the leading order QCD diagram is identical to the leading order QED diagram of $e^+e^- \rightarrow \mu^+\mu^-$, up to the trivial factor of the electric charge of the produced quarks, $Q_f$. Since in the cross section we must sum over all final states, the annihilation into hadrons is enhanced by an additional $N$-factor with respect to the annihilation into muons, because of the colour degrees of freedom. Therefore, the fraction

$$R = \frac{\sigma_{e^+ e^- \rightarrow \text{hadrons}}}{\sigma_{e^+ e^- \rightarrow \mu^+\mu^-}} = N \sum_f Q_f^2,$$

(91)

Where the sum runs over the number of active flavours, that is, the quarks with masses lighter than the Q. The experimental measurement of this quantity is shown in figure (5). At low
energies, the $R$ parameter shows a lot of structure, corresponding to the formation of hadronic vector resonances at different scales. In between these resonances, $R$ is approximately flat, according to the corresponding number of active quarks. At high $\sqrt{s}$ the measured ratio of cross section agrees well with Eq. (91) with all quarks, but the top quark, active. This measurement constitutes an experimental verification that there are indeed $N = 3$ colours. In the next subsection we will attempt to determine this asymptotic value of the $R$ parameter with higher accuracy in QCD by computing the next-to-leading correction to the annihilation cross section.

4.2 Infrared and collinear singularities

To compute the next-to-leading order correction to the annihilation cross section we must first identify the relevant QCD diagrams that we need to compute. Following what we have learned in the previous section, we only need to consider the cross section with additional coupling insertions. At the level of the amplitude, there are two graphs we must consider to determine the next-to-leading order amplitude. Expanding the amplitude as

$$M = M^{(0)} + M^{(1)} + M^{(2)} + ...$$

(92)

with the index indicating the order of the $g$ expansion, the diagrams contributing $M^{(1)}$ are shown in figure (6).

These correspond to the contribution of real emissions of final state gluons to the final
cross section. Using the QCD Feynman rules these diagrams are given by

\[
\mathcal{M}^{(1)} = \bar{u}(k) (igt^a) \epsilon(l) \frac{i}{k + l} (-ie) \gamma(q) (i) v(k') + \\
+ \bar{u}(k) (-ie) \gamma(q) (i) \frac{i}{-k' - l} (igt^a) \epsilon(l) v(k'),
\]

where \(\epsilon(l)\) and \(\epsilon(q)\) are the polarisation of the gluon and the virtual photon respectively.

This amplitude reveals something worrisome. In the massless limit, which as we have seen is the relevant limit for high energy processes, the amplitude has divergent denominators

\[
\frac{1}{k + l} = \frac{k + l}{2k \cdot l} = \frac{k + l}{2k^0 l^0 (1 - \cos \theta)},
\]

with \(\theta\) the angle between the quark and the gluon, which show apparent divergences when the external gluon is soft \(l^0 \ll 1\) or when the gluon is collinear with the quark \(\theta \sim 0\). Analogous divergences appear for the second diagram. To evaluate whether these indeed lead to a divergence in the cross section, let us focus on the infrared dynamics and perform and infrared approximation. This amounts to neglecting the emitted gluon momentum in numerators, while keeping it in denominators, since, as we have seen, in the quark propagator this leads to non trivial divergences. In this limit the amplitude can be simplified significantly by using the following identities:

\[
\begin{align*}
\bar{k}' v(k') &= 0, & \bar{u}(k) k &= 0, \\
\bar{k}' \gamma_\nu v(k') &= 2k'_\nu v(k'), & \bar{u}(k) \gamma_\nu k' &= 2\bar{u}(k) k_\nu
\end{align*}
\]

where the in the first line we have use Dirac’s equation and in the second line the anticommutation relations of \(\gamma_\nu\).

With this approximations the next-to-leading order amplitude is

\[
\mathcal{M}^{(1)} \approx \bar{u}(k) (-ie) \gamma(q) (i) g \left( \frac{k \cdot \epsilon}{k \cdot \epsilon(l)} - \frac{k' \cdot \epsilon(l)}{k' \cdot l} \right) t^a,
\]

\[
\approx \mathcal{M}^{(0)} g \left( \frac{k \cdot \epsilon}{k \cdot \epsilon(l)} - \frac{k' \cdot \epsilon(l)}{k' \cdot l} \right) t^a,
\]
where in the second line we have factored out the leading order matrix element. Because of this factorisation, in the infrared approximation the effect of gluon emission can be viewed as a multiplicative factor of the overall cross section. Taking the square of the matrix element and multiplying by the gluon phase space and corresponding flux factors, the contribution of this matrix element to the cross section is given by

$$\sigma_1^R = \sigma_0 C_F g^2 \int \frac{d^3l}{(2\pi)^3 2l^0} \frac{1}{(k \cdot l)(k \cdot l') \cdot (k' \cdot l')}$$

(99)

where we have used that $\text{tr} \{t^a t^b\} = C_F N$ and $N$, with absorbed in the leading order cross section, and that the sum over polarisation $\sum \epsilon_\mu \epsilon'^\nu = -g_{\mu\nu}$ for transverse structures.

The integral in Eq. (99) is divergent in several ways

1. UV divergence: as $l$ becomes large, the integral is divergent as

$$\frac{d^3l}{l} \sim \log$$

(100)

However, this is nothing to worry about, since we have performed an approximation which is valid only for soft momentum. Possible UV divergences of the true integral should have been dealt by the renormalisation procedure.

2. Infrared (IR) divergence: when the gluon momentum is soft $l$, the integral is also of the form above and also diverges logarithmically in the infrared. This is a problem, since this is a divergence in the region the field theory is not altered by renormalisation.

3. Collinear divergence: the polar angle integration $\theta$, with $\theta$ the angle between the gluon and the quark-antiquark axis in the centre of mass frame is

$$\int \frac{dx}{1 - x^2} = \frac{1}{2} \log \frac{1 + x}{1 - x}$$

(101)

with $x = \cos \theta$, which is also divergent whenever $\theta = 0, \pi$, i.e. whenever the emitted gluons are collinear with the quark or the antiquark. This is also a physical divergence, emerging in for Minkowski signature, and cannot, therefore, be removed via renormalisation.

This seems frustrating. As soon as we study our first radiative correction we find an uncontrolled divergence which apparently cannot be removed. Furthermore, this appears for a very simple observable, in which no kinematic cuts are applied and certainly something that can be measured and should be finite. Luckily, under close examination we can conclude that we have not computed all the contribution to the next-to-leading order cross section. In fact, terms there are contributions from the next-to-next-to leading order amplitude which also contribute to the next to leading order cross section. Indeed, squaring the matrix element

$$\mathcal{M} \mathcal{M}^\dagger = \mathcal{M}^{(0)} \mathcal{M}^{(0)\dagger} + \mathcal{M}^{(1)} \mathcal{M}^{(1)\dagger} + \mathcal{M}^{(2)} \mathcal{M}^{(0)\dagger} + \mathcal{M}^{(0)} \mathcal{M}^{(2)\dagger}$$

(102)
Therefore, there are contributions to the next-to-leading order cross section which depend on $\mathcal{M}^{(2)}$, those in which there are no gluons in the final state. These are the virtual correction shown in figure (7). To evaluate those we need to compute those loop corrections, which demand regularisation. The self energy corrections displayed in the right hand side of figure (7) vanish using dimensional regularisation in MS-type schemes, since for these scale-less divergent integrals vanish after regularisation. The vertex correction gives a finite contribution. Clearly, this diagram leads to similar IR&Coll divergences, since the quark and anti-quark propagators in the loop lead to terms of the type

$$\frac{1}{k \cdot l k' \cdot l}$$

(103)

which are also divergent.

Evaluating the vertex correction in $IR$-approximation one obtains

$$\sigma^V = -\sigma_0 C_F \frac{g^2}{2\pi^2} \int \frac{dl}{l} \frac{dx}{1-x^2},$$

(104)

which cancels the real contribution identically. Therefore the virtual correction in this observable cancel both the infrared and collinear divergences of the total cross section and the result is finite. Within the infrared approximation the next-to-leading order correction vanishes identically; however this is an artefact of the approximation, which is only valid to capture the IR&Coll divergent contribution. The full calculation yields

$$R = R_0 \left( 1 + \frac{\alpha_s}{\pi} \right).$$

(105)

The cancelation of divergences we have described is not specific of this fully inclusive observable. This is one example of the Kinoshita-lee-Nauenberg theorem which states that for suitably define inclusive cross section, infrared and collinear divergences cancel.
Note, for example, that if we would have like to determine the cross section of the fully exclusive quark-antiquark-gluon final state, the cross section would be divergent, since it would be only given by $\sigma_R$ and no virtual correction are present. For this reason, only measurements which are not fully exclusive are under good theoretical control in perturbative QCD. In the next lecture we will describe other less inclusive measurements that can be performed in $e^+e^-$ and that can be reliably computed.


5 Lecture 5

5.1 Jets in $e^+e^-$

In the last lecture we saw the cancelation of IR&Coll divergences in fully inclusive measurements in $e^+e^-$ collisions. We would like now to understand if there can be more differential measurements that we can perform in which there is a similar cancelation between real and virtual diagrams.

As we also saw, if we compute the probability rate of emission of a single gluon, this rate possesses both IR&Coll divergences. We may interpret the collinear divergence in physical terms: while the emission of a gluon from the quark is in principle suppressed by an additional power of $\alpha$, this emission rate is enhance by a log $\theta$ term, which make the radiation of almost collinear gluons non-suppressed. This means that in high-energy processes we expect that most of the energetic particles in hard annihilations will be produced in collimated sprays, since there are not suppressions for these collinear processes. These sprays are called jets and they are indeed observed in $e^+e^-$ annihilations. Similarly, the emission of soft particles at any angle is not suppressed either, since the soft divergence can compensate for the smallness of $\alpha_s$ and lead to soft radiation between the jets, or inter-jet radiation. An sketch of this structure is shown in figure (8).

The above discussion of a jet is simply descriptive. To be able to determine the production rate of those sprays of particles it is necessary to provide a precise definition of what we mean with a jet. This definition is not unique and, in fact, not all possible definitions can be reliably computed in QCD. In this lecture we will describe Sterman-Weinberg jets, which depend on two parameters, $\epsilon$ and $\delta$. In an $e^+e^-$ collision an event contributes to the 2-jet cross section if we can find 2 cones of opening angle $\delta$ such that the fraction of the total energy of the event contained in those cones is greater than $1-\epsilon$. These parameters are displayed in figure (8). Note that this definition is well suited for $e^+e^-$ collisions where the total energy in the event is known, but not so much for hadronic collisions. We will discuss other jet definitions in later lectures.

We will now compute the two jet cross section, integrating over all possible jet axis. At leading order, all the energy of the event goes into two quarks and, therefore, into two jets of (arbitrarily small) opening angle. As a consequence,

$$\sigma_{0\text{jets}}^2 = \sigma_0,$$

the total annihilation cross section at this order.

