TWO DIMENSIONAL FLOWS

Lecture 5: Limit Cycles and Bifurcations

5. Limit cycles

A limit cycle is an isolated closed trajectory ["isolated" means that neighbouring trajectories are not closed]

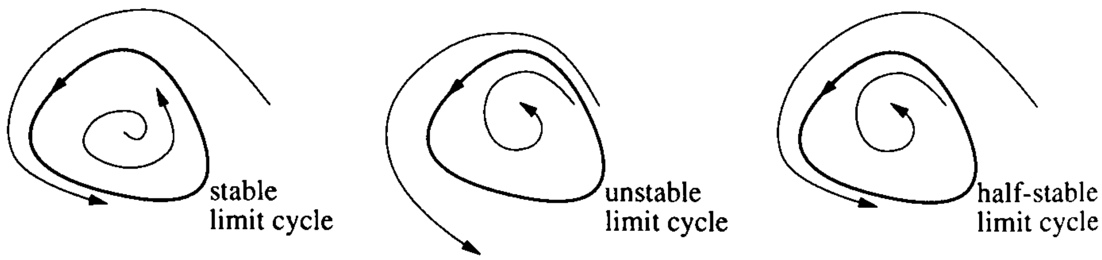


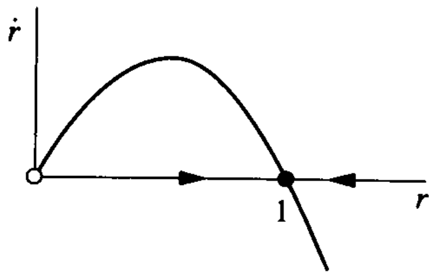
Fig. 5.1.1

- **Stable limit cycles** are very important scientifically, since they model systems that exhibit *self-sustained oscillations* i.e. systems which oscillate even in the absence of an external driving force (e.g. beating of a heart, rhythms in body temperature, hormone secretion, chemical reactions that oscillate spontaneously). If the system is perturbed slightly, it always returns to the stable limit cycle.

- Limit cycles *only* occur in **nonlinear systems** - i.e. a linear system $\dot{\mathbf{x}} = \underline{\mathbf{A}}\mathbf{x}$ can have closed orbits, *but they won't be isolated!*

Example 5.1.1 $\begin{cases} \dot{r} = & r(1-r^2); r \geq 0 \\ \dot{\theta} = & 1 \end{cases}$

$r^* = 0$ is an unstable fixed point and $r^* = 1$ is stable.



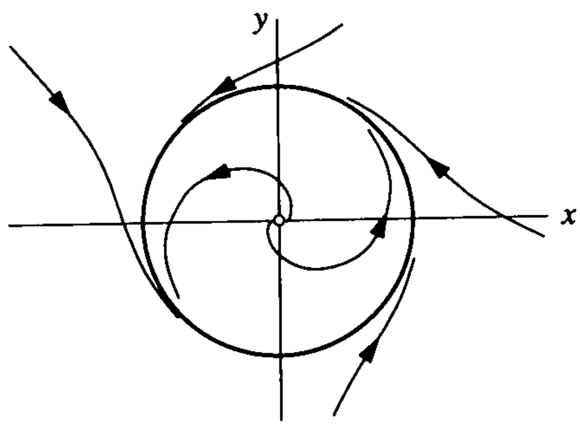


Fig. 5.1.2 Fig. 5.1.3

hence all trajectories in the phase plane (except $r^* = 0$) approach the unit circle $r^* = 1$ monotonically.

Example 5.1.2 Van der Pol oscillator

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \qquad (1)$$

where $\mu(x^2 - 1)\dot{x}$ is a nonlinear damping term whose coefficient depends on position: positive for $|x| > 1$ and negative for $|x| < 1$.

