CHAOS FROM MAPS

Lecture 7: 1-dimensional Maps

7. Chaos from Maps

Now we turn to a new class of dynamical system, in which time is *discrete* rather than continuous. These systems are known as difference equations, recursion relations, iterated maps or simply **maps**.

Consider $x_{n+1} = \cos x_n$ (try this on a calculator in radians mode!) This is an example of a 1-dimensional map - the sequence $x_0, x_1, x_2 ....$ is called the *orbit* starting from x_0.

Maps arise in various ways:

- As tools for analysing differential equations (e.g. Poincaré and Lorenz)

- As models of natural phenomena (inc. economics and finance!)

- As simple examples of chaos

Maps are capable of much wilder behaviour than differential equations because the points x_n hop discontinuously along their orbits rather than flow continuously.

7.1 Fixed points and Cobwebs

Consider $x_{n+1} = f(x_n)$, where f is a smooth function from the real line onto itself.

Suppose x^* satisfies $f(x^*) = x^* \Rightarrow x^*$ is a *fixed point* of the map.

Its stability is determined by considering a nearby orbit $x_n = x^* + \eta_n$. Thus

$$x^* + \eta_{n+1} = x_{n+1} = f(x^*) + f'(x^*)\eta_n + O(\eta_n^2).$$

Since $f(x^*) = x^*$,

$$\eta_{n+1} = f'(x^*)\eta_n$$

is the *linearized map* and $\lambda = f'(x^*)$ is the eigenvalue or *multiplier*.

- If $|\lambda| = |f'(x^*)| < 1$ then $\eta_n \to 0$ as $n \to \infty \Rightarrow x^*$ is linearly stable.

- If $|\lambda| = |f'(x^*)| > 1 \Rightarrow x^*$ is unstable.

- If $|\lambda| = |f'(x^*)| = 1$ then we need to consider the terms $O(\eta_n^2)$.

Example 7.1.1 $x_{n+1} = x_n^2$

Fixed points at $x^* = (x^*)^2 \Rightarrow x^* = 0, 1$.
$\lambda = f(x^*) = 2x^* \Rightarrow x^* = 0$ is stable and $x^* = 1$ unstable.

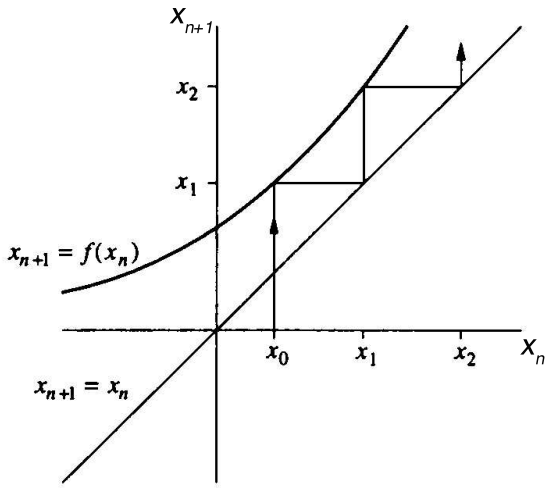


Fig. 7.1.1

Cobwebs allow us to see global behaviour at a glance.

Example 7.1.2 $x_{n+1} = \cos x_n$

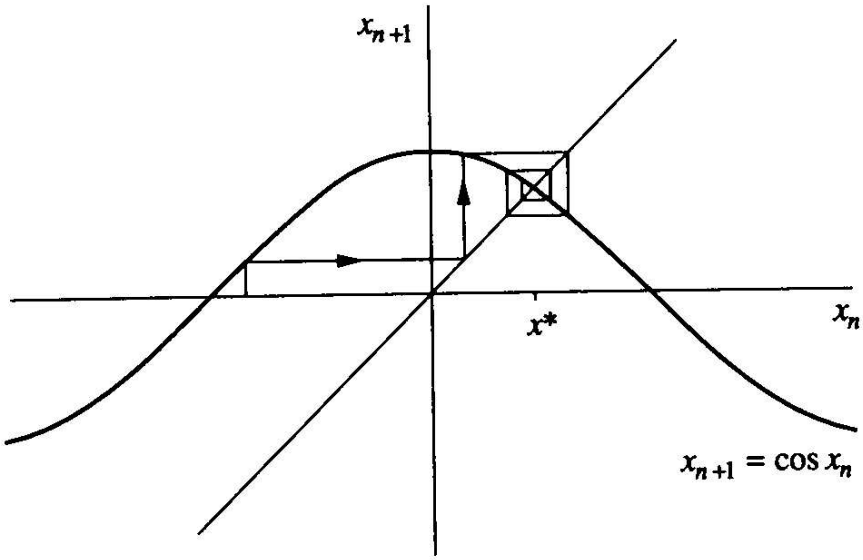


Fig. 7.1.2

$x_n \to x^*$ through damped oscillations.

7.2 The Logistic Map: numerics

$x_{n+1} = rx_n(1 - x_n)$ is the discrete-time analogue of the population growth model discussed in lecture 2.

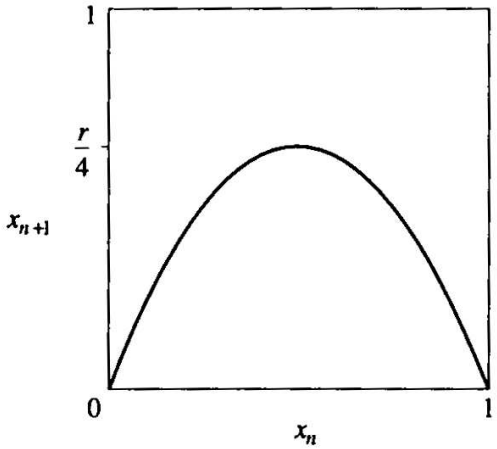


Fig. 7.2.1

Let $0 \leq r \leq 4, 0 \leq x \leq 1 \Rightarrow$ map is a parabola with maximum value of $r/4$ at $x = 1/2$.

- For $r < 1, x_n \to 0$ as $n \to \infty$ (proof by cobwebbing).

- For $1 < r < 3, x_n$ grows as n increases, reaching a non-zero steady state.