At next to leading order, the emission of gluons can take energy out of a given cone, and the cross section at this order differs from the total cross section. There are three distinct processes that contribute to the jet cross section:

- Emission of hard gluons with energy greater than $\epsilon E$ emitted at angle $\theta < \delta$ from either the quark or the antiquark. We will call this contribution $\sigma_1^H$, which in the infra-red
approximation is given by

\[ \sigma_1^H = C_F g^2 \frac{\pi}{2} \sigma_0 \int_0^E dE \frac{d \theta \sin \theta}{l_0} \left[ \int_0^\delta \frac{d \theta \sin \theta}{1 - \cos^2 \theta} + \int_\pi^{-\delta} \frac{d \theta \sin \theta}{1 - \cos^2 \theta} \right] \]  \quad (107)

- Emission of soft gluons with energy \( \delta E < \epsilon E \) at any angle with respect to the quark. Since these gluons are below the energy cut, they do not alter the jet cross section. We call this contribution \( \sigma_1^S \). In the infrared approximation this is given by

\[ \sigma_1^S = C_F g^2 \frac{\pi}{2} \sigma_0 \int_0^\epsilon E dE \frac{d \theta \sin \theta}{l_0} \int_0^\delta \frac{d \theta \sin \theta}{1 - \cos^2 \theta} \]  \quad (108)

- Virtual corrections. Since at this order virtual corrections do not have real gluon emission, all the virtual correction computed in the last chapter contributes to the jet cross section. In the infrared approximation

\[ \sigma_1^V = -C_F g^2 \frac{\pi}{2} \sigma_0 \int_0^\epsilon E dE \frac{d \theta \sin \theta}{l_0} \int_0^\pi \frac{d \theta \sin \theta}{1 - \cos^2 \theta} \]  \quad (109)

Putting all these terms together, the two jet cross section is given by

\[ = \sigma_1^H + \sigma_1^S + \sigma_1^Y \]  \quad (110)

\[ = -\sigma_0 \left( \frac{g^2}{2\pi^2} C_F \int_\epsilon^E dE \frac{d \theta \sin \theta}{l_0} \int_\delta^{\pi - \delta} \frac{d \theta \sin \theta}{1 - \cos^2 \theta} \right) \]  \quad (111)

As expected, the virtual corrections lead to the cancellation of the IR&Coll divergences. However, these leave an imprint into the cross section, which depends logarithmically on the jet definition parameters. In the \( \delta \ll 1 \) limit we obtain

\[ \sigma^{2\text{jets}} = \sigma_0 \left( 1 - \frac{g^2}{\pi} C_F \log \epsilon \log \delta \right) \]  \quad (112)
The two jet cross section decreases both as $\epsilon$ and $\delta$ decrease. The second effect is easy to understand, since requiring the angle of emission to be smaller and smaller reduces also the phase space for emissions. What is less intuitive is that the reductio of the energy cut also reduces the cross section, which means that demanding the observation of events with less and less soft radiation is rarer and rarer.

The inspection of Eq. (112) also shows that the accuracy of the perturbative computation also depend on the parameters used for the jet definition. For either small $\epsilon$ or $\delta$, the combination $\alpha \log \epsilon \log \delta$ can be of order one, which means that this leading order calculation is inaccurate and resummations of terms like those must be performed.

Finally from this computation, we can also easily determine the three jet cross section. Since at next to leading order the total cross section must be either formed by two or three final particle events, we must have

$$\sigma^{\text{tot}} = \sigma^{2\text{jets}} + \sigma^{3\text{jets}}, \quad (113)$$

we conclude

$$\sigma^{3\text{jets}} = \sigma_0 \frac{g^2}{\pi} C_F \log \epsilon \log \delta \quad (114)$$

### 5.2 Infrared and Collinear Safety

The bottomline of the analysis of the previous section is that not all observables that one can think of can be reliably computed in perturbative QCD. Good observables are those that, like the total cross section or the two jet cross section, include virtual correction that cancel infrared and collinear divergences. Otherwise the observable is sensitive to small non-perturbative scales in the processes, which go beyond perturbation theory. This does not mean that those observable cannot be measured; it simply means that it is not possible to obtain reliable predictions from QCD.

A set of observables that can be computed in perturbative QCD are what is known as infrared and collinear safe observables. An observable depending of the momenta of $n$ particles \(\{k_1, k_2, \ldots, k_n\}\) is infrared and collinear safe if for any pair of particles \(\{i, j\}\)

$$\mathcal{O}\{k_1, \ldots k_i, k_j \ldots k_n\} \to \mathcal{O}\{k_1, \ldots k_i + k_j \ldots k_n\}, \quad (115)$$

whenever either $k_i$ or $k_j$ become soft or the pair becomes collinear with one another. The first condition ensures that the observable is insensitive to soft emissions. The second one ensures that collinear splittings do not change the observable. These observables are, in fact, well defined to all orders in perturbation theory. We will come back to this point later in the course.

As a final remark, let us mention that the Sterman-Weinberg definition of jets we have analysed in the previous section is not, in fact, infra-red and collinear safe, since if the combination of a pair of hard gluons can take the resulting product outside of the angular region $\delta$. In fact, at order $\alpha^2$ these processes lead to a non cancellation of IR&Coll. divergences in that case.
6 Lecture 6

6.1 Deep Inelastic Scattering

Having analysed QCD processes with no hadrons in the initial state, we now move to the
next step in complexity and study high-energy lepton-hadron interactions. In particular, we
focus on the inelastic scattering of an electron off a hadron such that the hadron is totally
destroyed. The process is inclusive in the sense that the we will characterise the cross section
in terms of variables of the incoming $e$ and $p$ and on the scattered $e$ only, without specifying
the rest of the products of the scattering. This is a historically important process, since it led
to the development of the parton model and the discovery of quarks.

The diagrams representing this process to leading order in the electromagnetic coupling
$\alpha_{\text{em}}$ is shown in figure (9).

![Figure 9: Sketch of the DIS process.](image)

The cross section depends on the following invariant variables

\begin{align}
Q^2 & \equiv -q^2 = -(k' - k)^2, \quad x_{\text{Bj}} = \frac{Q^2}{2pq}, \\
y &= \frac{qp}{kp}, \quad \nu = pq.
\end{align}

(116) (117)

In the rest frame of the proton, the last two invariants admit a simple interpretation

\begin{align}
\nu &= M_p (E' - E) \quad \text{Energy lost by the electron,} \\
y &= 1 - \frac{E'}{E} \quad \text{fraction of energy lost by the electron.}
\end{align}

(118) (119)

In full generality, we can separate the QED and hadronic part of the process. To all others
in $\alpha_s$, we may write the amplitude of the process as

\[ \mathcal{M} = e\bar{u}(k')\gamma^\alpha u(k) \frac{1}{q^2} \langle X | j_\alpha(0) | p \rangle , \]

(120)
where \( j_\alpha(0) \) is the electromagnetic current on the strong fields evaluated at the origin of space

\[
j_\alpha(x) = \sum_f e Q f \bar{q}(x) \gamma_\alpha q(x),
\]

and the state \( |X\rangle \) is a generic hadronic state as a result of the scattering and \( |p\rangle \) is the incoming proton state. After squaring this matrix element

\[
\frac{d\sigma}{dQ^2 dx_{Bj}} \propto L_{\alpha\beta} W_{\alpha\beta},
\]

where we have separated the contributions of the cross section into a leptonic contribution, given by the leptonic tensor

\[
L_{\alpha\beta} \equiv e^2 Tr \left[ k\gamma_\alpha k'\gamma_\beta \right],
\]

which can be expressed in terms of products of \( k \) and \( k' \), and a hadronic tensor

\[
W_{\alpha\beta} = \frac{1}{4\pi} \sum_X \langle p| j_\alpha^+(0)|X\rangle \langle X| j_\alpha^+(0)|p\rangle (2\pi)^4 \delta(q + p - p_X),
\]

where \( p_X \) is the total four-momentum of the state \( X \) and we have also summed over all possible final hadronic states. This expression may be further simplified using the fact that \( X \) is a full set of states. It is easy to see that

\[
(2\pi)^4 \delta(q + p - p_X) = \int dz e^{i(q+p-p_X)z}.
\]

Similarly, using the momentum operator \( \hat{P} \)

\[
\langle p| j_\alpha^+(0)|X\rangle e^{i(q-p-p_X)z} = \langle p| e^{i\hat{P}z} j_\alpha^+(0)e^{-i\hat{P}z} |X\rangle = \langle p| j_\alpha^+(z)|X\rangle.
\]

Putting these relations together, we have

\[
W_{\alpha\beta} = \frac{1}{4\pi} \int dz e^{i q z} \langle p| j_\alpha^+(z) j_\alpha^+(0) |p\rangle,
\]

\( i. e. \), we have expressed the hadronic tensor as a correlator of electromagnetic currents evaluated on the proton. This expression is valid to all orders in the strong coupling constant and is sensitive both to perturbative and non-perturbative physics. The number of free functions in the hadronic tensor can be significantly reduced by using the conservation of the electromagnetic current

\[
\partial_\mu j^\mu(x) = 0,
\]

which implies that the hadronic tensor is transverse \( q^\alpha W_{\alpha\beta} = q^\beta W_{\alpha\beta} = 0 \). We can therefore constraint its tensorial structure to be

\[
W_{\alpha\beta} (p, q) = \left( g_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2} \right) F_1 (x_{Bj}, Q^2) + \left( p_\alpha + \frac{1}{2 x} q_\alpha \right) \left( p_\beta + \frac{1}{2 x} q_\beta \right) \frac{1}{\nu} F_2 (x_{Bj}, Q^2).
\]
Neglecting the mass of the hadron, at high energies the cross section reduces to
\[
\frac{d^{2}\sigma}{dx dQ^{2}} = \frac{4\pi\alpha_{em}^{2}}{Q^{4}} \left[ (1 + (1 - y)^{2}) F_{1}(x_{Bj}, Q^{2}) + \frac{1 - y}{x} (F_{1}(x_{Bj}, Q^{2}) - 2x_{Bj} F_{2}(x_{Bj}, Q^{2})) \right] \tag{130}
\]

This expression is general and it makes no assumptions on the structure of the proton. It parametrises the scattering cross section in terms of two functions, \(F_{1}\) and \(F_{2}\) that are properties of the scattered hadron. In the next section we will evaluate these function in a particular model for the proton.

### 6.2 The parton model

In the parton model, the proton is seen, at least at high energies, as a collection of almost free partons, quarks and gluons, which carry a given fraction of the proton momentum \(\hat{p} = \zeta \hat{p}\).

In this model, the DIS cross section at leading order is controlled by the scattering of the electron off a quark within the proton. For this microscopic process, the leading order cross section is easy to compute.

The matrix element of the \(e - q\) scattering at leading order is shown in figure (10). The amplitude is given by
\[
\mathcal{M}^{(0)} = \frac{e^{2}Q_{f}}{q^{2}} \bar{u}(\hat{p}') \gamma^{\mu} u(\hat{p}) \bar{u}(k') \gamma_{\mu} u(k) \tag{131}
\]

After averaging over initial states and summing over final states, the square matrix element is given by
\[
\frac{1}{2} \sum_{\text{pol}} \left| \mathcal{M}^{(0)} \right|^{2} = \frac{e^{4}Q_{f}^{2}}{4q^{4}} N \text{tr} [\bar{p}^{\gamma}_{\nu} \gamma_{\mu} \hat{p}] \text{tr} [k' \gamma_{\mu} l \gamma_{\nu}], \tag{132}
\]
\[
= 8 \frac{Q_{f}^{2} e^{4}}{q^{4}} N \left[ k \cdot \hat{p} k' \cdot \hat{p}' + k' \cdot \hat{p} k \cdot \hat{p}' \right] \tag{133}
\]
which, leads to the partonic cross section

\[
\frac{d\hat{\sigma}}{dQ^2} = \frac{1}{16\pi s^2} \frac{1}{2} \sum_{pol} |\mathcal{M}^{(0)}|^2
\]

\[
= \frac{2\pi\alpha^2 Q_f^2}{Q^4} (1 + (1 - y)^2)
\]

(134)

This simple finding has important consequences for the structure functions of electrohadron collisions in the parton model. Let us know remember that in the parton model the momentum of the quark is a fraction of the momentum of the incoming proton. In fact, the assumption that the proton is made of almost free quarks fixes the value of \(x_{\text{Bj}} = \zeta\), the fraction of momentum taken by the quark. Indeed,

\[
\hat{p}'^2 = 0 = (\hat{p} + q)^2 = 2\hat{p} \cdot q - Q^2 = 2\zeta p \cdot q - Q^2 = 2\nu (\zeta - x_{\text{Bj}})
\]