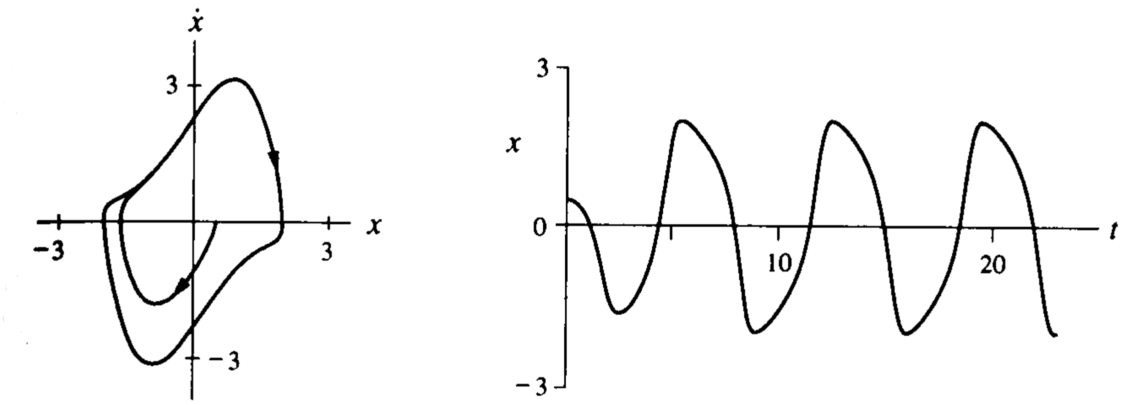


Fig. 5.1.4

Solution for $\mu = 1.5$, starting from $(x, \dot{x}) = (0.5, 0)$ at $t = 0$

Limit cycle is not generally a circle (signals are not sinusoidal), but must generally be found numerically.

The Poincaré-Bendixson Theorem says that the dynamical possibilities in the *2-dimensional* phase plane are very limited:

- **If a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory eventually must approach a closed orbit.**

- The formal proof of this theorem is subtle and requires advanced ideas from topology - so will not be developed further here!

- This theorem does not hold in higher-dimensional systems $n \geq 3$). Trajectories may then wander around forever in a bounded region *without* settling down to a fixed point or closed orbit.

- In some cases, the trajectories are attracted to a complex geometrical object called a **strange attractor**; a fractal set on which the motion is *aperiodic* and *sensitive to tiny changes in the initial conditions*. This makes the motion unpredictable in the long run **i.e. CHAOS!** - *but not in 2D*

5.2 Applications of Poincaré-Bendixson

How can we tell if a trajectory is indeed confined to a closed, bounded region?

This is essential if we want to show that the solution is a closed orbit and not a fixed point, for example. In order to do this we need to show that there exists a region R within which the trajectory will remain for all $t \to \infty$, and which excludes a nearby fixed point.

The trick is to construct a trapping region R, which is a closed, connected set such that the vector field points "inwards" everywhere on the boundary of R. Then all trajectories in R must be confined.

If we can also arrange that there are no fixed points contained inside R, then the Poincaré-Bendixson theorem ensures that R must contain a closed orbit.

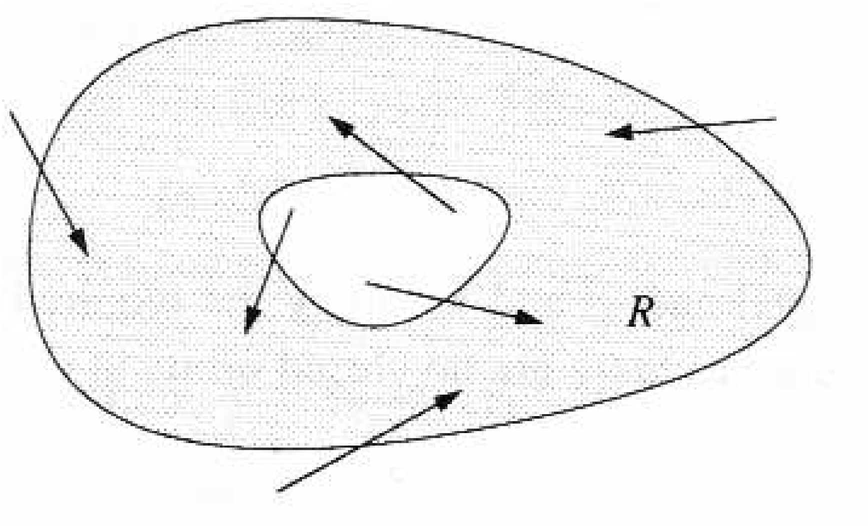


Fig. 5.2.1

Example in polar coordinates Consider the system

$$\dot{r} = r(1 - r^2) + \mu r \cos\theta$$
$$\dot{\theta} = 1.$$

When $\mu = 0$ there is a stable limit cycle at $r = 1$. Show that a closed orbit still exists for $\mu > 0$, as long as μ is sufficiently small.