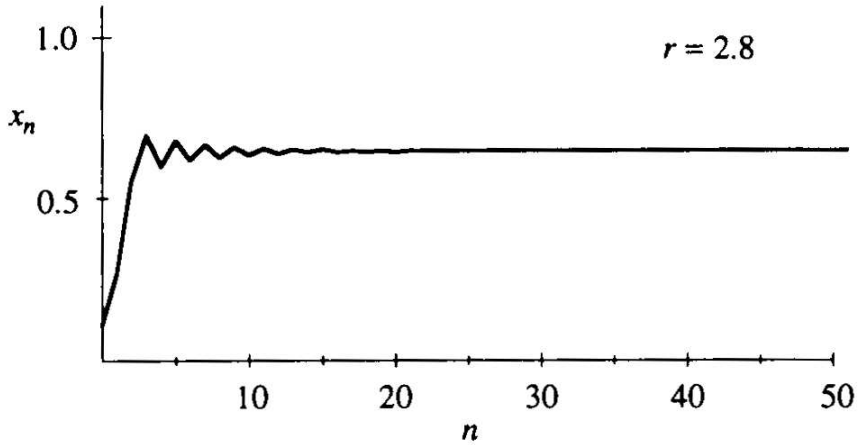


Fig. 7.2.2

For larger r (e.g. $r = 3.3$) x_n eventually oscillates about the former steady state $\Rightarrow$ *period-2 cycle*.

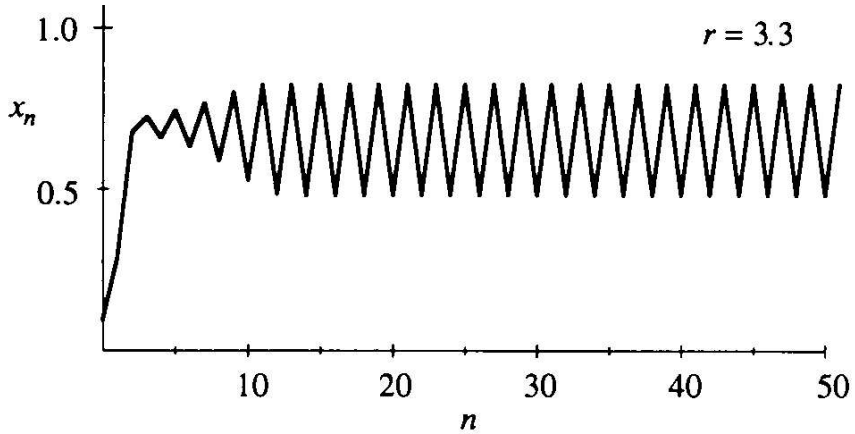


Fig. 7.2.3

At still larger r (e.g. $r = 3.5$), x_n approaches a cycle which repeats every 4 generations $\Rightarrow$ *period-4 cycle*.

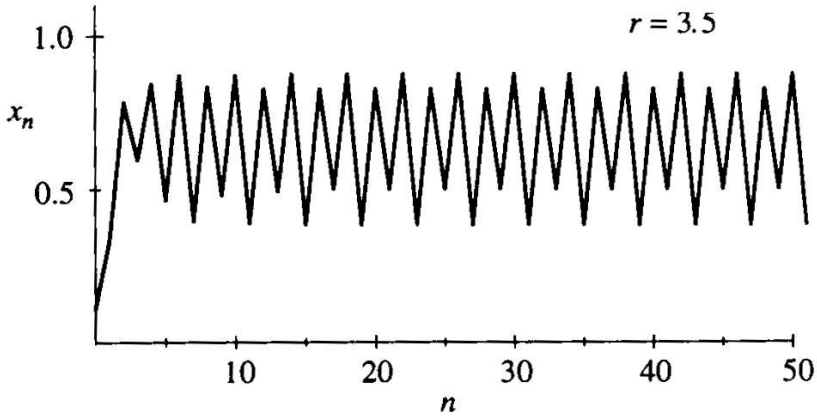


Fig. 7.2.4

Further *period doublings* to cycles of period 8, 16, 32... occur as r increases. Computer experiments show that

$$
\begin{aligned}
r_1 &= 3 \quad \text{(period 2 is born)} \\
r_2 &= 3.449... \quad \text{(period 4 is born)} \\
r_3 &= 3.54409... \quad \text{(period 8 is born)} \\
r_4 &= 3.5644... \quad \text{(period 16 is born)} \\
&\;\;\vdots \\
r_\infty &= 3.569946... \quad \text{(period } \infty \text{ is born)}
\end{aligned}
$$

- The successive bifurcations come faster and faster as r increases.

- The r_n converge to a limiting value r_∞.

- For large n, the distance between successive transitions shrinks by a constant factor

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$

Chaos and periodic windows

What happens for $r > r_\infty$? The answer is complicated! For many values of r, the sequence $\{x_n\}$ never settles down to a fixed point or a periodic orbit - the long term behaviour is *aperiodic*.

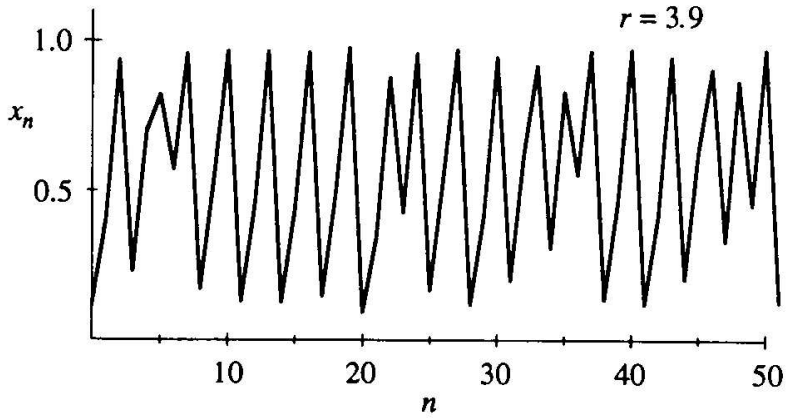


Fig. 7.2.5

This is a *discrete-time* version of the chaos found for the Lorenz Equations. The corresponding cobweb diagram is very complicated...

