Recap

- Capacitors store charge; they resist a change of voltage

$$V = \frac{Q}{C} \qquad W = \frac{1}{2}CV^2$$

- Inductors oppose a change in current

$$V = L\frac{dI}{dt} \qquad W = \frac{1}{2}LI^2$$

- Transient circuits

RC in series – decays to long-time values with $\tau = RC$

RL in series – decays to long-time values with $\tau = \frac{L}{R}$

LCR series – damped s.h.m. with $\omega_0 = \frac{1}{\sqrt{LC}}$ and $\gamma = \frac{R}{2L}$

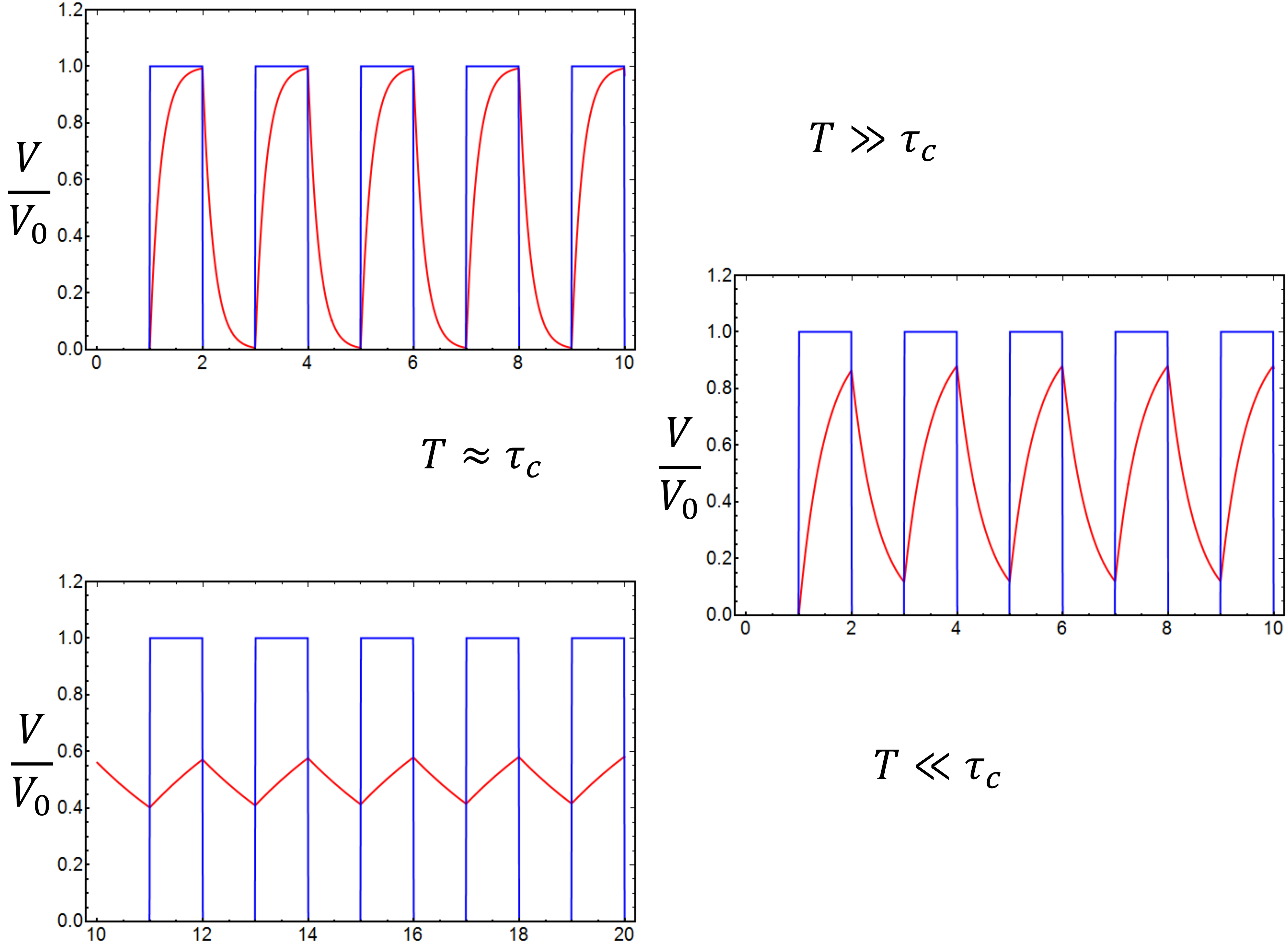
Over ($\gamma > \omega_0$), critical ($\gamma = \omega_0$) and under ($\gamma < \omega_0$) damped

