Quantum information basics
Lecture 1

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1 Qubits: 2-level quantum systems

A qubit is just a 2-level quantum system; we will call the lower level $|0\rangle$ and the upper level $|1\rangle$. The most general quantum state can thus be written:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{1}$$

where α and β are complex numbers. Due to normalisation ($|\alpha|^2 + |\beta|^2 = 1$) and the fact that the overall phase is unimportant we can represent this state as a point on the surface of a sphere of unit radius:

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2)|1\rangle, \tag{2}$$

where θ and ϕ are the usual polar and azimuthal angles respectively. In this context this sphere is called the Bloch Sphere and the vector going from the origin to the point on the surface is called the Bloch vector ($\mathbf{V}^{\mathrm{B}}$). The Bloch vector is the vector of expectation values of the Pauli matrices:

$$\mathbf{V}^{\mathrm{B}} = \langle\psi|\boldsymbol{\sigma}|\psi\rangle = \begin{pmatrix} \langle\psi|\sigma_x|\psi\rangle \\ \langle\psi|\sigma_y|\psi\rangle \\ \langle\psi|\sigma_z|\psi\rangle \end{pmatrix}. \tag{3}$$

1.1 Pauli matrices

For a 2-level system any operator can be written as a linear combination of the identity matrix (σ_0) and the Pauli matrices (σ_α) which are:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

The Pauli matrices are both unitary and Hermitian, they have eigenvalues of ± 1, and they obey the following useful identities:

$$\sigma_i^2 = \sigma_0, \qquad \sigma_i\sigma_{j\neq i} = i\epsilon_{ijk}\sigma_k, \qquad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k. \tag{5}$$

2 Gates: evolution of states and propagators

The evolution of a state under the influence of a Hamiltonian $\mathcal{H}$ is described by the time dependent Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}|\psi\rangle \tag{6}$$

which has the solution

$$|\psi(t)\rangle = U|\psi(0)\rangle \tag{7}$$

where the propagator U is given by

$$U = \exp\left(\frac{-i\mathcal{H}t}{\hbar}\right). \tag{8}$$

For a 2-level system any Hamiltonian can be represented as a linear combination of the Pauli matrices; for $\mathcal{H} = \frac{\hbar\Omega}{2}\sigma_\alpha$ we have

$$U = \exp\left(-i\frac{\Theta}{2}\sigma_\alpha\right) = \cos(\Theta/2)\sigma_0 - i\,\sin(\Theta/2)\sigma_\alpha \tag{9}$$

where $\Theta = \Omega t$. The effect of this propagator is to rotate $\mathbf{V}^{\mathrm{B}}$ around the α-axis by an angle of Θ. Some example gates are given below.

Gate	Hamiltonian, $\mathcal{H}$	rotation, Ωt	Propagator, U
Indentity	-	2π	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
NOT, X	$\frac{\hbar\Omega}{2}\sigma_x$	π	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
Y	$\frac{\hbar\Omega}{2}\sigma_y$	π	$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$
Z	$\frac{\hbar\Omega}{2}\sigma_z$	π	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
Phase, S	$\frac{\hbar\Omega}{2}\sigma_z$	$\pi/2$	$\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$
Small-angle Phase, T	$\frac{\hbar\Omega}{2}\sigma_z$	$\pi/4$	$\begin{pmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{pmatrix}$
Hadamard, H	$\frac{\hbar\Omega}{2}\frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$	π	$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$