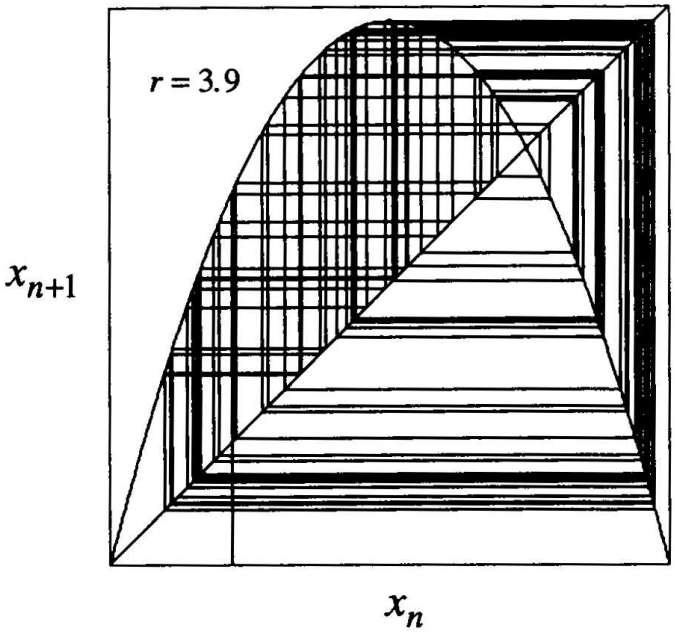
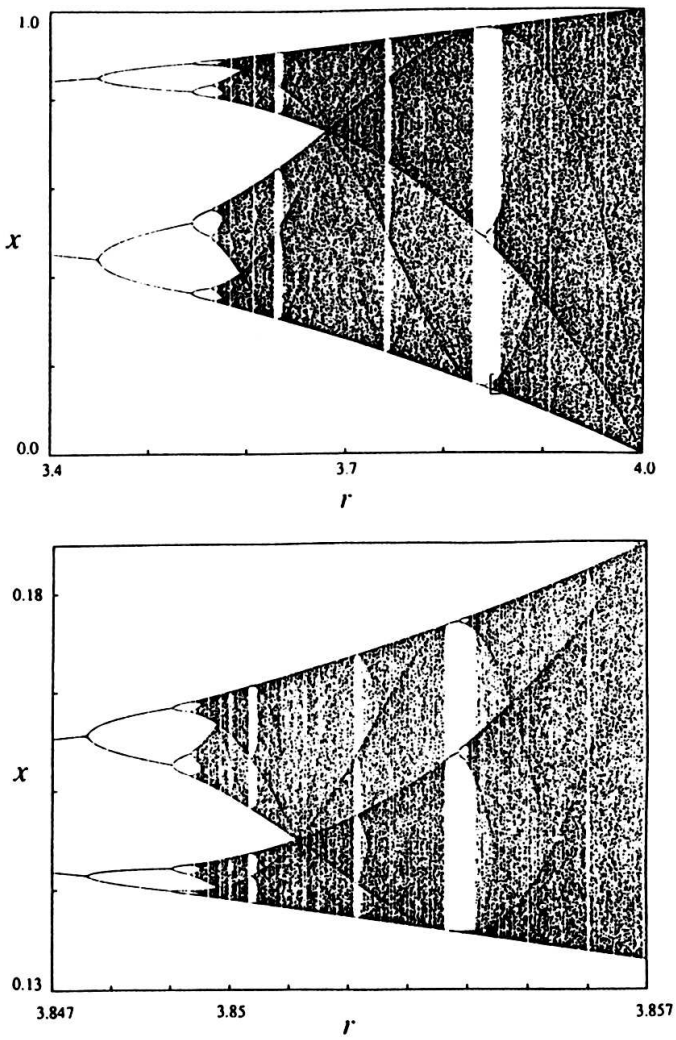


Fig. 7.2.6

One might think that the system would become more and more chaotic as r increases, but in fact the dynamics are more subtle...

The orbit diagram shows the long-term behaviour for *all* values of r at once...



blown-up version around $r = 3.85$

Fig. 7.2.7

- At $r = 3.4$ the attractor is a *period-2 cycle*.

- As r increases, both branches split, giving a *period-4 cycle* - i.e. a period-doubling bifurcation has occurred.

- A cascade of further period-doublings occurs as r increases, until at $r = r_\infty \simeq 3.57$, the *map becomes chaotic* and the attractor changes from a finite to an infinite set of points.

- For $r > r_\infty$, the orbit reveals a mixture of order and chaos, with periodic windows interspersed with chaotic clouds of dots.

- The large window near $r \simeq 3.83$ contains a stable period-3 cycle. A blow-up of part of this window shows that a *copy* of the orbit diagram *reappears in miniature!*

Logistic Map: analysis

Consider $x_{n+1} = rx_n(1 - x_n)$; $\quad 0 \leq x_n \leq 1$ and $0 \leq r \leq 4$.

Fixed points $x^* = f(x^*) = rx^*(1 - x^*)$
$\Rightarrow x^* = 0$ or $1 - 1/r$.

- $x^* = 0$ is a fixed point for all r
- $x^* = 1 - 1/r$ is a fixed point only if $r \geq 1$ (recall $0 \leq x_n \leq 1$).

Stability depends on $f'(x^*) = r - 2rx^*$

- $x^* = 0$ is stable for $r < 1$ and unstable for $r > 1$.
- $x^* = 1 - 1/r$ is stable for $-1 < (2 - r) < 1$, i.e. for $1 < r < 3$, and unstable for $r > 3$.

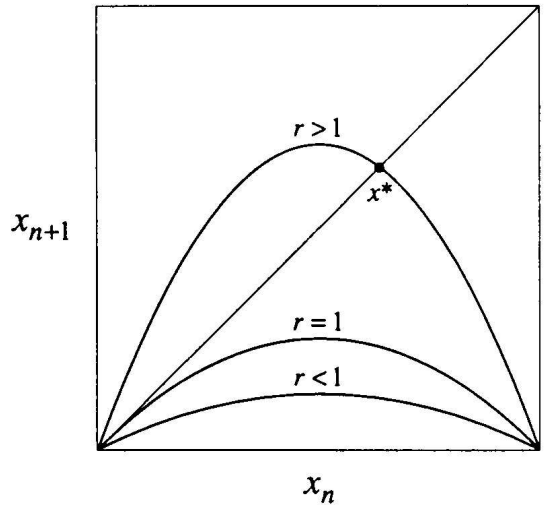


Fig. 7.3.1

- At $r = 1$, x^* bifurcates from the origin in a *transcritical bifurcation*.

- As r increases beyond 1, the slope at x^* gets increasingly steep. The critical slope $f'(x^*) = -1$ is attained when $r = 3$ - the resulting bifurcation is called a *flip bifurcation* $\Rightarrow$ 2-cycle!