Today: AC circuit analysis

- LCR circuits - what if we don’t have switches but have some other input instead?
  - E.g. sinusoidal input voltages (like the power from the wall)

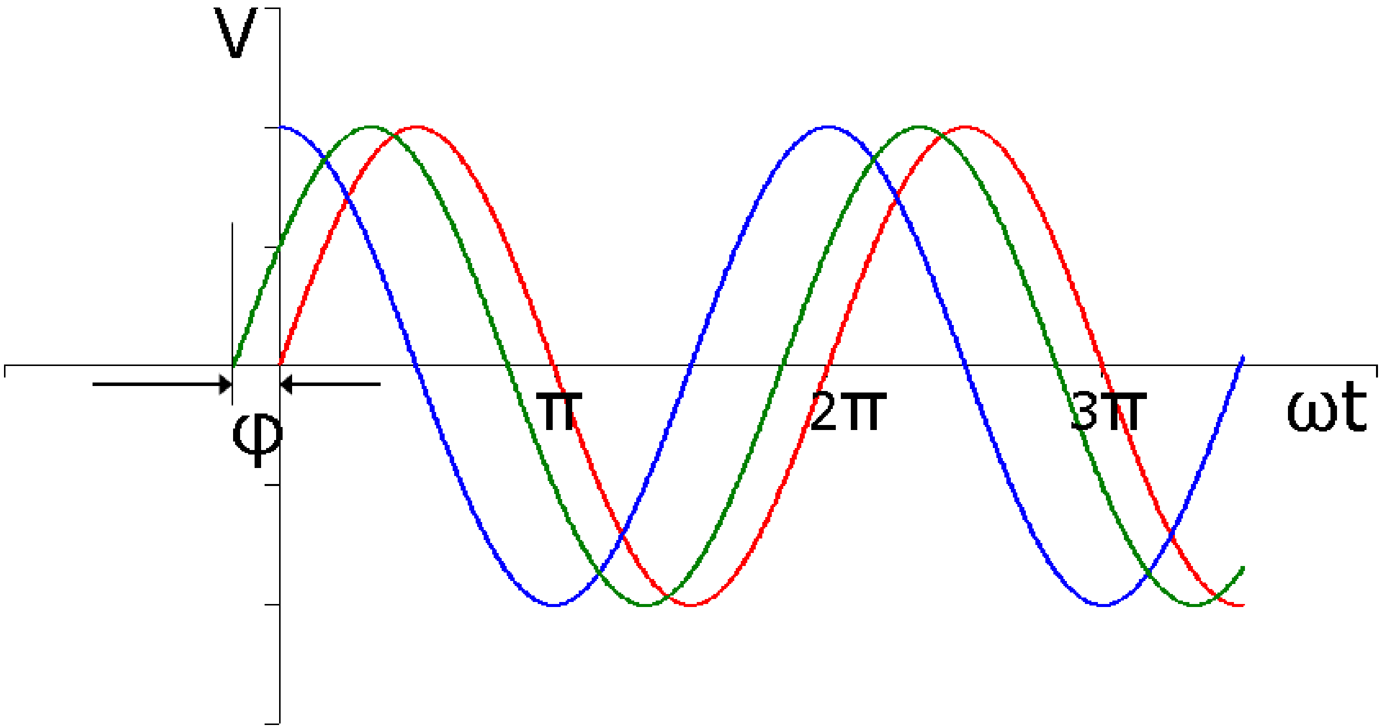
- Response of resistors, capacitors and inductors to AC sources

- Complex impedance



$\frac{V}{V_0}$
$T \gg \tau_c$
$T \approx \tau_c$
$T \ll \tau_c$
1.2 1.0 0.8 0.6 0.4 0.2 0.0
0 2 4 6 8 10
10 12 14 16 18 20