(135)

Using this result, we can express the cross section for \(e - p\) scattering

\[
\frac{d\sigma}{dQ^4 dx} = \frac{4\pi\alpha_{\text{em}}^2}{Q^2} \left(1 + (1 - y)^2 \frac{1}{2} Q_f^2 \delta(x_{\text{Bj}} - \zeta)\right)
\]

(136)

Which we can interpret as the scattering of an electron off a hadron made of a single quark which carries all the momentum of the quark. Therefore, if the hadron has a distribution of partons with a density of quarks carrying a momentum fraction \(f_q(\zeta)\) the cross section is

\[
\frac{d\sigma}{dQ^4 dx} = \frac{4\pi\alpha_{\text{em}}^2}{Q^2} \sum_f \left(1 + (1 - y)^2 \frac{1}{2} Q_f^2 f_f(x_{\text{Bj}})\right)
\]

(137)

The parton distribution function (PDF) \(f_f\) are non-perturbative objects which characterise the hadron target. These cannot be computed in perturbation theory and need to be determined experimentally. Nevertheless, given their interpretation as density of partons of a given species within the hadron, they must satisfy a number of integral constraints or sum rules. One of them is that summing over all partons the net momentum is the proton momentum, which implies

\[
\sum_f \int_0^1 dx xf_f(x) = 1
\]

(138)

Similarly, the total number of quarks must coincide with the known quark content of the hadron. For the particular case of the proton, the sum rules read

\[
\int_0^1 dx (f_u(x) - f_{\bar{u}}(x)) = 2
\]

(139)

\[
\int_0^1 dx (f_d(x) - f_{\bar{d}}(x)) = 1
\]

(140)

\[
\int_0^1 dx (f_s(x) - f_{\bar{s}}(x)) = 0
\]

(141)
Comparing Eq. (137) with Eq. (130), the parton model provides very strong predictions for the structure functions $F_1$ and $F_2$.

$$F_1(x, Q^2) = \sum_f \frac{1}{2} Q_f^2 f_f(x_{\text{Bj}}),$$

(142)

$$F_2(x, Q^2) = 2 x F_1(x, Q^2).$$

(143)

The second relation is known as Gallan-Gross relation, and it is a consequence of the fact that quarks possess spin 1/2. The first relation, known as Bjorken scaling implies that the function $F_1$ is only a function of $x_{\text{Bj}}$ and independent of $Q^2$. This is a consequence of the fact that the electro scatters of a point like object. Should the scattering proceed off an extended object of typical transverse size $\sigma_p$, we would have expected that the number of partons within a transverse area $1/Q^2$ would be given by the ratio $1/(\sigma_p Q^2)$, which shows a strong dependence

Figure 11: Measurements of $F_2$ from different experiments
on $Q^2$, in contrast to the parton model prediction.

Measurements of $F_2$ by different experiments are shown in figure (11). This structure functions is indeed very week functions of $Q$, and if the interval of $Q$ is not (logarithmically) large, $F_2$ can be well approximated by a constant. The early measurements of this functions performed in SLAC show indeed this behaviour which serve as a confirmation of Bjorken scaling and the parton model. However, for large ranges of $Q$ scaling violations are observed. In the next lecture we will see how these violations arise from radiative correction of the DIS process.
7 Section 7

7.1 Radiative corrections in DIS

We would like to be able to determine the leading order QCD correction of the predictions of the parton model for the value of the structure functions characterising DIS. As in the case of jets, radiative corrections lead to IR&Coll. divergences; but in this case, the cancellation of real and virtual corrections is more intricate than that in that case.

To simplify our analysis let us introduce some more notation. We will separate from the partonic matrix element the contribution of the external parton leg \( u(\hat{p}) \) and the rest of the diagram, such that

\[
\mathcal{M}^{(0)} \equiv \tilde{\mathcal{M}}^{(0)} u(\hat{p}),
\]

So that the partonic cross section is given by

\[
\sigma = \frac{N}{\hat{p}^0} \tilde{\mathcal{M}}^{(0)} \frac{1}{2} \sum \bar{u}(\hat{p}) u(\hat{p}) \tilde{\mathcal{M}}^{(0)\dagger} = \frac{N}{\hat{p}^0} \tilde{\mathcal{M}}^{(0)} \frac{\hat{p}}{2} \tilde{\mathcal{M}}^{(0)\dagger}
\]

We now study the radiation of a gluon from the incoming quark. The diagram is shown in figure (12) Using the notation introduced above, this diagram is

\[
\mathcal{M}^{(1)} = \tilde{\mathcal{M}}^{(0)} (\hat{p} - l) \frac{i(\hat{p} - I)}{(\hat{p} - I)^2} (tg) t^n \xi(l) u(\hat{p})
\]

We see that once again the quark propagator leads to a possible collinear divergence when \( l \cdot k = 0 \), that is, when the intermediate quark has very small virtuality. Since in DIS we
do not specify the gluon momentum \( l \) and we must integrate over all possible values. To focus on the most important region, we introduce the light-cone decomposition of the gluon momentum

\[
l = (1 - z) \hat{p} + l_\perp + \xi \eta,
\]

where \( \hat{p} \) is the (light-like) momentum of the incoming quark, \( \eta \) is another light-like momentum and \( l_\perp \) is a space-like momentum which satisfies

\[
\eta \cdot \hat{p} = 2 \hat{p}^0, \quad l_\perp \cdot \hat{p} = l_\perp \cdot \eta = 0
\]

In coordinates in which \( \hat{p} = (1, 0, 0, 1) \hat{p}^0 \), these four vectors are

\[
\eta = (1, 0, 0, -1), \quad l_\perp = (0, l_x, l_y, 0).
\]

In terms of these variables, the gluon phase space is

\[
d^3l (2\pi)^3 2l^0 \bigg/ \bigg( \frac{d^2l_\perp}{2(2\pi)^3} \bigg) dz = 1 \bigg/ l_\perp^2(1 - z),
\]

Note also that in the requirement that the gluon momentum is light-like fixes all the components of the four momentum once the transverse momentum with respect to the quark \( l_\perp \) and the fraction of the quark momentum taken by the gluon \((1 - z)\) are fixed

\[
\xi = \frac{l_\perp^2}{4 \hat{p}^0(1 - z)}.
\]

Similarly, the pole of the propagator is given by

\[
(l - \hat{p})^2 = -\frac{l_\perp^2}{1 - z},
\]

and, therefore, the collinear divergence occurs when \( l_\perp \rightarrow 0 \).

The apparent divergence in the cross section after squaring the amplitude is \( 1/l_\perp^4 \). However, this is not the case, since there is a cancelation in the numerator that makes the leading \( \hat{p} \) term vanish. This cancelation is a consequence of the on-shell conditions \( \epsilon \cdot l = 0 \) and \( \hat{p} u(\hat{p}) \). After a few commutation and using the decomposition Eq. (148) and neglecting terms proportional to \( \xi \) in the high energy limit we have

\[
(\hat{p} - l) \psi(\hat{p}) = (2 \epsilon \cdot p + \epsilon \hat{l}) u(\hat{p}) \approx \left( -2 \frac{\epsilon \cdot l_\perp}{1 - z} + \epsilon \hat{l} \perp \right) u(\hat{p}).
\]

Therefore, the numerator of the square matrix element is proportional to \( l_\perp^2 \), which leads to an overall \( 1/l_\perp^2 \) divergence. Putting everything together we find

\[
|\mathcal{M}^{(1)}|^2 = g^2 C_F \frac{2(1 + z^2)}{l_\perp^2} \tilde{\mathcal{M}}^{(0)}(p - \hat{l}) \hat{\psi}(p - \hat{l}),
\]

where

\[
\tilde{\mathcal{M}}^{(0)}(p - \hat{l}) = \frac{g^2}{2} \bar{\psi}(p - \hat{l}) \gamma_\mu p^\mu \frac{1}{2} \gamma_5 \psi(\hat{l}).
\]
We now multiply by the corresponding flux factor to compute the correction to the cross section

\[ \sigma_1^R = g^2 C_F \int \frac{d^2 l_\perp}{(2\pi)^2} dz \frac{1 + z^2}{1 - z} \frac{N}{p^0} \hat{\sigma}^{(0)}(p - l) \hat{p}^0 \hat{N}(p - l) \approx g^2 C_F \int \frac{d^2 l_\perp}{(2\pi)^2} dz \frac{1 + z^2}{1 - z} \frac{N}{(\hat{p} - l)^0} \hat{\sigma}^{(0)}(p - l) \hat{p}^0 \frac{\hat{N}(p - l)}{2} \hat{\sigma}^{(0)}(p - l) \]  

(156)

(157)

where in the second inequality we have assume the gluon is almost collinear with the quark, since we are focussing in the \( l_\perp \to 0 \) limit. Using Eq. (145)

\[ \sigma_1^R = \frac{\alpha_s C_F}{2\pi} \int d\hat{z} \frac{1 + \hat{z}^2}{1 - \hat{z}} \frac{d\hat{l}_\perp^2}{\hat{l}_\perp^2} \hat{\sigma}_0(\hat{z}\hat{p}) \]  

(158)

As in the case of jets, the emission of gluons factorises into a universal part, which only depends on the gluon kinematics and the leading order cross section, which depends of the process. However, the arguments of the leading order cross section depend on the gluon kinematics, unlike the \( e^+e^- \) case. Also, this real emission cross section is both collinear (\( l_\perp \to 0 \)) and soft (\( z \to 0 \)) divergent. However, as before, we need to consider virtual corrections.

Figure 13: Virtual corrections in DIS

The diagrams that contribute to those corrections are shown figure (13). The self-energy corrections in that figure vanish in dimensional regularisation. Therefore, the only non-vanishing correction is the vertex correction. However, this correction does not alter the kinematics of the scattered quark. As a consequence the correction can only be a divergent piece multiplying the leading order cross section for the incoming momentum \( \hat{p} \). After computing the integral we obtain

\[ \sigma_1 = \frac{\alpha_s C_F}{2\pi} \int d\hat{z} \frac{1 + \hat{z}^2}{1 - \hat{z}} \frac{d\hat{l}_\perp^2}{\hat{l}_\perp^2} [\hat{\sigma}_0(\hat{z}\hat{p}) - \hat{\sigma}^{\prime}(\hat{p})] \]  

(159)

The virtual correction cancels the soft divergence \( z \to 1 \). The way it does it is by subtracting the cross section value at the \( z = 1 \). In essence we can view the final result after taking into
account the virtual correction as transforming the divergent \((1 + z^2)/(1 - z)\) function in the interval \((0, 1)\) into a distribution, in which the function is regularised. To specify that this is indeed a distribution we introduce the notation

\[
P_{qq}(z) = \left(\frac{1 + z^2}{1 - z}\right)_+, \tag{160}
\]

such that when convoluted with any function

\[
\int_0^1 dz \left(\frac{1 + z^2}{1 - z}\right)_+ f(z) = \int_0^1 dz \left(\frac{1 + z^2}{1 - z}\right) (f(z) - f(1)). \tag{161}
\]

We therefore can split the process as

\[
\sigma_1 = \frac{\alpha_s C_F}{2\pi} \int dz \frac{d l^2_{\perp}}{\hat{t}_{\perp}} P_{qq}(z) \sigma_0(z\hat{p}). \tag{162}
\]

Note that the function \(P_{qq}\) is universal, in the sense that any process that will be initiated by a quark will lead to this function as a radiative correction. We have therefore factorised the process into a universal part and a process dependent cross section. This separation has a simple physical interpretation. In the collinear limit, the quark virtuality is small, which means that it has a very long life-time to emit the almost collinear gluon. This means that the process by which the quark emits this gluon must decouple from the rest of the diagram, which occurs at a much shorter time, given by the virtuality of the exchanged photon. This factorisation can be made much more precise, but this goes beyond the scope of this lectures.