We now go on to show that the logistic map has a 2-cycle for all $r > 3 \ldots$

A 2-cycle exists if and only if there are two points p and q such that $f(p) = q$ and $f(q) = p$. Equivalently, such a p must satisfy $f(f(p)) = p$ where $f(x) = rx(1-x)$. Hence, p is a fixed point of the *second iterate map* $f^2(x) = f(f(x))$. Since $f(x)$ is a quadratic map, $f^2(x)$ is a quartic polynomial. Its graph for $r > 3$ is...

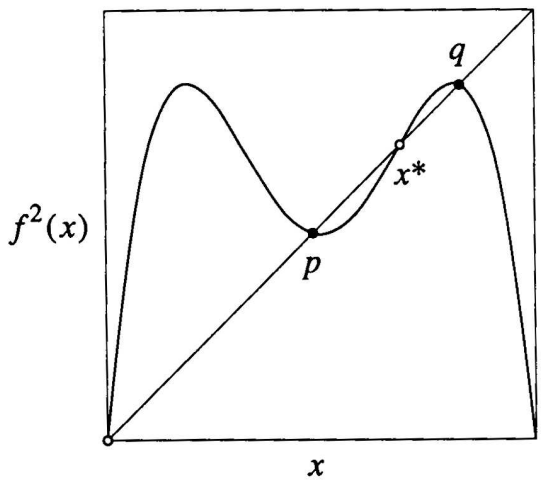


Fig. 7.3.2

We must now solve $f^2(x) = x$.

$x^* = 0$ and $x^* = 1 - 1/r$ are trivial solutions. The other 2 solutions are

$$p, q = \frac{r + 1 \pm \sqrt{(r-3)(r+1)}}{2r},$$

wich are *real* for $r > 3$.

Hence *a 2-cycle exists for all $r > 3$ as claimed!*

A *cobweb diagram* reveals how *flip bifurcations* can give rise to *period-doubling*. Consider any map f, and look at the local picture near a fixed point where $f'(x^*) \simeq -1 \ldots$

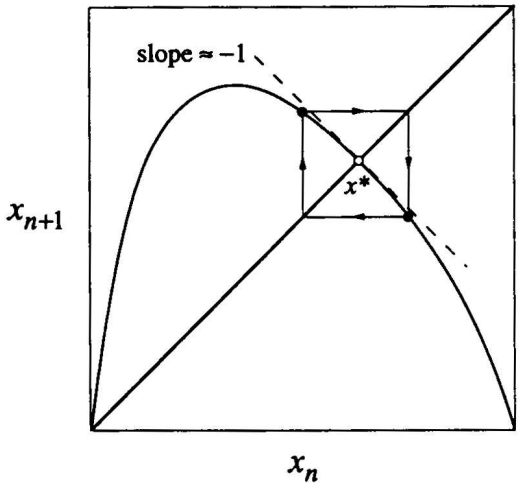


Fig. 7.3.3

If the graph of f is concave down near X^*, the cobweb tends to produce a small, stable 2-cycle near to the fixed point. But like pitchfork bifurcations, flip bifurcations can also be sub-critical, in which the 2-cycle exists below the bifurcation but is unstable.

A partial bifurcation diagram for the logistic map based mainly on the results so far look like ...

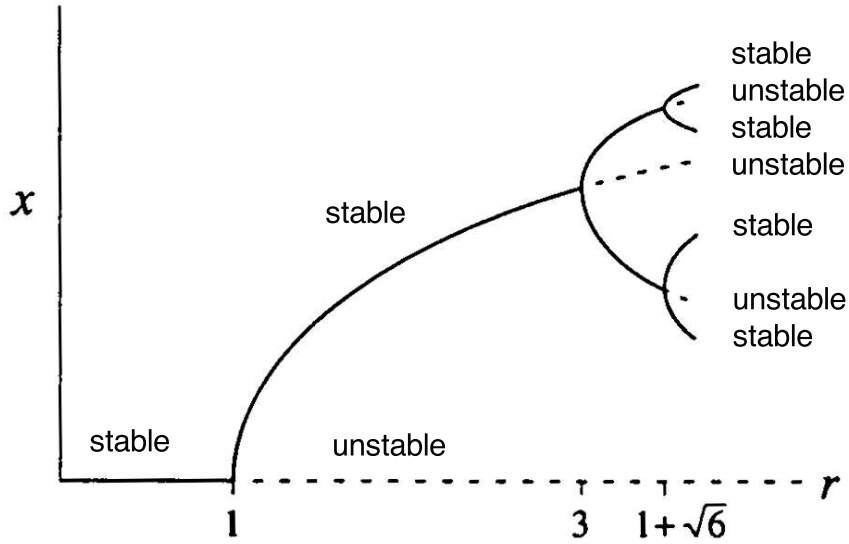


Fig. 7.3.4

Analytical analysis is getting difficult and complicated, so we will largely rely on graphical arguments from now on...

7.4 Periodic Windows

We now consider periodic windows for $r > r_\infty$ e.g. the period-3 window that occurs near $3.8284 \leq r \leq 3.8415$ [the same mechanism will account for the creation of all other similar windows!]

Let $f(x) = rx(1-x)$ so that the logistic map is $x_{n+1} = f(x_n)$. Then $x_{n+2} = f(f(x_n))$ or more simply, $x_{n+2} = f^2(x_n)$. Similarly $x_{n+3} = f^3(x_n)$. This *third-iterate map* is the key to understanding the birth of the period-3 cycle.

Any point p in a period-3 cycle repeats every three iterates, so such points satisfy $p = f^3(p)$. Consider $f^3(x)$ for $r = 3.835\ldots$

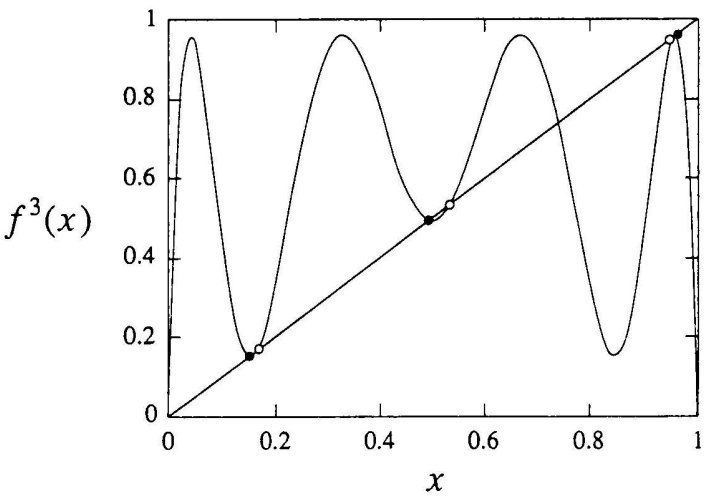


Fig. 7.4.1

Of the 8 solutions, two are period-1 points for which $f(x^*) = x^*$. The other six are shown on Fig. 7.4.1.