Sinusoidal voltages /currents



Amplitude

Phase

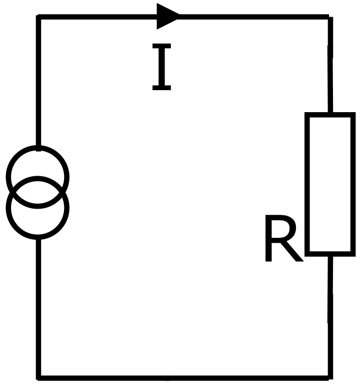
$V = V_0 \sin(\omega t)$

$V = V_0 \cos(\omega t)$

$V = V_0 \sin(\omega t + \varphi)$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

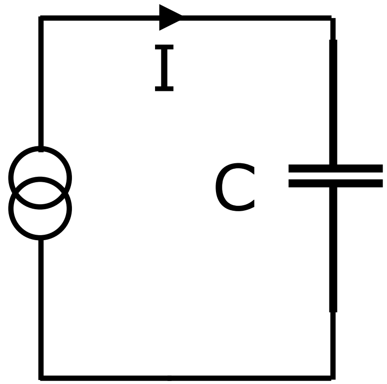
Apply AC current to a resistor…



$$I = I_0 \cos(\omega t)$$

$$V_R = RI = RI_0 \cos(\omega t)$$

…capacitor…



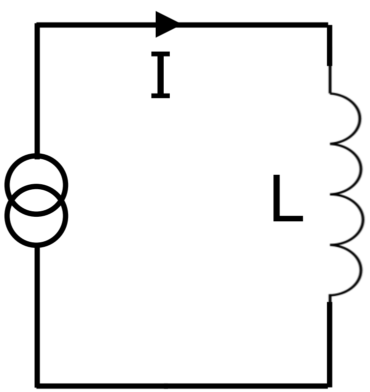
$$V_c = \frac{Q}{C} = \frac{1}{C}\int I_0 \cos(\omega t)dt$$

$$= \frac{1}{\omega C} I_0 \sin(\omega t)$$

$$= \frac{1}{\omega C} I_0 \cos\left(\omega t - \frac{\pi}{2}\right)$$

Voltage lags current by $\pi/2$

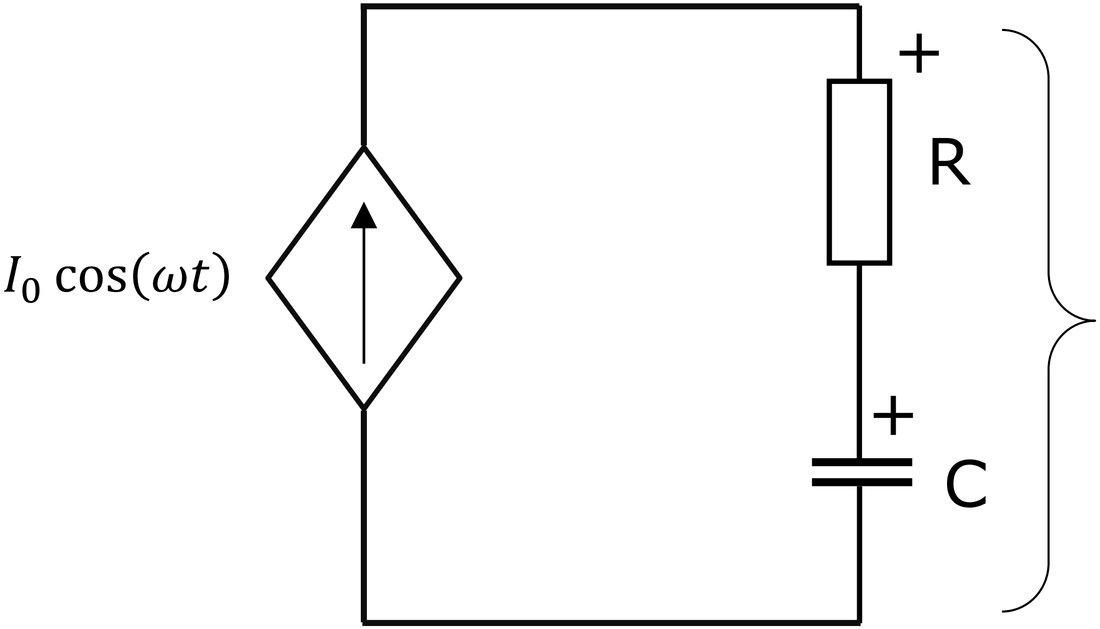
…and inductor



$$V_L = L\frac{dI}{dt} = -\omega L I_0 \sin(\omega t) = \omega L\, I_0 \cos\left(\omega t + \frac{\pi}{2}\right)$$

