C2: Quantum information basics

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Lecture 2

3 The density operator

Pure states can be represented with a state vector:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{10}$$

We can equivalently describe this state by a density operator (matrix),

$$\hat{\rho} = |\psi\rangle\langle\psi| = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}, \tag{11}$$

which can be used as an (equivalent) alternative to the state vector. So for example, the expectation value of an operator $\hat{M}$ is given by

$$\langle\psi|\hat{M}|\psi\rangle = \mathrm{Tr}(\hat{\rho}\hat{M}), \tag{12}$$

and evolution in the presence of a propagator is given by

$$\hat{\rho} \to U\hat{\rho}U^\dagger. \tag{13}$$

For mixed states the density operator is given by a sum of pure state density operators with appropriate classical probabilities P_n:

$$\hat{\rho} = \sum_n P_n|\psi_n\rangle\langle\psi_n| \tag{14}$$

which means that for a pure state (i.e. a state with only one non-zero probability) we have $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho) = 1$ whereas for any mixed state $\mathrm{Tr}(\rho^2) < 1$.

For a qubit it is clear that the density matrix must be a linear combination of the identity and Pauli matrices and in fact one can show that it is intimately related to the Bloch vector:

$$\hat{\rho} = \frac{1}{2}(\sigma_0 + V_x^{\mathrm{B}}\sigma_x + V_y^{\mathrm{B}}\sigma_y + V_z^{\mathrm{B}}\sigma_z). \tag{15}$$

Any mixed state has a Bloch vector with a norm $|\mathbf{V}^{\mathrm{B}}| < 1$ and the maximally mixed state has $|\mathbf{V}^{\mathrm{B}}| = 0$ and lies at the center of the Bloch sphere.

The density operator is particularly useful when we have (i) more than one qubit (ii) measurement (iii) decoherence (due to interaction with the environment).

4 Initialisation and measurement

So far we have considered unitary (reversible) gates (operations) but several important operations (i.e. measurement and initialization) are non-unitary. If we consider measuring an ensemble of qubits all in the state $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ then the density matrix after measurement will be given by

$$\rho = \alpha\alpha^*|0\rangle\langle 0| + \beta\beta^*|1\rangle\langle 1| = \alpha\alpha^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta\beta^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha\alpha^* & 0 \\ 0 & \beta\beta* \end{pmatrix}. \tag{16}$$

Note that all the off-diagonal elements have gone to zero which means we have lost coherence. This can also happen due to other (unplanned) interactions with the environment.

5 Physical implementation of qubits and single qubit gates

Any quantum system with two orthogonal states can be used for a qubit. Examples include:

- Atoms/ions.
- Spins in magnetic fields - e.g. nuclear spins (aka NMR) or solid state impurity spins (nitrogen vacancy centres in diamond).
- Artificial solid state systems - e.g. superconducting circuits and quantum dots.
- Photons - where the two orthogonal states are either encoded in position or polarisation.

5.1 Gate implementation: Atom/ion in a laser field

A two-level system (where the two levels $|g\rangle$ and $|e\rangle$ are eigenstates of the Hamiltonian $\mathcal{H}_0$ with energies $-\hbar\omega_0/2$ and $+\hbar\omega_0/2$) is subjected to an oscillating electric field $\boldsymbol{E} = E_0\cos(\omega t)\hat{\boldsymbol{z}}$ which results in the time-varying Hamiltonian

$$\mathcal{H}' = -\boldsymbol{E}.\boldsymbol{d} = \frac{1}{2}ezE_0(\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t}). \tag{17}$$

Making the rotating-wave approximation (RWA) gives

$$\mathcal{H}' = \frac{1}{2}ezE_0\mathrm{e}^{i\omega t}. \tag{18}$$

Applying the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = (\mathcal{H}_0 + \mathcal{H}')|\psi\rangle, \tag{19}$$

to the state

$$|\psi\rangle = \begin{pmatrix} c_g \\ c_e \end{pmatrix} \tag{20}$$

gives

$$i\hbar\begin{pmatrix} \dot{c_g} \\ \dot{c_e} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} -\omega_0 & V\mathrm{e}^{i\omega t} \\ V^*\mathrm{e}^{-i\omega t} & \omega_0 \end{pmatrix}\begin{pmatrix} c_g \\ c_e \end{pmatrix} \tag{21}$$

where $V = \langle g|ezE_0|e\rangle/\hbar$ and we have used the fact that $\langle g|ezE_0|g\rangle = \langle e|ezE_0|e\rangle = 0$ by symmetry. Now going into the rotating frame, by defining $c_g = d_g\mathrm{e}^{i\omega t/2}$ and $c_e = d_e\mathrm{e}^{-i\omega t/2}$, transforms Eq.(21) into

$$i\hbar\begin{pmatrix} \dot{d_g} \\ \dot{d_e} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} -\delta & V \\ V^* & \delta \end{pmatrix}\begin{pmatrix} d_g \\ d_e \end{pmatrix} \tag{22}$$

such that the the effective Hamiltonian (in the rotating frame) is

$$\tilde{\mathcal{H}} = \frac{\hbar}{2}\begin{pmatrix} -\delta & V \\ V^* & \delta \end{pmatrix} = \frac{\hbar}{2}\left(-\delta\sigma_z + V\sigma_x\right). \tag{23}$$

Adding a phase ϕ to the driving field modifies this to

$$\tilde{\mathcal{H}} = \frac{\hbar}{2}\left(-\delta\sigma_z + V\cos(\phi)\sigma_x - V\sin(\phi)\sigma_y\right). \tag{24}$$

We can thus, in principle, apply an arbitrary rotation (within the rotating frame) to the state.