1st-year Circuits
Lectures 5 to 7

Robert Smith

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10 Response of resistors, capacitors and inductors to AC sources

What happens if we apply an time varying current of the form $I = I_0 \cos(\omega t)$ to a resistor, capacitor or inductor?

- The voltage across a resistor will be given by $V_R = IR = I_0 R \cos(\omega t)$.
- The voltage across a capacitor will be $V_C = \frac{Q}{C} = \frac{1}{C}\int I dt = \frac{1}{\omega C} I_0 \sin(\omega t) = \frac{1}{\omega C} I_0 \cos(\omega t - \pi/2)$; the voltage *lags* the current by $\pi/2$ – the capacitor takes time to charge.
- The voltage across an inductor will be $V_L = L\frac{dI}{dt} = -\omega L I_0 \sin(\omega t) = \omega L I_0 \cos(\omega t + \pi/2)$; the voltage *leads* the current by $\pi/2$ or the current *lags* the voltage by $\pi/2$ – inductors resist a change in current.

We see that for inductors and capacitors, in addition to the components determining the magnitude of the voltage response they also determine the *phase*. We can encode this information in a *complex impedance* usually denoted Z. As Z is a complex number it can tell us both about the amplitude and phase of the voltage response.

11 Complex impedance

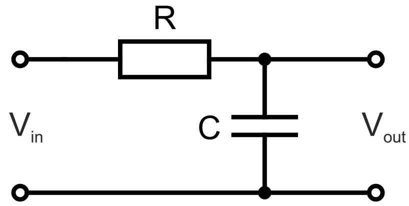
An applied current of $I = I_0 \cos(\omega t)$ can be written as $I = \mathrm{Re}(I_0 e^{i\omega t})$ [we can also write $I_0 \sin(\omega t)$ as $\mathrm{Im}(I_0 e^{i\omega t})$], we can then write the generalised version of Ohm's law $V = \mathrm{Re}(Z I_0 e^{i\omega t}) = |Z| I_0 \cos(\omega t + \phi)$ where Z is the complex impedance which determine both the magnitude and phase of the voltage response.

- For a resistor $Z_R = R$.
- For a capacitor $Z_C = \frac{1}{i\omega C}$. Note – it acts like a open circuit as $\omega \to 0$ and a short circuit as $\omega \to \infty$.
- For an inductor $Z_L = i\omega L$. Note – it acts like a short circuit as $\omega \to 0$ and a open circuit as $\omega \to \infty$.
- In a network impedances combine in the same way as resistors: in series $Z_T = \sum_n Z_n$ and in parallel $\frac{1}{Z_T} = \sum_n \frac{1}{Z_n}$.
- The real part of Z is called the resistance (R) and imaginary part the reactance (denoted X).

Note that the fact that we are concerned with the real (or imaginary) parts is usually taken to be implicit such that we often write, for example, $V = IZ$, but it should be remembered that in reality the voltage and currents always have real values. The complex 'currents' and 'voltages', in this approach (which can helpfully distinguished with tilde, i.e $\tilde{V} = \tilde{I}Z$) are called *phasors* and are effectively rotating vectors in the complex plane - the actual voltages and currents are given by the projection of the phasor onto the real (or imaginary) axis. Remembering we are dealing with phasors will become important when we come to consider power in AC circuits.

12 Some example circuits

12.1 Filters

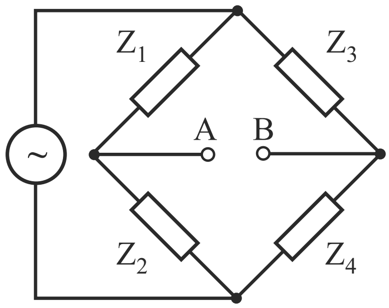


A network of passive components can be used as a filter in which an output voltage ($V_{\rm out}$) across some part of the circuit is a frequency dependent fraction of some input voltage $V_{\rm in}$ such that we can write $V_{\rm in}/V_{\rm out} = f(\omega)$. A simple example of this is the low-pass RC filter shown on the right, which is just a potential divider

$$\frac{\tilde{V}_{\rm out}}{\tilde{V}_{\rm in}} = \frac{\frac{1}{i\omega C}}{\frac{1}{i\omega C} + R} = \frac{1}{1 + i\omega RC}.$$

The filter is characterized by its amplitude ($|V_{\rm out}/V_{\rm in}|$) and phase ($\mathrm{Arg}(V_{\rm out}/V_{\rm in})$) response which are both functions of ω and usually represented in what are known as Bode plots which show the amplitude and phase response versus ω (often on a log-log plot). Note that we have assumed the input voltage source has no internal impedance and that no current is drawn from the output.

12.2 Bridge circuits



A bridge circuit (as shown to the right) is often used to measure an unknown impedance (e.g. Z_4). Generally, some known impedances (e.g. $Z_{1,2,3}$) are adjusted until the voltage (V_{AB}) across the bridge is nulled (= 0). The unknown impedance can be inferred using $V_A = Z_1/(Z_1 + Z_2)$, $V_B = Z_3/(Z_3 + Z_4)$ such that $V_{AB} = 0$ implies

$$\frac{Z1}{Z_1 + Z_2} = \frac{Z_3}{Z_3 + Z_4} \mapsto Z_1 Z_4 = Z_2 Z_3.$$

This expression can be used to find both the real and imaginary parts of the unknown complex impedance.

13 Power dissipation in AC circuits

The power flowing out of a circuit into an electrical component is given by $P = IV$. However, if we are using the complex representation for V and I then when calculating the power we need to be careful to use $P = \mathrm{Re}(\tilde{I}) \times \mathrm{Re}(\tilde{V})$. For AC circuits it is usual to consider the average power dissipated per cycle,

$$\langle P \rangle = \langle \mathrm{Re}(\tilde{I}) \times \mathrm{Re}(\tilde{V}) \rangle = \frac{1}{T} \int_0^T \mathrm{Re}(\tilde{I}) \times \mathrm{Re}(\tilde{V}) dt,$$

where $T = 2\pi/\omega$ is the cycle time. There are various equivalent ways to calculate and express this

$$\langle P \rangle = \langle I_0 \cos(\omega t) \times I_0 |Z| \cos(\omega t + \phi) \rangle = I_0^2 |Z| \left(\cos(\phi) \langle \cos^2(\omega t) \rangle - \sin(\phi) \langle \cos(\omega t) \sin(\omega t) \rangle \right)$$

where we have employed the compound angle formula. Performing the time averages gives:

$$\langle P \rangle = \frac{1}{2} I_0^2 |Z| \cos(\phi) = \frac{1}{2} V_0 I_0 \cos(\phi) = \frac{1}{2} I_0^2 \,\mathrm{Re}(Z).$$

Note that, using the fact that $V_0 = |Z| I_0$ we can also write

$$\langle P \rangle = \frac{1}{2} V_0 I_0 \cos(\phi) = V_{\rm rms} I_{\rm rms} \cos(\phi)$$

where $V_{\rm rms}$ and $I_{\rm rms}$ are the root mean squared values of the voltage and current and $\cos(\phi)$ is sometimes called the power factor. Note that for a sinusoidal voltage (and equivalently current) $V_{\rm rms} = \frac{V_0}{\sqrt{2}}$ (remember that V_0 is the *amplitude* of the voltage) but in general:

$$V_{\rm rms} = \sqrt{\frac{1}{T} \int_0^T V^2(t) dt}.$$