Voltage leads current by $\pi/2$

Adding voltages



$$V = V_R + V_c$$

voltages out of phase…

$$V = I_0\left(R\cos(\omega t) + \frac{1}{\omega C}\sin(\omega t)\right)$$

$$V = I_0 A \cos(\omega t + \phi)$$

$$A = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

$$\phi = -\arctan\left(\frac{1}{\omega R C}\right)$$

Complex Impedance

Tells you both the amplitude and phase of the voltage response when applying an oscillating current

$$Z = Ae^{i\phi} = |Z|e^{i\phi}$$

Resistor $Z_R = R$

Capacitor $Z_C = \frac{1}{i\omega C}$

Inductor $Z_L = i\omega L$

$$Z = R + iX$$

Resistance (R)   Reactance (X)

Generalised Ohm's law

$$I = I_0 \cos(\omega t) = \mathrm{Re}\left(I_0 e^{i\omega t}\right) = \mathrm{Re}(\tilde{I})$$

$$V = I_0 |Z| \cos(\omega t + \phi) = \mathrm{Re}\left(I_0 |Z| e^{i(\omega t + \phi)}\right)$$

$$V_0 \cos(\omega t + \phi) = \mathrm{Re}\left(I_0 |Z| e^{i\phi} e^{i\omega t}\right)$$

$$\mathrm{Re}\left(V_0 e^{i(\omega t + \phi)}\right) = \mathrm{Re}\left(I_0 Z e^{i\omega t}\right)$$

$$\mathrm{Re}\left(\tilde{V}\right) = \mathrm{Re}(\tilde{I} Z)$$

$$\tilde{V} = \tilde{I} Z$$

$\tilde{V}$ and $\tilde{I}$ are phasors

Phasors

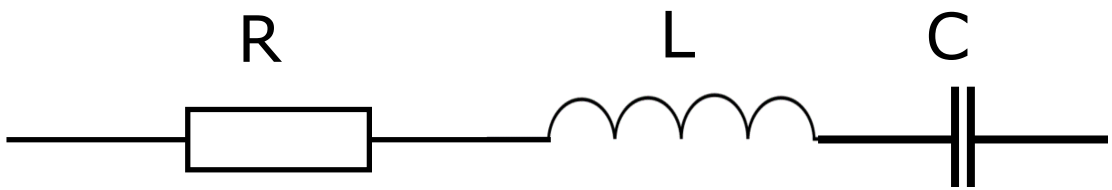
- Rotate around in complex plane
- Actual voltage is projection onto real axis

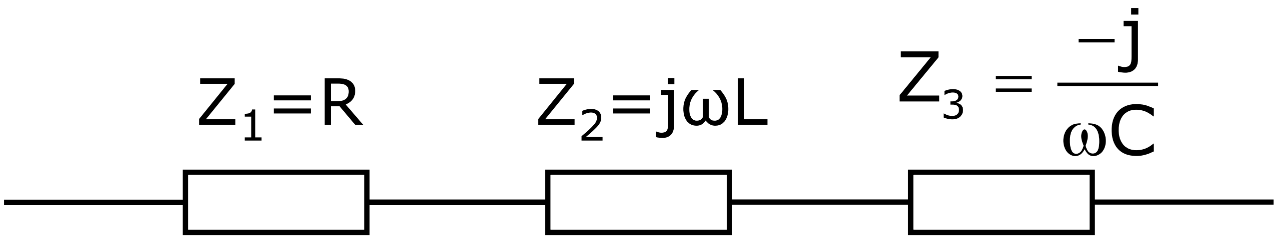
$$\tilde{V} = \tilde{I}Z \qquad \tilde{I} = \frac{\tilde{V}}{Z}$$