There is one remaining problem: the virtual correction does not cancel the collinear divergence. This is a consequence of the fact that the virtual corrections cannot alter the parton kinematics. This non-cancelation implies that our naive parton model cannot be correct, since it cannot predict the cross section beyond leading order. In the next section we will study which modifications we need to introduce to the parton model to render the cross section finite.

### 7.2 Scaling violation

To deal with the uncanceled collinear divergence we will follow an strategy very similar to what it is done for renormalisation. There, the UV divergences are absorbed into the parameters of the Lagrangian via multiplicative constants, such that the bare parameters are divergent, while the renormalised ones are finite. Here, the object which we can use to renormalise is the parton distribution function, which, as it was the case for the parameters of the Lagrangian, is an object that we cannot compute (in perturbation theory), but we need to determine experimentally.

The calculation we analysed in the previous subsection is the cross section at partonic level. To obtain the cross section at hadronic level we must convolute the partonic cross section with the (bare) PDF

\[
\sigma = \int dx f^{(0)}(x)\tilde{\sigma}(xp), \tag{163}
\]
Using our perturbative computation, this expression is still collinear divergent, since convoluting with the (bare) PDF does not change the collinear kinematics. Introducing a collinear regulator $\lambda$, which is a minimum transverse momentum the gluon can have, we obtain

$$\sigma = \int dx dz f_q^{(0)}(x) \left( \delta(z - 1) + \frac{\alpha_s C_F}{2\pi} \log \left( \frac{Q^2}{\lambda^2} P_{qq}(z) \sigma_0 \right) \right)$$

(164)

where we have cut off the small $l_\perp$ limit of the integral by the infrared regulator. The upper cut-off is of order of the transverse momentum exchange $Q$, as can be easily estimated by demanding that the final scattered parton is massless. The full cross section is obtained taking the cut-off to zero.

As for renormalisation we would like to "multiplicatively" renormalise the bare PDF such that it absorbs the logarithmic divergence. But since this is a function of the momentum, instead of multiplying by a (divergent) constant, we convolute the PDF with a (divergent) correction. As in renormalisation, we introduce an arbitrary scale, called the factorisation scale $\mu_F$ to separate the contribution of the cutoff. Using the trivial fact that

$$\log \frac{Q^2}{\lambda^2} = \log Q^2 \mu_F^2 + \log \mu_F^2 \lambda^2$$

(165)

we redefine

$$f_q(x, \mu_F^2) = \int dy dz f_q^{(0)}(y) \left( \delta(1 - z) + \frac{\alpha_s C_F}{2\pi} P_{qq}(z) \log \frac{\mu_F^2}{\lambda^2} \right) \delta(x - yz)$$

(166)

This redefinition absorbs the collinear divergence to leading order: additional factors may be found order by order in perturbation theory. Similarly, we may redefine the partonic cross section as

$$\hat{\sigma}(\hat{p}, \mu_F^2) = \hat{\sigma}_0(\hat{p}) + \frac{\alpha_s C_F}{2\pi} \log \frac{Q^2}{\mu_F^2} \int dz P_{qq}(z) \sigma_0(z\hat{p})$$

(167)

This expression coincides with the partonic cross section provided we only allow emission of gluons with transverse momentum $l_\perp < \mu_F$. The rest is absorbed into the PFD.

An heuristic explanation of what the procedure we have just described is the following: the PDF provides the distribution of partons within the hadron. But the distribution of those partons can be easily changed by almost collinear emissions, which can alter the kinematics and flavour of the parton we are interacting with. Similarly, the process of interaction can be altered by collinear emissions off the incident quark. What the factorisation scale provides is a separation to which radiative processes we include as part of the proton structure and which ones contribute to the real emissions in the cross section. Once this is determined we can factorise the cross section between the properties of the target, as codified by the PDF, and the partonic interactions. The scale separating these assignments is $\mu_F$.

To determine the full cross section we must therefore choose a value of $\mu_F$ to compute the partonic cross section Eq. (167) and convolute it with the PDF at that same $\mu_F$. Since $\mu_F$ is arbitrary, all choices should give the same answer in a complete computation. However, as it was the case for the running coupling constant, in a fixed order calculation some choices
reduce the uncertainty in the computation. Examining Eq. (167) it becomes apparent that we must choose \( \mu_F = \xi Q \), with \( \xi \) an order one number, to avoid the presence of a large log. Therefore, we must also evaluate the PDF in the same scale \( \mu_F \sim Q \). As a consequence of radiative corrections the PDF depends then in both \( x_{Bj} \) and \( Q \), although the dependence on \( Q \) is weak, since its only logarithmically. The dependence of the PDF on the scale is known as scaling violation, since radiative corrections break Bjorken scaling.

Although the PDF must be determined experimentally, we do not need to determine them at all possible scales. We can relate the dependence from one scale to another via an evolution equation. Indeed, taking a derivative with respect to the scale of Eq. (166) we obtain

\[
\mu^2 \frac{\partial}{\partial \mu^2} f_q(x, \mu^2) = \frac{\alpha_s C_F}{2\pi} \int dy dz f_q^{(0)}(x) P_{qq}(z) \delta(x - yz). \tag{168}
\]

Integrating out one of the variables and noticing that to the accuracy that we have performed, we can replace the bare partition function with the full partition function, we obtain

\[
\mu^2 \frac{\partial}{\partial \mu^2} = \frac{\alpha_s C_F}{2\pi} \int x^1 \frac{dy}{y} f_q(y, \mu_F^2) P_{qq} \left( \frac{x}{y} \right). \tag{169}
\]

This equation has a simple interpretation. As we have discussed, \( \mu \) is the maximum value of the collinear gluons that are included in the PDF. If we now increase this scale by a small amount, the PDF will change, since we allow for additional radiation off the quarks. The new PDF at this bigger scale is given by the convolution of the PDF at the smaller scale times the provability of emission. Since the emission takes away some energy, the PDF at the smaller scale will be evaluated at a different value of the momentum fraction \( y \), such that after emission of a gluon with momentum \( 1 - z \) the final quark possesses a momentum \( x = yz \).

Our calculation up to now has been oversimplified, since we have only considered processes in which the scattered quark comes from a quark that emits a gluon. However, we could have also included processes like \( g \rightarrow q\bar{q} \). This makes the evolution also sensitive to the gluon distribution, since, following the previous interpretation, the quarks at one given scale can...
also come from the conversion of gluons from a smaller scale into quark pairs. In general, all PDF are coupled and the evolution equations take the generic form

$$\mu^2 \frac{\partial}{\partial \mu^2} f_i(x, \mu) = \sum_{ij} \int_x^1 \frac{dy}{y} P_{ij} \left( \frac{x}{y} \right) f_j \left( y, \mu^2 \right),$$

(170)

where the splitting $P_{ij}$ can be determined in perturbation theory and they are shown in figure (14). These are important equations, which are know as the DGLAP (Dokshitzer, Gribov, Lipatov, Altarelli, Parisi) equations. These are the basis to determine the parton distribution functions from experimental data on $e - p$ collisions (and also hadron hadron collision).

In essence, the procedure to determine the PDF is to start with some parametrization of the distribution function at some reference scale $\mu_F^2 \sim 10\text{GeV}^2$ and use the evolution equation to constraint data at different values of $x$ and $Q$. The initial parametrisation may be of the form of some prescribed functional form or, in more modern approaches, chosen via a neural network trained to fit the data. A state-of-the-art determination of the PDF is shown in figure (15)
8 Lecture 8

8.1 QCD in hadronic collisions

We now move to the description of QCD processes in hadron-hadron collisions. A clear difference with respect to \( e^+ - e^- \) is that now the partonic energies involved in the process are unknown, since the kinematics of the final states do not fix the parton kinematics. The factorisation formula for a process reads

\[
\sigma_{pp}(\sqrt{s}) = \sum_{ij} \int dx_i dx_j f_i(x_i) f_j(x_j) \hat{\sigma}_{ij}(\hat{p}_i, \hat{p}_j),
\]  

(171)

where \( \hat{p}_i = x_i p \). This means, in particular, that the centre of mass of the elementary partonic collision is unknown and can be boosted by an arbitrary factor in the beam direction (\( \hat{z} \)). The final state kinematics do not fix the kinematics of the final state.

Because of this fact, it is helpful to work with kinematical variables with nice transformation properties under boosts along the longitudinal axis. A common parametrisation of the momentum of a particle is

\[
p = (m_T \cosh y, p_T \cos \phi, p_T \sin \phi, m_T \sinh y)
\]  

(172)

where we have introduced

- \( p_T = \sqrt{p_x^2 + p_y^2} \equiv \text{transverse momentum} \)
- \( \phi \equiv \text{azimuthal angle}, \ i.e. \ the \ angle \ of \ the \ particle \ in \ the \ plane \ transverse \ to \ the \ beam. \)
- \( y = \frac{1}{2} \log \frac{E + p_z}{E - p_z} \equiv \text{rapidity}. \)
- \( m_T = \sqrt{m^2 + p_T^2} \equiv \text{transverse mass}. \)

The variables \( p_T, m_T, \) and \( \phi \) are invariant under boosts (along the beam axis). The variable \( y \) is not invariant under boosts, but it is additive. This means that, for example, the rapidity in the lab frame \( y \), is related to the rapidity in the c.o.m frame \( \hat{y} \) via the rapidity of the c.o.m. \( y^* \) as

\[
y = y^* + \hat{y}.
\]  

(173)

This is particularly useful, since the partonic computations are typically computed in the centre of mass frame of the partons, which then needs to be translated back to the lab frame.

Another useful variable is pseudorapidity, which is defined as

\[
\eta = \frac{1}{2} \log \frac{1 + \cos \theta}{1 - \cos \theta},
\]  

(174)

The advantage of this variable is that it is easily measurable in experiments, since it is given by the angle of the particle with the beam axis. Note that for massless particles, \( y = \eta \) but
this is not the case in general, although when the $p_T$ of the particle is much greater than the mass these are also approximately the same.

Another important property of these kinematic variables is that they also have nice expression for phase space integration. It is easy to see that

$$\frac{d^3p}{E} = d^2p_Tdy,$$

i.e. the phase space is homogeneous in these variables. We will now explore a few simple example to illustrate the use of these variables.

8.2 Drell-Yan

We will first study the production of a massive vector boson in a hadronic collisions, known as Drell-Yan production. This is almost identical to the production of electro-weak bosons $W$ and $Z$, up to the fact that, as we will see, they have a nontrivial axial structure. The leading order diagram at partonic level (assuming that the boson couples to quarks) is shown in figure (16), where we have included also the subsequent decay of the boson into lepton pairs, but these do not affect the on-shell production rate.

The leading order partonic matrix element is given by

$$\mathcal{M} = g_v v(\hat{p}_2)\bar{u}(\hat{p}_1),$$

where $\epsilon$ is the polarisation vector of the massive vector particle. The cross section is then given by

$$\sigma = \frac{1}{2s} \frac{1}{49} \int d\phi_1 \sum_{pol} |M|^2,$$
where we have included the flux factor for the cross section, the average over colour and initial polarisations. The matrix element squared is
\[ \sum_{pol} |M|^2 = 3g_v^2 tr \left[ \hat{p}_1 \gamma^\mu (-\hat{p}_2) \gamma_\mu \right] = 12g_v^2 \hat{s}, \]
(178)
where we have neglected the mass of the partons and taken into account that massive bosons have three polarisations.

The phase space is also very simple
\[ d\pi_1 = \int \frac{d^3q}{2g^0 (2\pi)^3} (2\pi)^4 \delta^{(4)} (\hat{p}_1 + \hat{p}_2 - q) = 2\pi \delta ((\hat{p}_1 + \hat{p}_2)^2 - M_v^2). \]
(179)
Convoluting with the PDF we find
\[ \sigma_M = \int dx_1 dx_2 \sum_{q_1 q_2} f_{q_1}(x_1) f_{q_2}(x_2) \frac{\pi}{3} g_v^2 \delta (sx_1 x_2 - M_v^2) \]
(180)
Note that have suppressed the appearance of the factorisation scale. Since this process is characterised by one single hard scale, the mass of the heavy boson, a convenient choice is \( \mu_F \sim M_v \), to ensure the absence of large logarithmic in the cross sections.