Black dots are *stable* period-3 cycles
Open dots are *unstable* period-3 cycles

Now suppose we decrease r towards the chaotic regime...
Consider $r = 3.8\ldots$

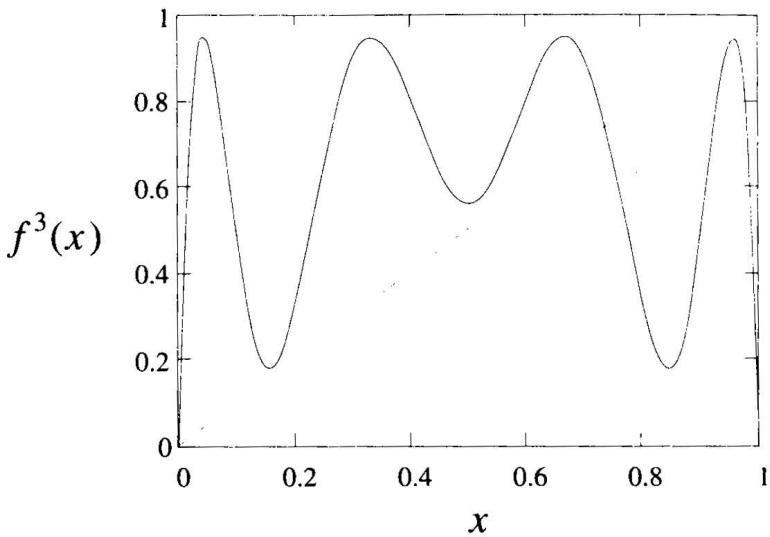


Fig. 7.4.2

The 6 solutions have vanished! [only the 2 period-1 points are left]

Hence for some r where $3.8 < r < 3.835$ the graph of $f^3(x)$ must have become a tangent to the diagonal $\Rightarrow$ stable and unstable period-3 cycles coalesce and annihilate in a *tangent bifurcation*. This transition defines the beginning of the periodic window.

Intermittency

For r just below the period-3 window one finds...

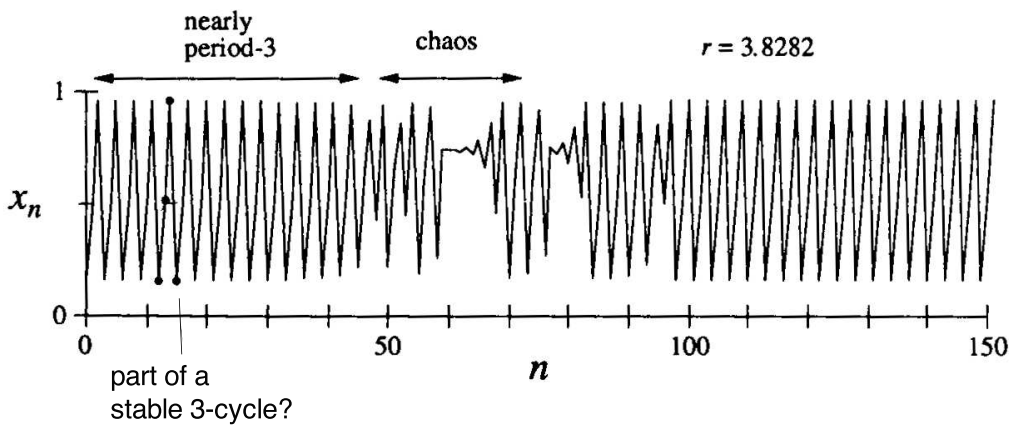


Fig. 7.4.3

where black dots indicate part of the orbit which looks like a stable 3-cycle. This is spooky, since the 3-cycle no longer exists...? We are seeing the "ghost" of the 3-cycle... since the tangent bifurcation is essentially just a *saddle-node bifurcation*.

The new feature is that we have intermittent behaviour of nearly period-3 → chaos → nearly period-3 because...

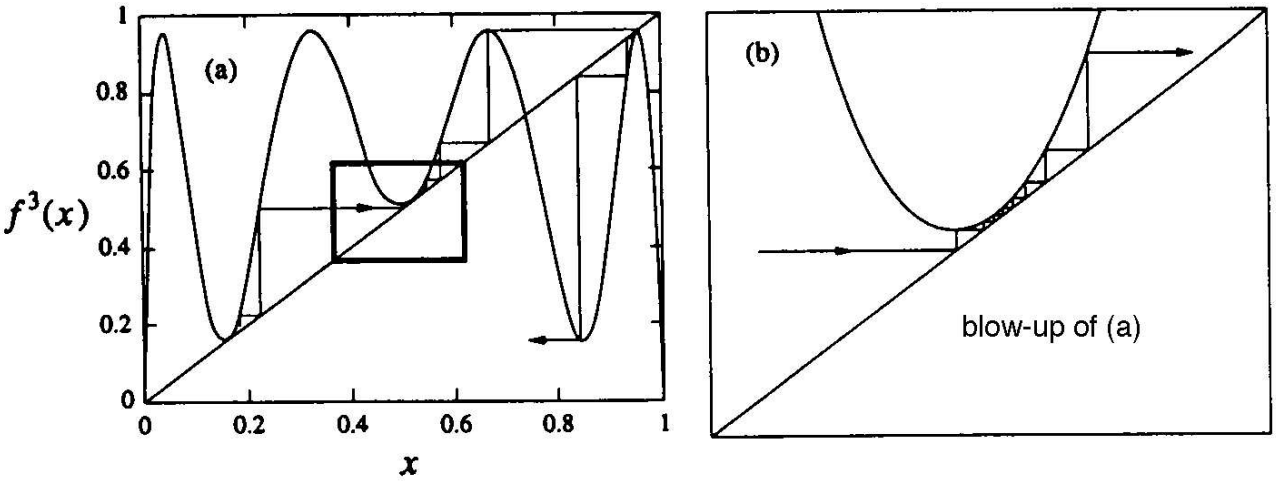


Fig. 7.4.4

Such intermittency is fairly common. The time between irregular bursts in experimental systems is statistically distributed, much like a random variable, even though the system is completely deterministic! As the control parameter is moved further away from the periodic window, the irregular bursts become more frequent until the system is fully chaotic. This progression is known as *the intermittency route to chaos*.