- *Solution:* We need to find two concentric circles with radii r_{min} and r_{max} such that $\dot{r} < 0$ on the outer circle and $\dot{r} > 0$ on the inner circle. Then the annulus $0 < r_{min} \leq r \leq r_{max}$ is the trapping region.

- For $r > 0$ there are no fixed points, since $\dot{\theta} = 1$, so no fixed points inside the annulus.

- Hence if r_{max} and r_{min} exist, then Poincaré-Bendixson implies the existence of a closed orbit.

- For r_{min} we require $\dot{r} = r(1-r^2)+\mu r\cos\theta > 0$ for all θ. Since $\cos\theta \geq -1$, any $r_{min} < \sqrt{1-\mu}$ will work.

- For r_{max} we need $\dot{r} = r(1-r^2)+\mu r\cos\theta < 0$ for all θ. By a similar argument, any $r_{max} > \sqrt{1+\mu}$ will work.

- So provided $\mu < 1$ then both r_{min} and r_{max} can be found and a closed orbit exists.

- In fact the closed orbit can exist for $\mu > 1$ too, though this is more complicated to show.

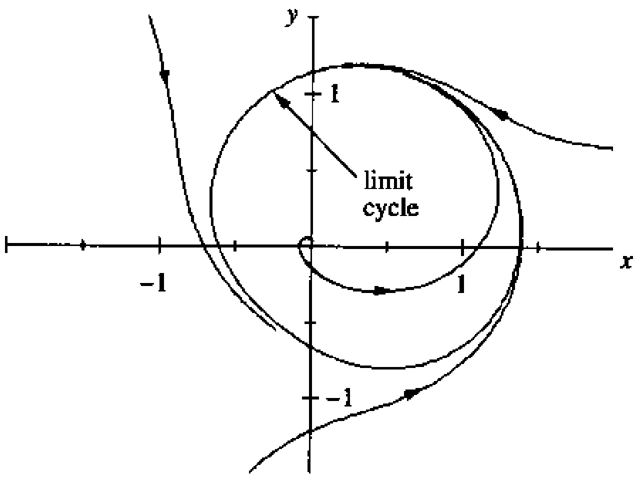


Fig. 5.2.2

Example using nullclines Consider the system

$$\begin{aligned} \dot{x} &= -x + ay + x^2 y \\ \dot{y} &= b - ay - x^2 y \end{aligned}$$

representing a biochemical process called glycolysis which cells use to obtain energy from sugar. x and y are concentrations of the compounds ADP and F6P, and $a, b > 0$ are parameters. Construct a trapping region for this system.

- *Solution:* We cannot use simple coordinates as for the first example above, but can first find the nullclines, defined as the curves on which either $\dot{x}$ or $\dot{y} = 0$.

- $\dot{x} = 0$ on $y = x/(a + x^2)$.

- $\dot{y} = 0$ on $y = b/(a + x^2)$.

- Then use the fact that the trajectory is exactly horizontal on the $\dot{y} = 0$ nullcline and exactly vertical on the $\dot{x} = 0$ nullcline, to sketch the vector field.