It is also instructive to study the kinematics of the process. From momentum conservation
\[ q_0 = \sqrt{s}(x_1 + x_2) \quad q_z = \sqrt{s}(x_1 - x_2), \]
(181)
therefore, the rapidity of the boson is
\[ y = \frac{1}{2} \log \frac{q_0 + q_z}{q_0 - q_z} = \frac{1}{2} \log \frac{x_1}{x_2}. \]
(182)
Together with the constraint \( sx_1 x_2 = M_v^2 \) the momentum fractions of each parton is given by
\[ x_1 = \frac{M}{\sqrt{s} e^y} \quad x_2 = \frac{M}{\sqrt{s} e^{-y}}. \]
(183)
Putting this together, and performing a simple change of variables
\[ \frac{d\sigma_M}{dy} = \frac{1}{3} \pi g_v^2 \sigma_{pol} f_{q_1}(x_1) f_{q_2}(x_2) \]
(184)
with the above expressions for \( x_1 \) and \( x_2 \). As we can see, this differential cross section is sensitive directly to the PDF at specific values of the momentum fractions controlled by the rapidity of the boson. Having a good determination of the PDF, this process can be used to do precision studies of the properties of electro-weak bosons, as well as searching for new particles. Similarly, these measurements can also help constraint the parton distribution function. It is interesting to observe the correlation of the rapidity and the value of \( x \). The large rapidity coverage of the LHC allows us to explore the PDF to very small values of \( x \) by studying the production of massive bosons at large rapidities.
The computation we have performed is, certainly, an oversimplification since it does not take into account radiative corrections. In general we expect radiative corrections which lead to IR&Coll. divergences which can be subtracted and absorbed into the PDF, as in the DIS case. Note also that the leading order computation implies that the heavy boson can only be produced at zero momentum transfer. Radiative processes give a transverse momentum to the boson, and constraining the $p_T$ spectrum of those bosons provide also important information to constraint the effect of those radiative processes.

8.3 Inclusive Jet Production

We now consider the production of jets as we discussed in $e^+e^-$. Jets are the objects most frequently produced at the LHC, which play a central role in all physics analyses. The factorised expression for the cross section is

$$d\sigma_{\text{jet}} = \sum_{ijkl} \int dx_1 dx_2 f_i(x_i) f_j(x_j) \frac{d\tilde{\sigma}_{ij\rightarrow kl}}{d\phi_2} d\phi_2.$$  \hspace{1cm} (185)

The partonic cross section can be determined via perturbation theory at increasing order of accuracy. Since jets correspond to final states partons of any type, including gluons, which also arise from primordial scatterings of any partons, many microscopic processes occur. The list of possible processes and the corresponding leading order cross section as a function of the usual Mandelstam variables are shown in figure (17). As before, radiative corrections lead to collinear and infrared divergences that need to be treated either by absorbing them into the parton distribution function (initial state) or by defining sufficiently inclusive observables.

Depending on the value of the rapidities of the two jets, $y_1$ and $y_2$, the values of the explored momentum fractions change. From energy momentum conservation it is easy to

<table>
<thead>
<tr>
<th>Process</th>
<th>$\frac{d\tilde{\sigma}}{d\phi_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$qq' \rightarrow qq'$</td>
<td>$\frac{1}{2} \frac{4}{9} \left( \frac{4}{9} \frac{s^2+u^2}{t^2} + \frac{s^2+u^2}{u^2} \right) - \frac{8}{27} \frac{u^2}{u^2}$</td>
</tr>
<tr>
<td>$qq \rightarrow qq$</td>
<td>$\frac{1}{2} \frac{4}{9} \left( \frac{4}{9} \frac{t^2+u^2}{s^2} + \frac{t^2+u^2}{s^2} \right) - \frac{8}{27} \frac{u^2}{u^2}$</td>
</tr>
<tr>
<td>$q\bar{q} \rightarrow q\bar{q}'$</td>
<td>$\frac{1}{2} \frac{4}{9} \left( \frac{4}{9} \frac{u^2+\bar{u}^2}{s^2} + \frac{t^2+\bar{u}^2}{s^2} \right) - \frac{8}{27} \frac{u^2}{u^2}$</td>
</tr>
<tr>
<td>$q\bar{q} \rightarrow gg$</td>
<td>$\frac{1}{2} \frac{4}{9} \left( \frac{4}{9} \frac{u^2+\bar{u}^2}{s^2} + \frac{t^2+\bar{u}^2}{s^2} \right) - \frac{8}{27} \frac{u^2}{u^2}$</td>
</tr>
<tr>
<td>$gg \rightarrow q\bar{q}$</td>
<td>$\frac{1}{2} \frac{4}{9} \left( \frac{4}{9} \frac{u^2+\bar{u}^2}{s^2} + \frac{t^2+\bar{u}^2}{s^2} \right) - \frac{8}{27} \frac{u^2}{u^2}$</td>
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</tr>
</tbody>
</table>

Figure 17: Partonic processes contributing to di-jet productions
determine that

\[
x_1 = \frac{p_T}{\sqrt{s}} (e^{y_1} + e^{y_2}) ,
\]

\[
x_1 = \frac{p_T}{\sqrt{s}} (e^{-y_1} + e^{-y_2})
\]

Therefore, unlike the case of Drell-Yan the observation of a jet at a particular rapidity is sensitive to a wide range of \(x\) values, since the rapidity of the opposite jet can be at any value, given the fact that the centre of mass of the partonic collision also can change its rapidity with respect to the lab frame. Also, this expression allows us to estimate the maximum transverse momentum that can be observed in a collision.
9 Lecture 9

9.1 Parton Showers and MonteCarlo Simulations

Up to now we have focussed on fixed order QCD calculations, which, in principle, can be extended to higher and higher order by continuing the perturbative series. However, in practice, the number of diagrams grows fast and it makes it very hard to go beyond NNNLO. Furthermore these fixed order computations are good for sufficiently inclusive measurements or for particularly simple final states. However, a generic hadron-hadron collisions event has a lot of activity which make it very hard or even impossible to describe them in fixed order pertubative computation.

To describe those complicated events we need to perform approximations, which concentrate in regions of phase space that are kinematically enhanced. As we have seen, the presence of collinear and infrared divergences tells us that the dominant momentum regions in physical cross sections are precisely in the regions where these divergences occur.

To make the discussion concrete, let us consider a diagram in which an almost on shell parton with (small) virtuality $t = p_a^2$ splits into two partons with virtualities $p_c^2, p_b^2 \ll t$. The relevant scale to compare the virtuality of the parent parton is the energy of the parton, $E_a \gg p_a^2$. Similarly, we will assume that the virtualities of the decay products are much larger than $\Lambda_{QCD}$, so that the coupling constant is small. A sketch of this branching process is shown in figure (18)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Fig18.png}
\caption{Sketch for the splitting dynamics.}
\end{figure}

Let us examine the kinematics of the branching. After fixing the four momenta of the decay products, the virtuality of the parent parton is given by

$$t = p_b^2 + p_c^2 + 2E_bE_c \left(1 - v_bv_c \cos(\theta)\right) \approx z(1-z)E_a^2 \theta^2,$$

where $v_i = p_a/E_a$ and in the right hand side we have neglected the virtualities of the decay products and assume that the angle between the particles is small. In this approximation it is clear that the virtuality of the parent parton is small when either one of the decay products is soft or when the splitting angle is small. In this case, the $1/t$ dependence of the parton
propagator in the diagram will lead to an enhancement of the cross section, reminiscent of the infrared and collinear singularities.

In the almost on-shell limit, the propagator of the intermediate particle simplifies significantly. Indeed, using the completeness relation of the external polarisation factors, it is easy to show that close to the pole the numerator of the propagator can be expressed as the sum over external polarisations of the on-shell particles

\[
\begin{align*}
\frac{-g_{\mu\nu}}{k^2} \bigg|_{k^2 \to 0} & \to \sum_{\text{pol}} \frac{\epsilon(k)\epsilon^*(k)}{k^2}, \\
\frac{\phi}{p^2} \bigg|_{p^2 \to 0} & \to \sum_{\text{pol}} \frac{u(p)\bar{u}(p)}{p^2},
\end{align*}
\]

which allows us to separate the contribution of the production of the almost on-shell particle from the subsequent splittings. In particular, assuming that the process in figure (18) is mediated by a gluon, we may written as

\[
M_{n+1} = M_n \frac{1}{t} ig f^{abc} \epsilon^\alpha \epsilon^\beta \epsilon^\gamma [g_{\alpha\beta}(-p_a - p_b)\gamma + g_{\beta\gamma}(p_b - p_c)\alpha + g_{\gamma\alpha}(p_c + p_a)\beta],
\]

where we have used that the momentum flows into the vertex and, as before, \(M_n\) is the diagram producing the on-shell gluon with polarisation \(\epsilon^\alpha\). As mentioned, the amplitude already splits into the production of the gluon times and an additional factor encoding the gluon splitting. Taking the square of the amplitude and summing over polarisations one can obtain (see for example Ellis, Stirling and Webber)

\[
|M_{n+1}|^2 = |M_n|^2 \frac{4g^2}{t} \hat{P}_{gg}(z)
\]

where \(\hat{P}_{gg}(z)\) is the unregularised gluon splitting function

\[
\hat{P}_{gg}(z) = C_A \left[ \frac{1 - z}{z} + \frac{z}{1 - z} + z(1 - z) \right].
\]

Note that this splitting function is divergent at \(z = 1\) and no “+”-prescription appears (yet). For this reason we call this object the unregularised splitting function. Similar computations can be done for all other channels, where a similar factorisation of the square amplitudes between the production of the n-particle state and the subsequent splitting occurs. Note that this also means that, at least in the almost on-shell limit, the splitting distribution is universal, independent of the process that leads to the production of the parent parton.

To show that the full cross section also factorises, we must also show that the phase space for the process can be separated in a similar fashion. The phase space of production of \(n\) particles will be given by

\[
d\phi_n = \ldots d^3p_a \frac{1}{(2\pi)^3 2E_a},
\]
while the phase space after splitting is given by

\[ d\phi_{n+1} = \ldots \frac{d^3p_b}{(2\pi)^3} \frac{1}{2E_b} \frac{d^3p_c}{(2\pi)^3} \frac{1}{2E_c}. \]  \hspace{1cm} (195)

Fixing the momentum of particle \( p_b \) we can use momentum conservation \((p_a = p_b + p_c)\) to trade \( d^3p_c = d^3p_a \). We may now introduce unity into the integral as

\[ 1 = \int dz \delta(t - z(1 - z)E_a^2\theta^2). \]  \hspace{1cm} (196)

Using simple kinematics, demanding that the momentum of the decay products transverse to \( p_a \) vanishes, in the small angle approximation we obtain

\[ \theta_b = (1 - z)\theta \]  \hspace{1cm} (197)

and change variables to integrate over \( \theta_b \). Eliminating the \( p_b \) and \( \theta_b \) integrals via the introduced \( \delta \)-functions we end up with

\[ \frac{d^3p_b}{(2\pi)^3} \frac{1}{2E_b} \frac{d^3p_c}{(2\pi)^3} \frac{1}{2E_c} = \frac{d^3p_a}{(2\pi)^3} \frac{1}{2E_a} \frac{dtdzd\phi}{4(2\pi)^3}. \]  \hspace{1cm} (198)

And, as claimed, the phase space factorises. Putting all together, we find a factorised expression for the cross section

\[ d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha_s}{2\pi} \hat{P}_{ab} \]  \hspace{1cm} (199)

which separates the cross section into a process-independent universal factor, and the cross section to produce an on-shell particle. Note that the presence of \( dt/t \sim d\log t \) signals once again the collinear singularity, since, as we have seen, \( t \) vanishes at small angle. In a given process, this singularities are regulated by the fact that the virtualities of the decay product do not vanish identically, leading to terms or the order

\[ \alpha_s \log \frac{p_b^2}{p_a^2}, \]  \hspace{1cm} (200)

Therefore, if one examines processes with very different initial and final virtuality, this ratio of logs of scales indicates that the multiple radiations are likely, since the smallness of \( \alpha_s \) is compensated by the large \( \log \). These factors are also behind the emergence of virtuality ordered sowers. If at each step in the shower there is a jump in the virtuality, the factorisation of the process we have described before appears at each splitting with enhancing factors as the above. Emissions in which the virtualities are not ordered are possible, but their rate is much smaller than those of a strong ordering as a consequence of the above logarithmic enhancement.