Period-doubling in the window

Recall Fig. 7.2.7 where a copy of the orbit diagram appears in miniature in the period-3 window. The same mechanism operates here as in the original period-doubling cascade, but now produces orbits of period $3 \cdot 2^n$. A similar period-doubling cascade can be found in *all* of the periodic windows.

7.5 Lyapunov Exponents

We have seen that the logistic map can exhibit aperiodic orbits for certain parameter values, but how do we know that this is really chaos?

To check the sensitivity to initial conditions required of a chaotic system, we can extend our definition of Lyapunov exponents to 1-dimensional maps.

Given some initial condition x_0, consider a nearby point $x_0 + \delta_0$, where the initial separation δ_0 is extremely small. Let δ_n be the separation after n iterates.

If $|\delta_n| \simeq |\delta_0| e^{n\lambda}$, then λ is called the Lyapunov exponent. A positive Lyapunov exponent is a signature of chaos. More precisely, the Lyapunov exponent for the orbit starting at x_0 is defined to be

$$\lambda = \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \right).$$

Note that λ depends on x_0. However, it is the same for all x_0 in the basin of attraction of a given attractor.

- For stable fixed points and cycles $\Rightarrow$ λ negative

- For chaotic attractors $\Rightarrow$ λ positive.

Example 7.5.1

Consider the so-called *tent map*

$$f(x) = \begin{cases} rx & 0 \leq x \leq 1/2 \\ r - rx & 1/2 \leq x \leq 1 \end{cases}$$

(for $0 \leq r \leq 2$ and $0 \leq x \leq 1$).

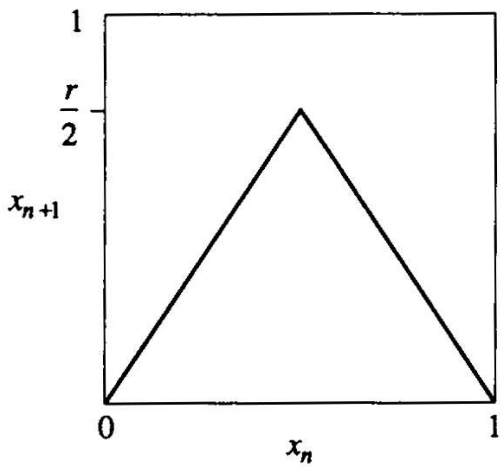


Fig. 7.5.1

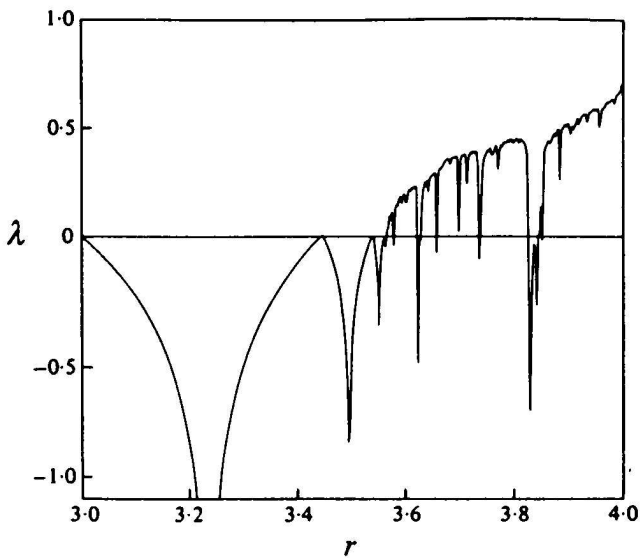
This looks similar to the logistic map, but is much easier to analyse!

Since $f'(x) = \pm r$ for all x, we find

$$\begin{aligned}
\lambda &= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|\right) \\
&= \lim_{n\to\infty}\left(\frac{\ln r}{n}\sum_{i=0}^{n-1}1\right) \\
&= \ln r
\end{aligned}$$

This suggests that the tent map has chaotic solutions for all $r > 1$, since $\lambda = \ln r > 0$.

In general one needs a computer to calculate λ!



e.g. λ for the Logistic Map

7.6 Universality

Consider the *sine map* $x_{n+1} = r \sin \pi x_n$ for $0 \leq r \leq 1$ and $0 \leq x \leq 1$.

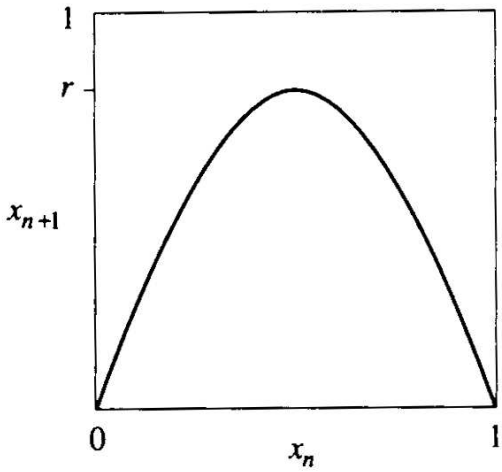


Fig. 7.6.1

It has qualitatively the same shape as the logistic map - such maps are called *unimodal*.

We now compare the orbit diagrams for the sine map and the logistic map...

the resemblance is quite amazing...

