1st-year Circuits
Lectures 3 and 4

Robert Smith

Michaelmas 2019

6 Capacitors

- Capacitors store charge. The capacitance C is defined as the charge stored per unit voltage $C = \frac{Q}{V}$; it is measured in Farads (F) or more usually in µF, nF or pF.
- Capacitors in series add as $\frac{1}{C_\mathrm{T}} = \sum_n \frac{1}{C_n}$ and in parallel as $C_\mathrm{T} = \sum_n C_n$.
- The energy stored on a capacitor is given by $W = \frac{1}{2}CV^2$. The energy is stored in the electric field.

7 Inductors

- Inductors oppose a change in current. They are usually coils of wire; a current passing through them results in a linked magnetic field flux Φ; changing the current through the circuit means changing the magnetic flux which (via Faraday's law) results in an induced voltage: $V = \frac{d\Phi}{dt}$ which opposes the change in current. The inductance is defined via $V = L\frac{dI}{dt}$, it tells you the voltage that will be induced if the current is changed at 1 Ampere per second. Inductance is measured in Henrys (H).
- Inductors in series add as $L_\mathrm{T} = \sum_n L_n$ and in parallel as $\frac{1}{L_\mathrm{T}} = \sum_n \frac{1}{L_n}$.
- The energy stored on a inductor is given by $W = \frac{1}{2}LI^2$. The energy is stored in the magnetic field.

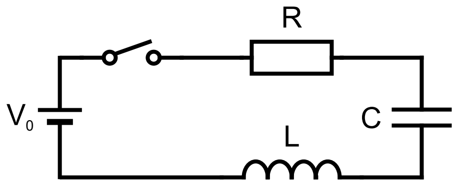
8 Transients in RC and RL circuits

A sudden change in a RC (resistor/capacitor) or RL (resistor / inductor) circuit, such as closing a switch, results in a time-varying transient response. Such problems can be tackled as follows:

- Define current directions and apply the passive sign convention as normal.
- Apply Kirchoff's laws, remembering that for a capacitor $V = \frac{Q}{C}$ and for an inductor $V = L\frac{dI}{dt}$.
- This will lead to a (or a set of) differential equation(s). Solve these equations subject to the appropriate boundary conditions (ask what is the situation at $t = 0$ and at $t = \infty$) taking into account that:
  - it is impossible to change the voltage on a capacitor instantly (because charge has to build up and so that would require infinite current),
  - it is impossible to change the current on a inductor instantly (because that would result in infinite voltage).
- For a series RC and RL circuit the transient response is characterised by a decay/growth of current/voltage with a time constants of $\frac{1}{RC}$ and $\frac{L}{R}$ respectively.

9 LCR series circuit

Consider the circuit below. The capacitor is initially uncharged and at time $t = 0$ the switch is closed.



Applying KVL we find:

$$V_0 = IR + \frac{Q}{C} + L\frac{dI}{dt} \tag{1}$$

where Q is the charge on the capacitor and $I = \frac{dQ}{dt}$ the current in the circuit. This can be re-written as

$$V_0 = L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q, \tag{2}$$

or

$$0 = L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I. \tag{3}$$

These are equations for damped simple harmonic motion; note there is a direct equivalence with a mechanical oscillator, with the displacement $x \mapsto Q$, the velocity $v \mapsto I$, the mass $m \mapsto L$ and the spring constant $k \mapsto 1/C$, R defines the damping. The general solution of Eq. (3) is $I = A\mathrm{e}^{\lambda_1 \mathrm{t}} + \mathrm{B}\mathrm{e}^{\lambda_2 \mathrm{t}}$ where $\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 + \omega_0^2}$ with $\gamma = \frac{R}{2L}$ and $\omega_0^2 = \frac{1}{LC}$. The nature of the solution depends on γ/ω_0 leading to three different types of solution. In this instance, using the $t = 0$ boundary conditions: $I = 0$ (because of the inductor) and $L\frac{dI}{dt} = V_0$ (because initially $Q = 0$), we have the following solutions:

- Over-damped ($\gamma > \omega_0$)

$$I = \frac{V_0}{2L\sqrt{\gamma^2 - \omega_0^2}} \left(\mathrm{e}^{-(\gamma - \sqrt{\gamma^2-\omega_0^2})t} - \mathrm{e}^{-(\gamma + \sqrt{\gamma^2-\omega_0^2})t} \right). \tag{4}$$

- Critically damped ($\gamma = \omega_0$)

$$I = \frac{V_0}{L} t\mathrm{e}^{-\gamma t}. \tag{5}$$

- Under-damped ($\gamma < \omega_0$)

$$I = \frac{V_0}{L\sqrt{\omega_0^2 - \gamma^2}}\, \mathrm{e}^{-\gamma t} \sin\left(\sqrt{\omega_0^2 - \gamma^2}\; t \right). \tag{6}$$

Example solutions are shown below, where we have chosen $C = 1\,\mu\mathrm{F}$, $L = 10\,\mathrm{mH}$ and $R = 33$, 200 and $600\,\Omega$ leading to a fixed $\omega_0 = 10^4$ rad s^{-1} and $\gamma/\omega_0 = 1/6, 1$ and 3 for the under-, critical- and over-damped cases respectively.