This branching picture is very powerful and can provide us with an intuitive explanation for evolution equations. In the above we have focussed on a final state shower, in which the virtuality is positive and it decreases towards the final state. Identical arguments can be done
in an initial state shower, in which the virtuality of the partons is negative and increases towards the elementary vertex, as it is the case of DIS. We may describe the subsequent emissions of gluons in the initial state as a factorised process, see figure (19). This emission processes changes the kinematics of the parton that is going to interact with the virtual photon, and, therefore, modifies the PDF distribution.

Let us consider the change of the distribution of partons $f(x, Q^2)$ after an emission process which changes infinitesimally the value of $t$. There are two possible contributions to this change. On the one hand, the number of parton at a given $x$ increases by the emission of gluons from partons carrying a larger fraction of the proton momentum. This is given by

$$
\delta f_{\text{in}}(x,t) = \frac{\delta t}{t} \int_x^1 dx' dz \frac{\alpha_s}{2\pi} \hat{P}(z) f(x', t) \delta (x - zx').
$$

(201)

In addition, there is a loss term by the emissions of partons carrying a fraction $x$ of any gluon. This is given by

$$
\delta f_{\text{out}}(x,t) = \frac{\delta t}{t} f(x, t) \int_0^1 dz \frac{\alpha_s}{2\pi} \hat{P}(z).
$$

(202)

The total change in the distribution of partons is then

$$
\delta f(x, t) = \delta f_{\text{in}} - \delta f_{\text{out}}
$$

(203)

$$
= \frac{\delta t}{t} \int_0^1 dz \frac{\alpha_s}{2\pi} \hat{P}(z) \left[ \frac{1}{z} f \left( \frac{x}{z}, t \right) - f(x, t) \right].
$$

(204)

The $\delta f_{\text{out}}$ term is the responsible for the second term in the bracket. Note that this term produces the same effect as the $+$ prescription, where the negative term in the bracket is nothing else than the first term evaluated at $z = 1$. This provides an intuitive origin to the "$+$"-prescription and, in fact, we can express this equation in terms of the (regularised) splitting function

$$
t \partial_t f = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f \left( \frac{x}{z}, t \right),
$$

(205)

which is nothing else than the DGLAP equation we derived before, choosing $\mu = t$.

The expression of evolution equations with the regularised splitting function is suitable for analytical computations but hard to evaluate numerically. For this reason we look for a
simplified expression which does not require the regularised splitting function. To do so, let us introduce the Sudakov form factor

\[ \Delta(t) = \exp \left\{ -\int_{t_0}^{t} \frac{dt'}{t'} \int dz \frac{\alpha_s}{2\pi} \hat{P}(z) \right\} . \] (206)

It is easy to see that if \( f \) satisfies the DGLAP equation, then

\[ t \partial_t \left( \frac{f(x, t)}{\Delta(t)} \right) = \int dz \frac{\alpha_s}{z} \frac{\hat{P}(z)f(x/z, t)}{\Delta(t)} . \] (207)

This equation has a formal solution given by

\[ f(x, t) = \Delta(t)f(x, t_0) + \int_{t_0}^{t} dt' \frac{\Delta(t)}{\Delta(t')} \int dz \frac{\alpha_s}{z} \hat{P}(z)f(x/z, t') \] (208)

This expression admits a simple interpretation. The first term may be interpreted as the contribution of processes in which no radiation occurs, since the distribution in \( x \) remains the same between the two scales. The prefactor \( \Delta(t) \) may be then interpreted as the probability of no radiation between the initial and final scale. This interpretation also help us to understand the second term. The PDF at scale \( t \) is related to the PDF at scale \( t' \) via the radiation of a parton that changes the kinematics of the initial parton; the ratio of Sudakov form factors can be interpreted as the probability that there is no further radiation between \( t' \) and \( t \).

This interpretation is the basis for Monte-Carlo event generators. Given some initial virtuality, the distribution of branchings at some other scales follows the distribution \( \Delta(t)/\Delta(t_0) \). The value of \( t \) may be found randomly, by generating an arbitrary number \( R \in (0, 1) \) and solving

\[ \frac{\Delta(t)}{\Delta(t_0)} = R . \] (209)

After finding the scale, we can determine the fraction of momentum of the emitted parton \( z \) according to the splitting distribution \( \hat{P} \). This procedure generates a shower up to a non-perturbative value of \( t \sim \Lambda_{QCD} \). At this point, current Monte-Carlo generators connect with different models for hadronisation which convert the partonic distribution at the end of the shower into hadrons, either via clustering or via classical string dynamics. We will not describe those models in these lectures. For a brief description see Ellis, Stirling and Webber.
10 Lecture 10
11 Lecture 11

11.1 Introduction to Electroweak Theory

The electroweak theory describes both the weak force and electromagnetism in a unified framework. As we have already discussed, both electromagnetism and the strong force are described by gauge theories with either abelian (the former) or non-abelian (the latter) gauge groups. In the standard model, the weak force is also described via a gauge theory. However, there are two properties of the weak force that make the description of this force different from the other two.

• The weak force is short range. This can be inferred by the fact that the effective four fermion interaction that describes weak decays can be understood as mediated by a heavy exchange particle.

• The weak force violates parity. This is unlike QED or QCD. This implies that the weak bosons will couple differently to the different fermion chiralities.

In the next lectures we will see how these two features can be described via a SU(2) gauge theory.

11.2 The gauge sector

As we have mentioned, the Electroweak theory describes simultaneously the weak interaction and electromagnetism. For reasons that will become apparent later, we will start writing the gauge Lagrangian of a SU(2) theory and a U(1) theory together as

\[ L_{\text{gauge}} = -\frac{1}{4} W_{\mu\nu} W^{i\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \]

where \( W \) and \( B \) are the field strengths of a SU(2) and a U(1) gauge fields. As it will become apparent soon, these are not yet the ElectroWeak bosons. The non-abelian gauge field carries an adjoint index \( i=1,2,3 \). In terms of the gauge fields, these field strengths are given by

\[
B_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu,
\]

\[
W^{i}_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g\epsilon^{ijk} A^j_\mu A^k_\nu,
\]

where \( g \) is the weak coupling constant and \( \epsilon^{ijk} \) are the structure constant of SU(2). Following the standard normalisation convention, the generators in the fundamental representation are given by the Pauli matrices as

\[
t^i = \frac{1}{2} = \sigma^i
\]

which satisfy

\[
[t^i, t^j] = \epsilon^{ijk} t^k,
\]
The Lagrangian Eq. (210) alone cannot describe the EW interaction, since gauge symmetry demands that Bosons are massless. The way to generate those massive bosons is via spontaneous symmetry breaking. In the Standard Model this proceeds by the coupling of weak bosons with a scalar field, the Higgs field.

We will describe the dynamics of this scalar field in subsequent lectures. For the moment, let us simply consider the relevant properties which enter spontaneous symmetry breaking. We will consider a complex scalar field charged both under the SU(2) and U(1) gauge fields. In principle, we do not need to couple the scalar to the U(1) field, since we know that photons are massless, but we will see why this is necessary below.

The coupling of the scalar field to the gauge fields proceeds via minimal coupling through the covariant derivative.

\[ D_\mu \phi = \left( \partial_\mu - igA^i_\mu t^i - ig'Y_0 V_\mu \right) \phi , \]  

(215)

where \( g' \) is the gauge coupling of the U(1) field and \( Y_0 \) is the charge of \( \phi \) under this symmetry, called hypercharge. Since \( \phi \) couples to the SU(2) field, it must be charged under SU(2). In the Standard Model, the field is in the fundamental representation, and therefore, it is given by a doublet

\[ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} . \]  

(216)

Let’s assume that, by some mechanism, \( \phi \) acquires a vacuum expectation value. i.e. the lowest energy state is the theory possesses a non-vanishing modulus of \( \phi \), \( |\phi| = v/\sqrt{2} \). We may choose a state which satisfies this condition

\[ \phi_{\text{vac}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} . \]  

(217)

We do not want to break the gauge symmetry explicitly in the theory. Therefore, such lowest energy state must be degenerate, since any SU(2) rotation of the state must also be a minimum energy state. By selecting one particular direction in gauge theory space, the state (and not the theory) breaks the symmetry. This implies that the effective potential of the Higgs field must have a degenerate minimum. We will see how this is achieved in the standard model in later lectures.

Without specifying the dynamics of the field, we can still understand how this expectation function affects the dynamics of gauge fields. Independently of the interaction, the scalar field must posses a kinetic term of the form

\[ \mathcal{L}_{\text{kin}} = (D_\mu \phi)^\dagger (D^\mu \phi) , \]  

(218)

Evaluating this on the vacuum state we have

\[ \mathcal{L}_{\text{kin} \phi_{\text{vac}}} = \frac{1}{2} \begin{pmatrix} 0 & v \end{pmatrix} (g t^i A^i_\mu + Y_0 g' V_\mu)(g t^j A^j_\mu + Y_0 g' V_\mu) \begin{pmatrix} 0 \\ v \end{pmatrix} , \]  

(219)
which, using the explicit form of the Pauli matrix, we have

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} v^2 \left[ g^2 (A^1_\mu)^2 + g^2 (A^2_\mu)^2 + (-gA^3_\mu + g'2Y_0 V_\mu)^2 \right].$$  \hspace{1cm} (220)$$

We may re-write this interaction term as matrix

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} \begin{pmatrix} A^1 & A^2 & A^3 & V \end{pmatrix} \mathcal{M} \begin{pmatrix} A^1 \\ A^2 \\ A^3 \\ V \end{pmatrix},$$  \hspace{1cm} (221)$$

with

$$\mathcal{M} = \begin{pmatrix} \frac{g^2v^2}{4} & 0 & 0 & 0 \\ 0 & \frac{g^2v^2}{4} & 0 & 0 \\ 0 & 0 & \frac{g^2v^2}{4} & -\frac{gg'v^2}{4}2Y_0 \\ 0 & 0 & -\frac{gg'v^2}{4}2Y_0 & \frac{g^2v^2}{4}(2Y_0)^2 \end{pmatrix}$$  \hspace{1cm} (222)$$

This matrix has zero determinant, which implies that it must have at least a zero eigenvalue. The other eigenvalues are given by

$$m_W = g \frac{v}{2}, \quad m_Z = \sqrt{g^2 + g'^22Y_0^2 \frac{v^2}{2}}, \quad m_\gamma = 0.$$  \hspace{1cm} (223)$$

The first eigenvalue is degenerate and is associated to two eigenvectors. For convenience, we perform a unitary transformation of those two and express those eigenvectors as

$$W^\pm_\mu = \frac{1}{\sqrt{2}} (A^1_\mu \mp iA^2_\mu).$$  \hspace{1cm} (224)$$

Similarly, the (normalised) eigenvector corresponding to $m_Z$ is given by

$$Z^0_\mu = \frac{1}{\sqrt{g^2 + g'^22Y_0^2}} \left( gA^3_\mu - g'2Y_0 V_\mu \right).$$  \hspace{1cm} (225)$$

Finally, the eigenvector with zero eigenvalue is given by

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^22Y_0^2}} \left( g'2Y_0 A^3_\mu + gV_\mu \right).$$  \hspace{1cm} (226)$$

After forming this unitary transformation, the contribution of the scalar condensate $\mathcal{L}_{\text{kin}}$ becomes a mass term for the gauge fields, in which two of those fields $W^\pm$ acquire the same mass, while $Z$ acquires a different mass, while $A$ remains massless. It now becomes clear why we have treated simultaneously both the non-abelian and the abelian gauge fields. If the coupling of the scalar to the U(1) field would be zero, then we would have the same mass for all Bosons. By giving a hypercharge to the scalar, we have introduced a mixing between the gauge fields, which yields this mass difference. In addition there is a massless mode, the photon, which is also a mixture of the non-Abelian and U(1) fields.
Note also that in the $gg' \to 0$ (keeping the mass fixed), the unitary transformation we have performed implies that the kinetic terms of the gauge fields are also diagonal in the new fields $W^\pm$, $Z$ and $\gamma$. This implies that the masses in the above matrix are the masses that appear in the propagators of the gauge fields.