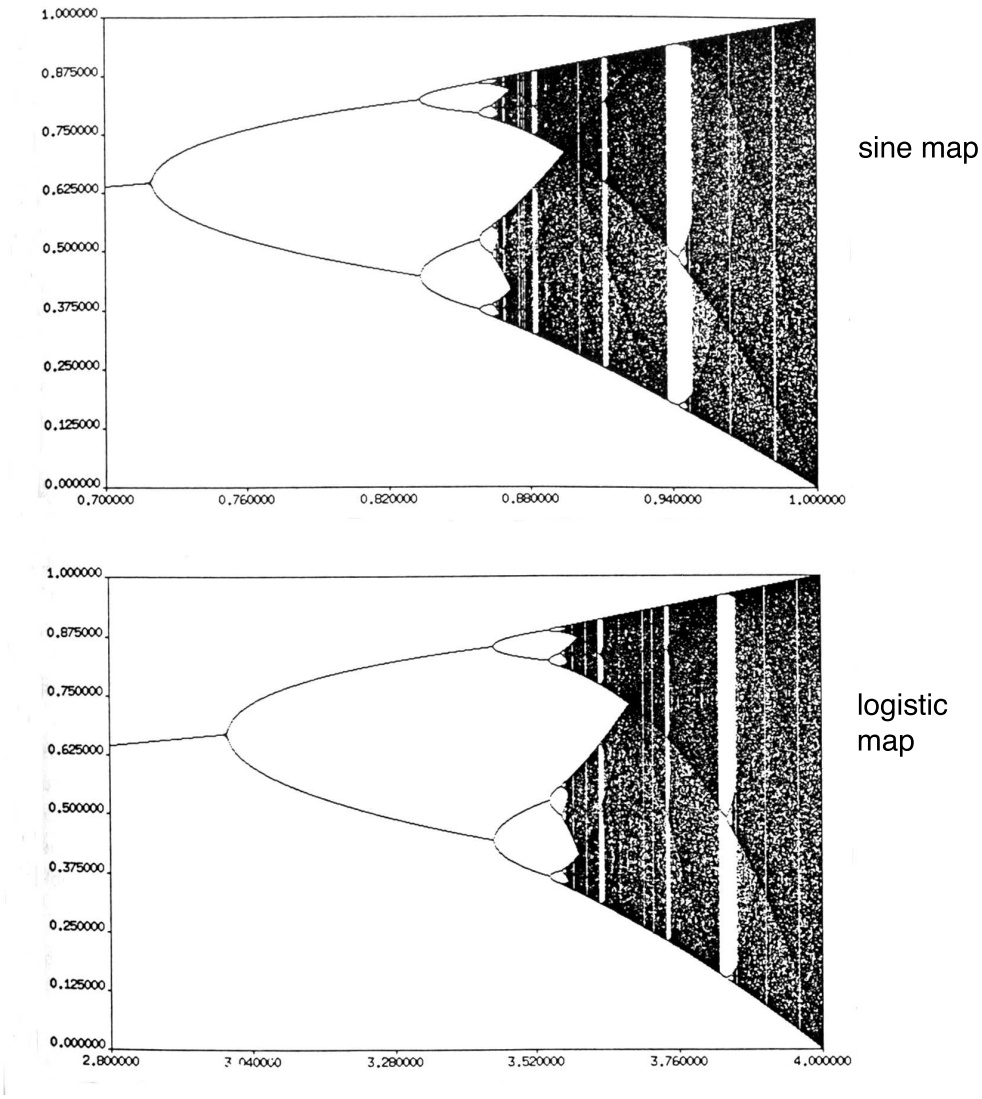


Fig. 7.6.2

The *qualitative* dynamics of the two maps are identical! Metropolis (1973) proved that all unimodal maps have periodic attractors (i.e. stable periodic solutions) occurring in the same sequence. This implies that the algebraic form of the map $f(x)$ is irrelevant - only its overall shape matters!

There is an even more amazing *quantitative* universality in 1-dimensional maps...

In 1975, Mitch Feigenbaum was trying to develop a theory to predict r_n, the value of r where a 2^n-cycle first appears. He found that, no matter what unimodal map is iterated, the same convergence rate appears!

$$\text{i.e. } \delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$

is universal! It is a new mathematical constant, as basic to period-doubling as π is to circles.

He explained why δ is universal, based on the idea of *renormalization* from statistical physics. He thereby found an analogy between δ and the universal exponents observed in experiments on *second-order phase transitions* in magnets, fluids and other physical systems. This has been confirmed in experiments...

What do 1-D maps have to do with science?

Real systems often have tremendously many degrees of freedom. How can all that complexity be captured by a 1-dimensional map? To try and answer this, we start by considering the so-called *Rössler model*...

The *Rössler model* is a set of 3 differential equations designed to exhibit the *simplest possible strange attractor*

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned}$$

where a, b, c are parameters. The term zx is the only nonlinear term (recall that Lorenz has two!)

We consider the Rössler system with $a = b = 0.2$ held fixed, and vary c

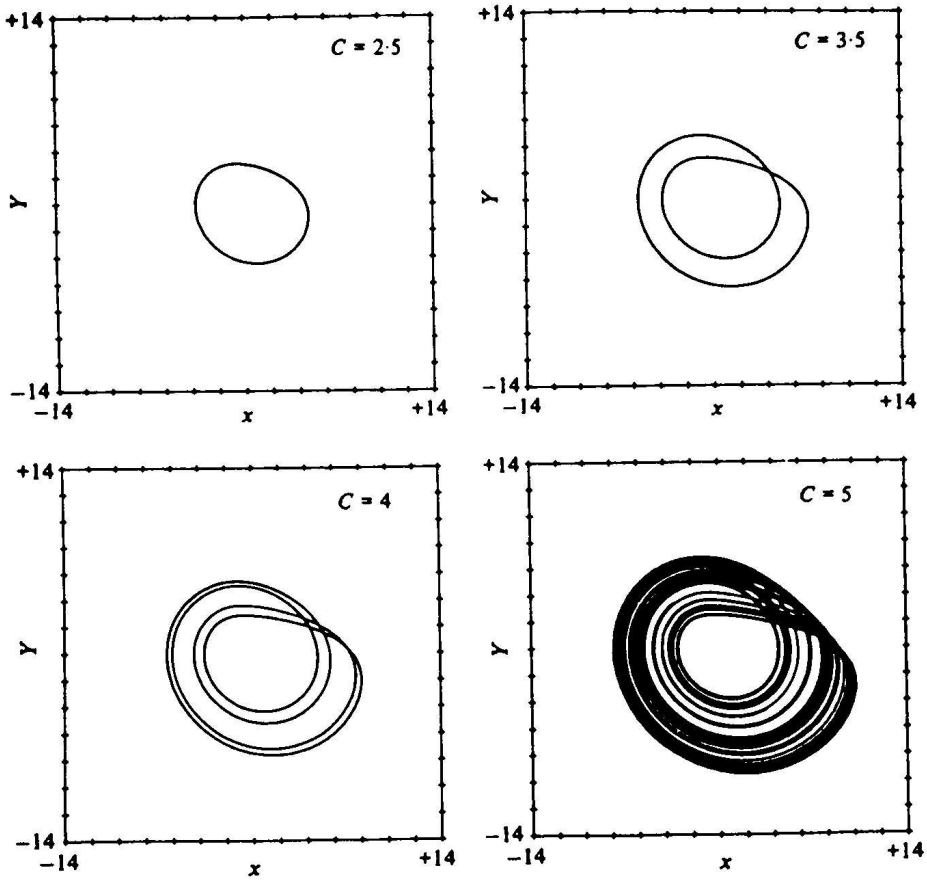


Fig. 7.6.3

- $c = 2.5$ - attractor is a simple limit cycle

- $c = 3.5$ - period-doubling in a continuous-time system! Hence, a *period-doubling bifurcation of cycles* must have occurred somewhere between 2.5 and 3.5

- $c = 4$ - another period-doubling bifurcation creates the 4-loop shown at $c = 4$

- $c = 5$ - after an infinite cascade of further period-doublings, one obtains the strange attractor shown at $c = 5$.