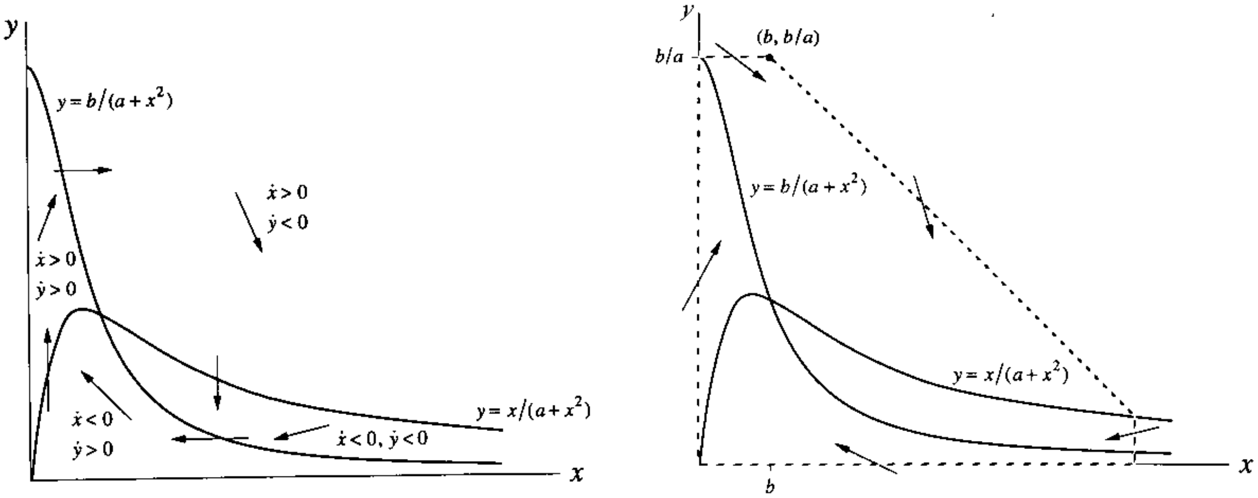


Fig. 5.2.3 & 4

- We can construct the dashed boundary in Fig. 5.2.4 using information from Fig. 5.2.3, such that all trajectories are inwards $\rightarrow$ a trapping region. By showing that the fixed point where the nullclines cross is a repeller, the remaining region must contain a closed orbit.

5.3 Bifurcations revisited

Just as for 1-dimensional systems, we find in *2-dimensional* systems that fixed points can be created or destroyed or destabilized as parameters are varied - but now the same is true of *closed orbits* as well. Hence we can begin to describe the ways in which oscillations can be *turned on or off*.

Saddle-node, transcritical and pitchfork bifurcations

The bifurcations of fixed points discussed above for 1-dimensional systems have analogues in *all* higher dimensions, including $n = 2$.

Nothing really new happens when more dimensions are added - all the action is confined to a 1-dimensional subspace along which the bifurcations occur, while in the extra dimensions the flow is either simple attraction to or repulsion from that subspace.

e.g. $\begin{cases} \dot{x} &= -ax + y \\ \dot{y} &= \frac{x^2}{1+x^2} - by \end{cases}$ as a model for a genetic control system $(a, b > 0)$

Phase portrait

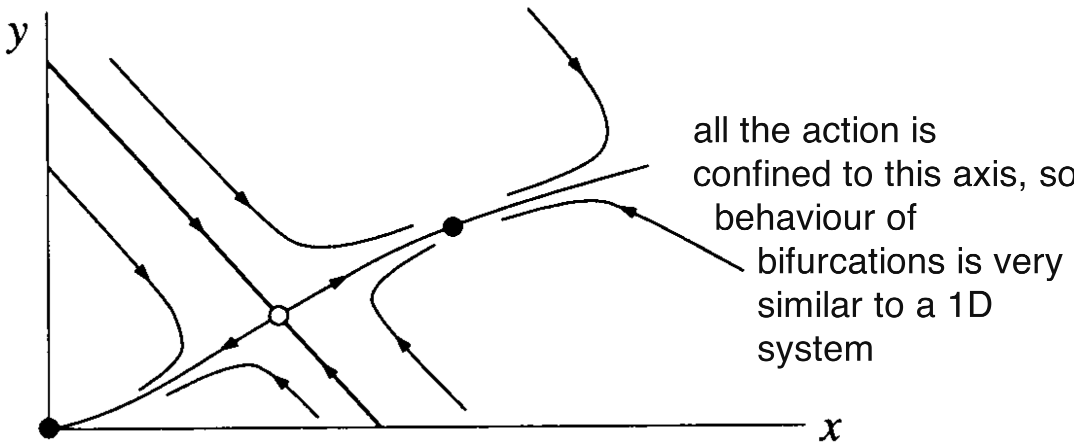


Fig. 5.3.1

Keep a constant and vary $b \Rightarrow$ bifurcations occur along the line $y = ax$.

Hopf bifurcations

Suppose a 2-dimensional system has a *stable* fixed point. What are all the possible ways it could lose stability as a parameter μ varies?