Performing the rotation in the covariant derivative, we can calculate how the different fields couple to the gauge bosons. A simple calculation yields

$$D_\mu = \partial_\mu - i \frac{g}{\sqrt{2}} (W^\pm_\mu T^\pm) - i \frac{1}{\sqrt{g^2 + g'^2 Y_0^2}} Z^0_\mu (g^2 T^3 - (g')^2 Y) - i \frac{gg'}{\sqrt{g^2 + g'^2 Y_0^2}} A_\mu (T^2 + Y)$$  \hspace{1cm} (227)$$

where $Y$ is the hypercharge of a generic field, while $Y_0$ is the hypercharge of the Higgs field and we have defined

$$T^\pm = t^1 \pm it^2. \hspace{1cm} (228)$$

This allows us to identify the electric charge operator in terms of the hypercharge and SU(2) generator as

$$Q = T^3 + Y. \hspace{1cm} (229)$$

This identification allows us to fix the hypercharge of the Higgs field. Demanding that the vacuum is neutral,

$$Q \begin{pmatrix} 0 \\ v \end{pmatrix} = 0 \hspace{1cm} (230)$$

We obtain

$$Y_0 = \frac{1}{2}. \hspace{1cm} (231)$$

Note that this value for the hypercharge depends on the representation of the scalar field. We have assumed that $\phi$ is in the fundamental representation of SU(2). But if instead the Higgs field would be in any other representation, neutrality of the vacuum would imply a different value for $Y_0$. Note also that $t^3 + Y$ is the generator of transformation that leave the vacuum invariant.

Having fixed $Y_0$ we can express the parameters in the EW-lagrangian in terms of parameters already determined by experiments. In particular, we can recognise the electric charge, which is related to the coupling constants as

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} \hspace{1cm} (232)$$

We can also rewrite the mixing between the non-abelian and abelian photon in a convenient way as

$$\begin{pmatrix} Z^0 \\ A \end{pmatrix} = \begin{pmatrix} \cos \theta_w & -\sin \theta_w \\ \sin \theta_w & \cos \theta_w \end{pmatrix} \begin{pmatrix} A^3 \\ V \end{pmatrix}, \hspace{1cm} (233)$$

where we have determine the Weinberg angle as

$$\cos \theta_w = \frac{g}{\sqrt{g^2 + g'^2}}, \quad \sin \theta_w = \frac{g'}{\sqrt{g^2 + g'^2}} \hspace{1cm} (234)$$
At classical level, this mixing between states also imposes a relation between the week coupling constant and the ratio of masses of the weak bosons, which is particularly simple

\[ g = \frac{e}{\sin \theta_w}, \quad m_W = m_Z \cos \theta_W \] (235)

Finally, it is also convenient to write the covariant derivative in terms of these quantities, which can be easily determined experimentally as

\[ D_\mu = \partial_\mu - i \frac{g}{\sqrt{2}} (W_\mu^\pm T^\pm) - i \frac{g}{\cos \theta_W} Z_\mu^0 (T^3 - \sin^2 \theta_w Q) - ieA_\mu Q \] (236)

This expression determines how all fields couple to the weak bosons, once the charge and SU(2) representation of the field is determined. In the next lecture we will focus on the particular case of fermions.
12 Lecture 12

12.1 Coupling to fermions

Another specific feature of the Weak interaction is parity violation. For massless fermions this means that the two helicity states couple differently to electroweak bosons. As we saw when we discussed chiral symmetry breaking, for Dirac fermions, the two helicity states may be found by multiplying the fermion field by the projectors

\[ P_L = \frac{1}{2} (1 - \gamma_5), \quad P_R = \frac{1}{2} (1 + \gamma_5), \]

such that

\[ \psi_L = P_L \psi, \quad \psi_R = P_R \psi. \]

The kinetic term of free fermions nicely splits into left and right-handed contribution as

\[ \bar{\psi} \partial \psi = \bar{\psi}_L \partial \psi_L + \bar{\psi}_R \partial \psi_R. \]

Since the minimal coupling of fermions to the gauge fields proceed by replacing the derivatives in the free kinetic term by covariant derivatives, a simple way in which a gauge theory can break parity is by acting with different covariant derivatives on the left and the right hand fields. This may be achieved by assigning different charges under the different interactions to the left and right-handed fields.

This is precisely how parity violations arises in the Standard Model. In particular, in the standard model left-handed fields are in the fundamental representation of the SU(2) gauge theory, while the right-handed fields are singlets (i.e. are decoupled) of SU(2). Similarly, the hypercharge assignments of left and right-handed fields are different, as we will see.

The fact that left-handed fermions are in the fundamental representation of SU(2) implies that they must come in doublets. Instead of adding additional quantum numbers to the fields, the existence of flavour changing currents of the weak interaction indicates that these doublets are made of left-handed fermions of different particle species. In particular, the lepton and quark doublets are given by

\[ E = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \]

where we have only referred two the SU(2) structure and suppressed all other internal indexes of the fields. Similar doublets can be constructed with the rest of the lepton and quark species. The SU(2) coupling implies that quarks, in particular, must come in pairs and that for each (charged) lepton there must be an associated neutrino. SU(2) rotations mix the species within the doublet.

Using the general expression for the covariant derivative, Eq. (236) the minimal coupling Lagrangian for left-handed fields is then

\[ \mathcal{L}_{\text{m.c.}} = \sum_{\text{flavours}} i \bar{E} \slashed{D} E + i \bar{Q} \slashed{D} Q, \]
Since the right handed fields are singlets under SU(2) we do not have to group them. We may write an identical Lagrangian for those fields, remembering that they are singlets of SU(2). Nevertheless, to make this assignment apparent, we can write this part of the Lagrangian explicitly as

\[ \mathcal{L}_{\text{m.c. R}} = i \bar{e}_R (\not{\partial} - ie^{R}_Y A) e_R + i \bar{\nu}_R (\not{\partial} - ie^{R}_Y A) \nu_R + i \bar{u}_R (\not{\partial} - ie^{R}_Y A) u_R + i \bar{d}_R (\not{\partial} - ie^{R}_Y A) d_R + \ldots \] (242)

where we have added also a possible right-handed neutrino. We will come back to this point later.

To completely specify the Lagrangian, we must determine the values of the hypercharge for the different fields. For the right-handed particles this is easy, since the minimal coupling Lagrangian we have written must coincide with the right-handed part of the QED Lagrangian which fixes the hypercharge of each species to be equal to the charge:

\[ Y^R_e = -1, \quad Y^R_\nu = 0 \quad Y^R_u = \frac{2}{3}, \quad Y^R_d = -\frac{1}{3}. \] (243)

Note, in particular, that the hypercharge assignments for the right-handed partners of the left-handed fields in a doublet are different. This must be different for the left-handed fields, since both partners in the double must have the same hypercharge assignment. This is another manifestation of parity breaking.

To fix the hypercharge of the left-handed fields we use the expression of the electric charge operator in terms of the algebra elements of the gauge group.

\[ Q \left( \begin{array}{c} 0 \\ e_L \\
\end{array} \right) = - \left( \begin{array}{c} 0 \\ e_L \\
\end{array} \right), \quad Q \left( \begin{array}{c} \nu_L \\ 0 \\
\end{array} \right) = 0, \] (244)

\[ Q \left( \begin{array}{c} 0 \\ d_L \\
\end{array} \right) = -\frac{1}{3} \left( \begin{array}{c} 0 \\ d_L \\
\end{array} \right), \quad Q \left( \begin{array}{c} u_L \\ 0 \\
\end{array} \right) = \frac{2}{3} \left( \begin{array}{c} u_L \\ 0 \\
\end{array} \right), \] (245)

with \( Q = t^3 + Y \). This implies that

\[ Y^L_E = -\frac{1}{2}, \quad Y^L_\nu = \frac{1}{6}, \] (246)

and similarly for all other lepton and quark species. These assignments fix completely the coupling of the fermion fields to electro-weak bosons.

However, there is a complication with this construction. We know that fermions are not massless. This means that the helicity states we have based our construction upon are not eigenstates of the hamiltonian. Furthermore, since an explicit mass term mixes helicity states

\[ m \bar{\psi} \psi = m \bar{\psi}_L \psi_R + m \bar{\psi}_R \psi_L \] (247)

such term is not gauge invariant if we assign different quantum numbers to the left and right-handed fields.
As it was the case for the EW bosons, the way in which fermions acquire a mass is via the vacuum expectation value of the Higgs field, which breaks the gauge symmetry spontaneously. Taking, for example, the leptonic fields, we can construct a gauge-invariant interaction term in the Lagrangian which involves fermions and the Higgs field

$$\mathcal{L}_{EY} = -\lambda_e (\bar{e}_R \phi^\dagger \cdot e_L + \text{c.c.}) ,$$

which is a Yukawa-type interaction of the Higgs field with the leptonic field. This coupling introduces a new (Yukawa) coupling constant, which are additional parameters of the Standard Model. Evaluating this term on the vacuum state $\phi_{\text{vac}}$, we obtain a mass-like term in the Lagrangian with a mass given by

$$m_e = -\lambda_e \frac{v}{\sqrt{2}} .$$

This mechanism of the generation of masses links the coupling of the different fermions to the Higgs field.

Let’s come back now the issue of the right-handed neutrino. In Eq. (242) we included a right-handed neutrino field $\nu_R$, which is not coupled to any of the gauge bosons, since it is a singlet of SU(2) and $Y_{\nu R} = 0$. Furthermore, if neutrinos were massless, the corresponding neutrino Yukawa $\lambda_\nu = 0$. This would mean that the right handed neutrinos, if they existed, would be completely decoupled from the standard model, which is consistent with the fact that a right-handed neutrino has never been observed. In the original formulation of the Standard Model, it was assumed that, indeed $\lambda_\nu = 0$. However, today we know that neutrinos have mass, which means that the original formulation of the standard model is incomplete. A possibility to include this fact is to add a non-zero Yukawa, but this means that a right-handed neutrino must exist and should be detectable. We will come back to this point in a few lectures.