$$\left|\tilde{V}\right| = |Z||\tilde{I}|$$

$$\text{Arg}\left(\tilde{V}\right) = \text{Arg}(Z) + \text{Arg}(\tilde{I})$$

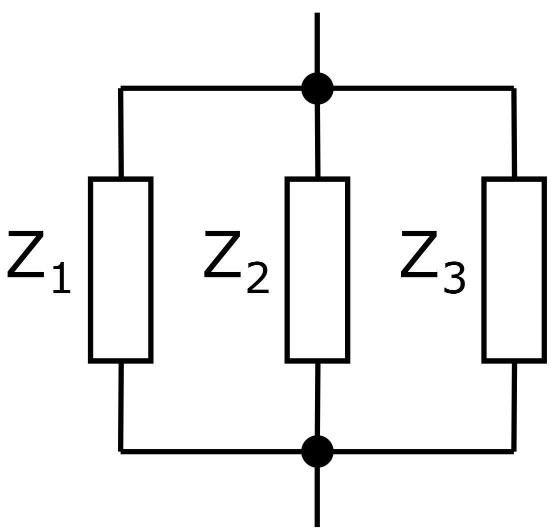
Series impedances





$$Z_T = \sum_n Z_n$$

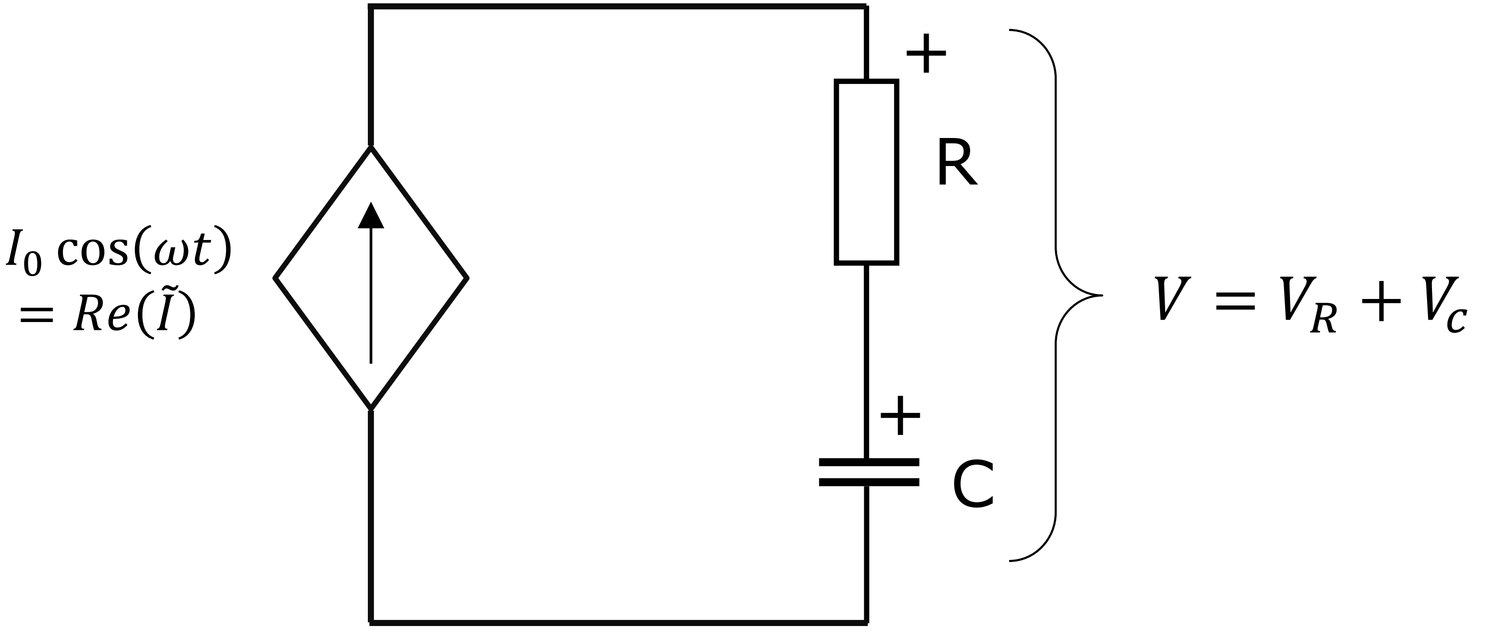
Parallel impedances



$$\frac{1}{Z_T} = \sum_n \frac{1}{Z_n}$$

$$= \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \cdots$$

Example



$$\tilde{V} = \tilde{I}Z = \tilde{I}\left(R - \frac{i}{\omega C}\right)$$

$$V = Re(\tilde{V}) = I_0|Z|\cos(\omega t + \phi)$$

$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

$$\phi = -\arctan\left(\frac{1}{\omega RC}\right)$$