Consider the eigenvalues of the Jacobian λ_1 and λ_2

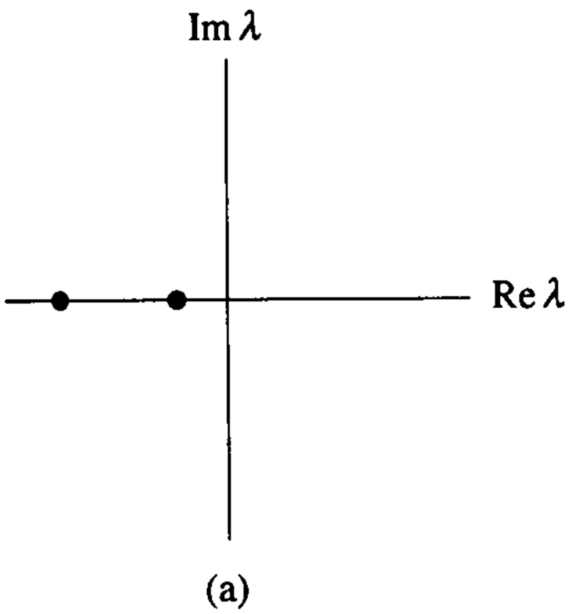


Fig. 5.3.2a

λ_1 or λ_2 passes through $\lambda = 0$ on varying $\mu \Rightarrow$ saddle-node, transcritical and pitchfork bifurcations.

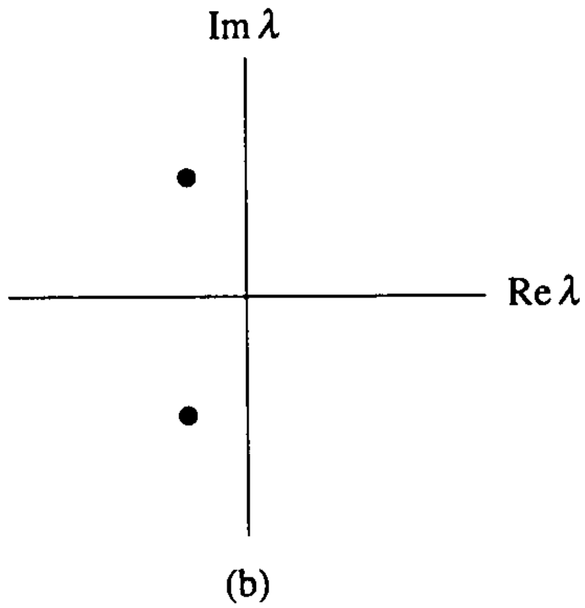


Fig. 5.3.2b

λ_1 and λ_2 cross Im(λ) axis onto Re(λ) > 0 plane as μ is varied $\Rightarrow$ Hopf bifurcation

Figs 5.3.2a and b are the only two possible scenarios for λ_1 and λ_2, since the λs satisfy a quadratic equation with real coefficients.

NB The Hopf Bifurcation has no counterpart in 1-dimensionsal systems

Supercritical Hopf bifurcation Leads from a decaying oscillation to growth and saturation of a sustained oscillation.

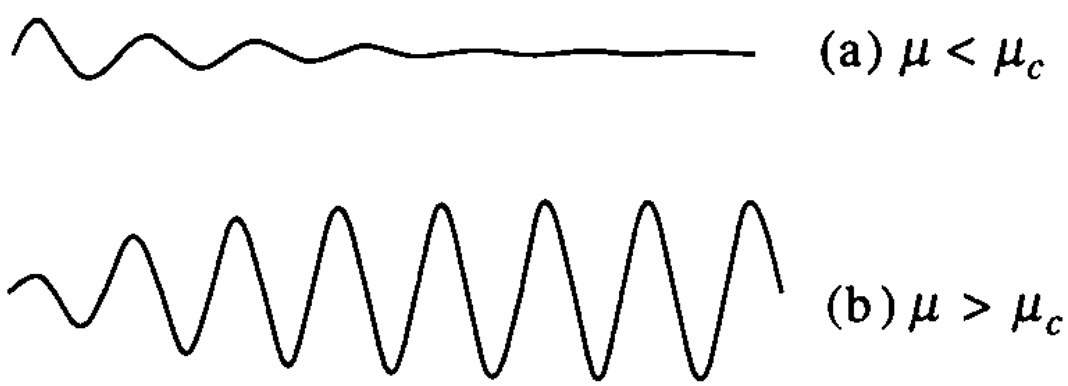


Fig. 5.3.3

Example: $\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + b r^2 \end{cases}$