For quark fields, a similar interaction can be written. For the down-type quarks, writing this interaction is simple

$$\mathcal{L}_{dY} = -\lambda_d \bar{Q} \cdot \phi d_R + \text{c.c.}$$

For u-quark we cannot write an identical term, since the VeV of the Higgs field only contains down components. To construct this mass term we can take advantage of the fact that the fundamental representation of SU(2) is pseudoreal, which implies that if we have two vectors, $\xi$ and $\zeta$, transforming under SU(2) in this representation then the combination

$$\xi_i \zeta_j \epsilon^{ij} ,$$

with $\epsilon$ an anti-symmetric tensor with $\epsilon^{12} = 1$, is an invariant under SU(2). Therefore we can write

$$\mathcal{L}_{dY} = -\lambda_d \bar{Q} \phi \xi^j \epsilon^{ij} u_R + \text{c.c.}$$

To finish this lecture, let’s examine more closely the properties of the weak interaction under discrete symmetries. Let us first start by performing a parity transformation

$$x^\mu \rightarrow A^\mu_\nu x^\nu \quad A^\mu_\nu = \text{Diagonal \{1, -1, -1, -1\}}$$
Under this transformation the Weak bosons and Dirac Matrices and derivatives transform as (see for example Peskin & Schroeder)

\[
W^0 \rightarrow W^0 \quad \partial_0 \rightarrow \partial_0 \quad \bar{\psi} \gamma^0 \psi \rightarrow \bar{\psi} \gamma^0 \psi \quad (254)
\]

\[
W^i \rightarrow -W^i \quad \partial_i \rightarrow -\partial_i \quad \bar{\psi} \gamma^i \psi \rightarrow -\bar{\psi} \gamma^i \psi \quad (255)
\]

Similarly, the pseudoscalar current under parity transforms as

\[
\psi \gamma^0 \gamma^5 \psi \rightarrow -\bar{\psi} \gamma^0 \gamma^5 \psi \quad \bar{\psi} \gamma^i \gamma^5 \psi \rightarrow \bar{\psi} \gamma^i \gamma^5 \psi \quad (256)
\]

An, therefore, the coupling of the W filed to fermions violates parity since, under parity

\[
W^\mu \bar{\psi} \gamma^\mu (1 - \gamma_5) \psi \rightarrow W^\mu \bar{\psi} \gamma^\mu (1 + \gamma_5) \psi \quad (257)
\]

If we now perform a charge conjugation transformation on the Lagrangian, only the fermion bilinears change as

\[
\psi \gamma^\mu \psi \rightarrow -\bar{\psi} \gamma^\mu \psi \quad \psi \gamma^\mu \gamma^5 \psi \rightarrow +\bar{\psi} \gamma^\mu \gamma^5 \psi \quad (258)
\]

And, as a consequence, the interaction of quark fields with the W boson change as

\[
W^\mu \bar{\psi} \gamma^\mu (1 - \gamma_5) \psi \rightarrow -W^\mu \bar{\psi} \gamma^\mu (1 + \gamma_5) \psi \quad (259)
\]

where the last minus sign can be understood as a consequence of the change in the charge. Therefore, since the effect of these two transformations is identical, when summing over both \( W^\pm \) terms, the Lagrangian is invariant under CP symmetry, which, in turn, implies that the Lagrangian is invariant under time-reversal symmetry. This is a property of the ElectroWeak Lagrangian for only one species of fermions. In later lectures we will see how having several species of fermions may lead to CP-violation terms.
13 Lecture 13

13.1 The Higgs Field

We now turn to study the dynamics of the Higgs field. In previous lectures we have seen that, in order to spontaneously break SU(2) gauge symmetry, the field must acquire a non-vanishing expectation value in vacuum. Furthermore, under the assumption that the Higgs field is in the fundamental representation of SU(2), neutrality of the vacuum fixes the hypercharge of the Higgs field. We now would like to write the Lagrangian. Since SU(2) is a gauge symmetry, we would like to write the most general Lagrangian consistent with this symmetry

\[ L_H = (D_\mu \phi)^\dagger (D^\mu \phi) - V (\phi^\dagger \phi) , \]  

where the potential must only depend on the modulus of the field to ensure gauge invariance.

We may now expand the potential in power series of the modulus of the field as

\[ V (\phi^\dagger \phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \]  

We may consider further terms in the power expansion. However, each additional order we add would correspond to a higher dimension operator, starting from D=6. This means that the highest order renormalisable term that we can add in the Lagrangian is the $\phi^4$ term. All additional terms would mean that the Standard Model would be incomplete, since these non-renormalisable terms come only arise as the low energy description of an underlying UV-complete theory. Note also that even if these higher dimensional terms would be present, they are irrelevant operators, in the renormalisation group sense, which means they cannot alter the infra-red dynamics of the theory.

To be consistent with spontaneous symmetry breaking the above potential must develop a non-trivial minimum. Indeed, the extrema of the potential are given by

\[ |\phi| = \frac{1}{2} v \quad v = \sqrt{\frac{\mu^2}{\lambda}} \]  

The parameter $\lambda$ must be positive, otherwise the potential is unbounded, which would mean that there would not be stable vacuum solution. Therefore, to have a non-trivial vacuum expectation value $\mu^2$ must be positive. This means that in terms of $\phi$ the potential has an apparent negative mass term.

We would now like to study the excitations of the Higgs field. A complex scalar double has in total four degrees of freedom. Without loss of generality we can parametrise the fluctuations of the scalar field as

\[ \phi(x) = \exp \left\{ i \pi_i (x) t^i \right\} \frac{1}{\sqrt{2}} \begin{pmatrix} \ 0 \ v + h(x) \end{pmatrix} \]  

The field $h$ can be understand as fluctuation of the modulus of the field while the $\pi^i$ can be understood as a SU(2) rotation.
If the scalar field would not be charged under the gauge fields, the field \( \pi^i(x) \) would correspond to massless excitations, since they only mean a rotation of the vacuum state, which does not change the energy. These are known as goldstone bosons. However, in the standard model the Lagrangian is SU(2) gauge invariant and, therefore, the fields \( \phi^i(x) \) can be gauged away from the Lagrangian. Therefore, the \( \pi^i(x) \) excitations cannot be part of the spectrum and the only physical fluctuations is \( h(x) \).

We may now choose a gauge which makes the physical degrees of freedom apparent. Note that since gauge transformations affect both the gauge fields an the scalar field, specifying a particular form of the scalar is also fixing a gauge. Choosing to parametrise the (physical) fluctuations as

\[
\phi_U(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}
\]

we ensure that only physical fluctuations in the scalar field appear. This choice of gauge is called the Unitary gauge.

The fact that the goldstone modes disappear from the spectrum does not mean that the Standard Model possesses less degrees of freedom. In the unbroken gauge theory gauge fields posses only two physical polarisations. However, when the symmetry is spontaneously broken and these bosons acquire a mass, they must also have an additional physical polarisation. Since three bosons become massive, these means that there must be three additional degrees of freedom in the gauge sector. These are precisely the degrees of freedom that are missing in the scalar sector. To support this interpretation, it is easy to see that by performing a gauge transformation from the unitary gauge, the goldstone-like excitations only affect the longitudinal polarisation of the gauge bosons.

In the unitary gauge, the potential for the field \( h(x) \) is given by

\[
V(h) = \mu^2 h + \lambda v h^3 + \frac{1}{4} \lambda h^4,
\]

\[
\equiv \frac{1}{2} m_H^2 h^2 + \sqrt{\frac{\lambda}{2}} m_H h^3 + \frac{1}{4} \lambda h^4
\]

where we have introduced \( m_H \), the mass of the Higgs field, which it was recently measured at the LHC to be \( m_H = 125 \text{ GeV} \). In writing this form we have identified the Higgs mass with the parameters of the potential as

\[
m_H = \sqrt{2} \mu = \sqrt{2} \lambda v
\]

Note now that this is a positive mass term. Unlike the fluctuations \( \phi \), the fluctuation \( h \) occur around a minimum in the potential. This potential also include 3-point and 4-four point interactions of the Higgs field. The 3-point interaction are proportional to the Higgs mass while the 4-point interaction are solely controlled by the \( \lambda \)-coupling.

We can now look into the kinetic term

\[
\mathcal{L}_K = (D_\mu \phi)^\dagger (D^\mu \phi) = \frac{1}{2} \partial_\mu h \partial^\mu h + \left[ m_W^2 W^+ W^- + \frac{1}{2} m_Z^2 Z^2 \right] \left( 1 + \frac{h}{v} \right)^2
\]

(268)
which again includes triple Higgs-gauge boson vertices and four point vertices. All these interactions are also proportional to the mass of the gauge boson square. We may also rewrite the interaction strength

$$\frac{1}{v} = \frac{\sqrt{2\lambda}}{m_H}$$  \hspace{1cm} (269)

Finally, the Yukawa-like terms that give masses to the fermions also control the interactions of the gauge fields

$$\mathcal{L}_f = -m_f \bar{\psi} \psi \left(1 + \frac{h}{v}\right) = -m_f \bar{\psi} \psi \left(1 + \frac{\sqrt{2\lambda}}{m_H} h\right)$$  \hspace{1cm} (270)

which is also a triple vertex term controlled by the mass of the fermion.

The triple vertices inferred from the Lagrangian are displayed in figure (20) taken from Peskin and Schroeder.

Figure 20: Interactions of the Higgs field with itself and the rest of the particles of the Standard Model taken from Peskin and Schroeder.
13.2 Custodial symmetry

By specifying the scalar potential we have fully specified the standard model at classical level. We are now in a position to start considering quantum effects. The first correction we would like to address is self-energy corrections to the masses of the Electro Weak bosons.

In particular, let us concentrate on the classical relation between the $m_Z$ and $m_W$ we have derived

$$\rho_{\text{classical}} = \frac{m_W}{m_Z \cos \theta} = 1$$

(271)

However, when we start to think about mass corrections we realise that, since the coupling of the gauge fields to the Higgs and all other fermions are not identical, we would expect self-energy corrections to break this relation. In fact, if we think about the mass matrix for the electroweak bosons after quantum correction we would expect that it would be of the generic form

$$\mathcal{M} = \begin{pmatrix}
M_1^2 & 0 & 0 & 0 \\
0 & M_1^2 & 0 & 0 \\
0 & 0 & M_3^2 & -m^2 \\
0 & 0 & -m^2 & M_0^2
\end{pmatrix}$$

(272)

Only if $M_1 = M_3$ the relation between $m_W$ and $m_Z$ quoted above is satisfied, but we would expect from quantum corrections that this would not be true. However, the experimentally measured parameter $\rho$ is very close to 1, which means that there must be some additional symmetry that protects this ration. This symmetry is called custodial symmetry.

To understand the origin of this symmetry in the Standard Model we may represent the Higgs field as a four-vector of real scalar fields as

$$\Phi = \begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\varphi_4
\end{pmatrix}$$

(273)

Written in this way, we may consider global rotations of these real fields, which are given by elements of the group $O_4$. The kinetic term of the Lagrangian

$$\partial_\mu \phi \dagger \partial^\mu \phi \equiv \partial_\mu \Phi^i \partial^\mu \Phi,$$

(274)

is invariant under this transformation. And since

$$\phi^i \phi \equiv \Phi^i \Phi,$$

(275)

so is the Higgs potential.

After the Higgs field acquires a VeV, this global symmetry is also broken. However, since we can view the condensation of the Higgs field as only one of these real field acquiring an
expectation value
\[
\Phi_{\text{vac}} = \begin{pmatrix}
v \\
0 \\
0 \\
0 
\end{pmatrix},
\] (276)

a \( O(3) \sim SU(2) \) global symmetry is left, which involves rotating the three not condensed directions among themselves. The effect of this residual symmetry on the mass matrix is to impose \( M_1 = M_3 \), and for this reason this relation survives at quantum level.

Note however, that this \( SU(2) \) symmetry is not exact symmetry of the full Standard Model Lagrangian. Two type of terms break the symmetry: the coupling of the Higgs field to the hyper charge field \( V_\mu \); and the Yukawa-like coupling, unless the Yukawa couplings of the u and d-type quarks in the doublets are identical \( \lambda_u = \lambda_d \). Therefore, quantum corrections indeed break the tree level relation but only by terms suppressed by the small \( g' \) constant and with terms proportional to the difference of masses.