To compare these results to those for 1-dimensional maps, we use Lorenz's trick for obtaining a map from a flow (see Lecture 6). For a given c, record successive local maxima of $x(t)$ for a trajectory on the strange attractor. Then plot x_{n+1} vs x_n where x_n denotes the nth local maximum.

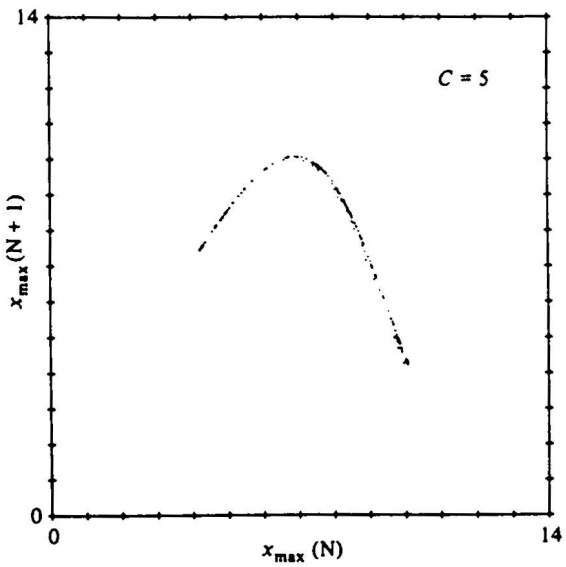


Fig. 7.6.4

Data points fall very nearly onto a 1-D curve - note uncanny resemblance to the (unimodal) logistic map!

To compute an orbit diagram for the Rössler model, we allow c to vary. Above each c we plot *all* the local maxima x_n on the attractor for that value of c. The number of different maxima tells us the "period" of the attractor...

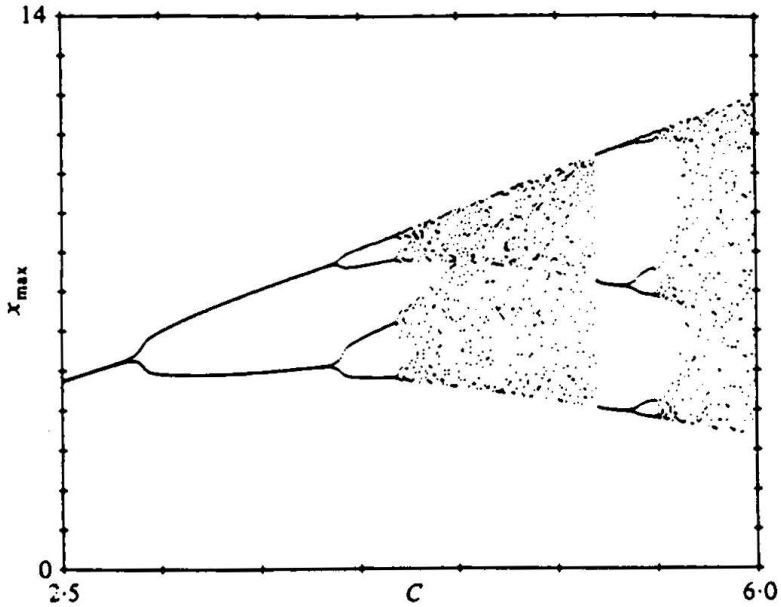


Fig. 7.6.5

Now we can see why *certain* physical systems are governed by Feigenbaum's universality theory - *if the system's Lorenz map is nearly one-dimensional and unimodal, then the theory applies!*

For the Lorenz map to be almost 1-dimensional, the strange attractor has to be *very flat* i.e. only slightly more than 2-dimensional. This requires the system to be *highly dissipative*; only 2 or 3 degrees of freedom are truly active - the rest follow on slavishly.

7.7 Renormalization

Renormalization theory is based on self-similarity - the orbit diagram can look like a figtree *[note: Feigenbaum = "figtree" in German!]* which has the self-similarity property that twigs look like earlier branches.

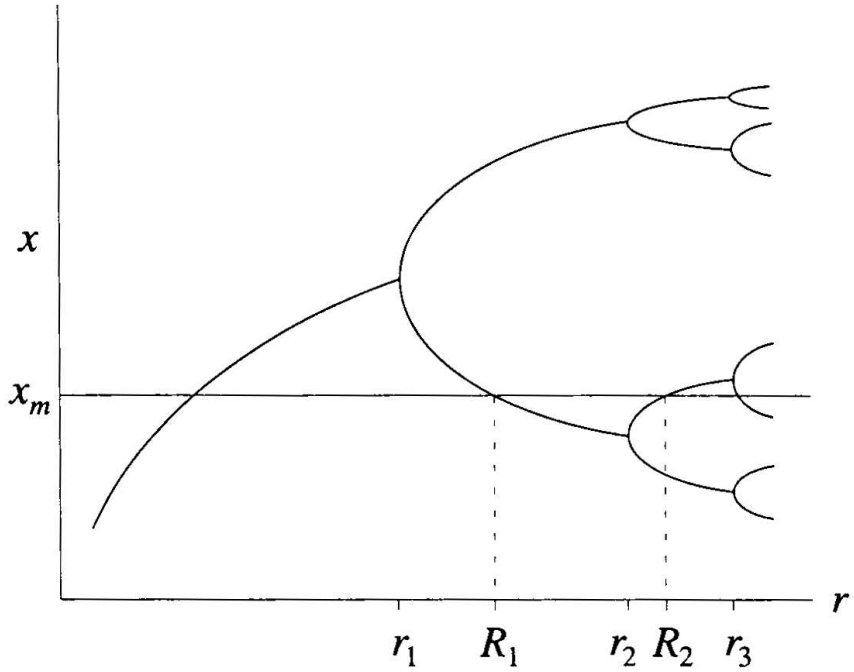


Fig. 7.7.1

This structure reflects the endless repetition of the same dynamical processes: a 2^n cycle is born, and then it loses stability in a period-doubling bifurcation.

See May & Oster (1980) *Phys. Lett. A*, **78**, 1.