Phase portraits

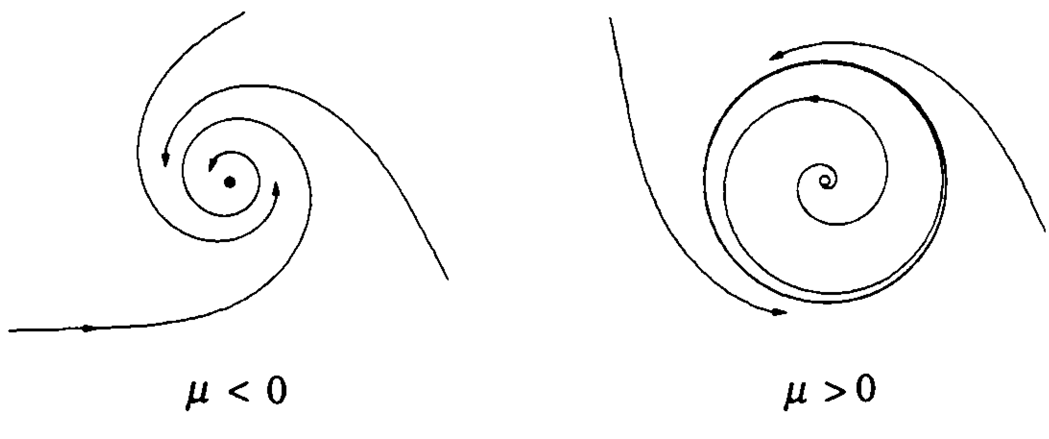


Fig. 5.3.4

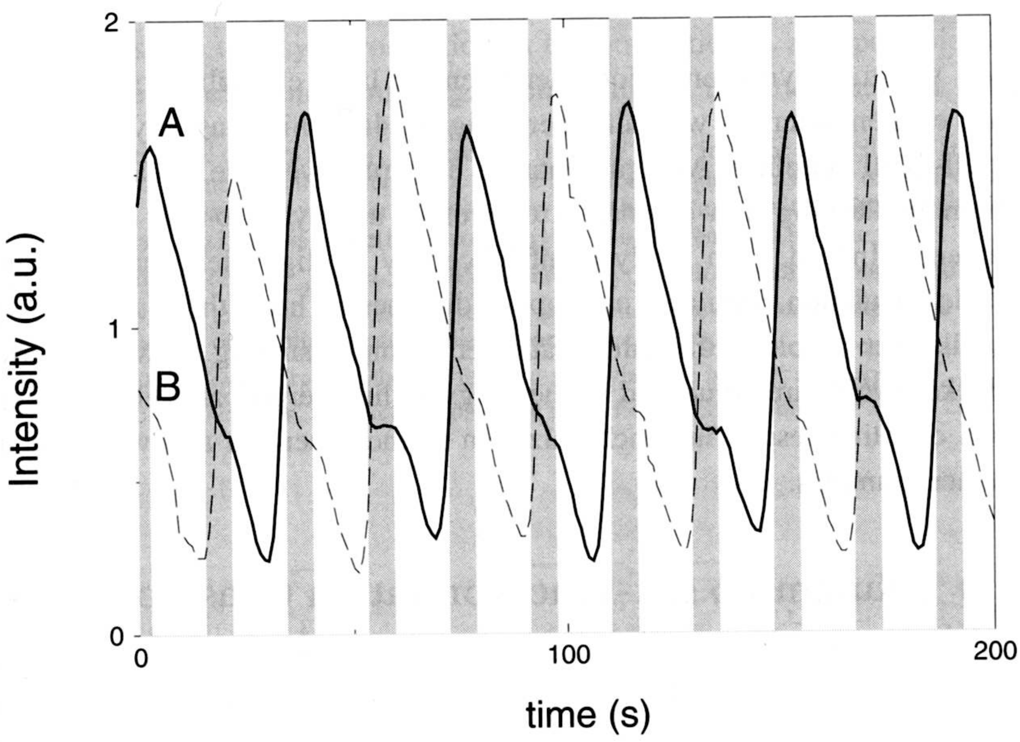
Subcritical Hopf bifurcation

Much more dramatic...and potentially dangerous in engineering! After the bifurcation, the trajectories *jump* to a *distant attractor*, which could be a fixed point, another limit cycle, infinity or - for $n \geq 3$ - a chaotic attractor (e.g. the Lorenz equations in Lecture 6).

The question as to whether a Hopf bifurcation is sub- or super-critical is difficult to answer without an *accurate* numerical solution e.g. via a computer.

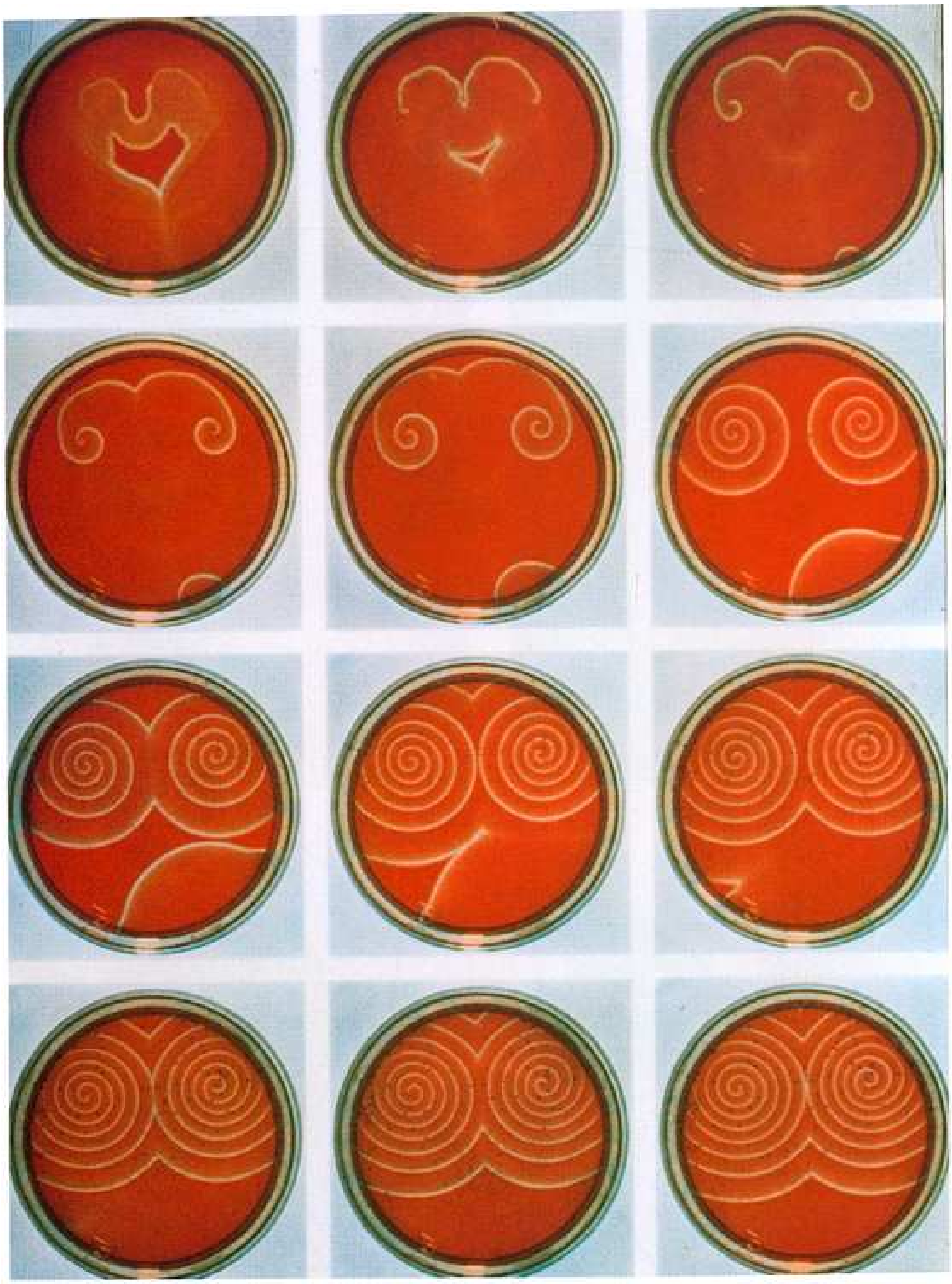
Oscillating chemical reactions

These are a famous example of the occurrence of Hopf bifurcations: the Belousov-Zhabotinsky reaction. This is a complex set of more than 20 reactions amongst reagents which include malonic acid, bromate ions, sulphuric acid and a cerium catalyst. The system undergoes periodic oscillations which appear as changes in colour of the solution.



The kinetic rate equations for the chemical reactions can be written in terms of the concentrations c_i of the i^{th} reagent as

$$\frac{dc_i}{dt} = f_i(c_1, c_2...c_N)$$



Example

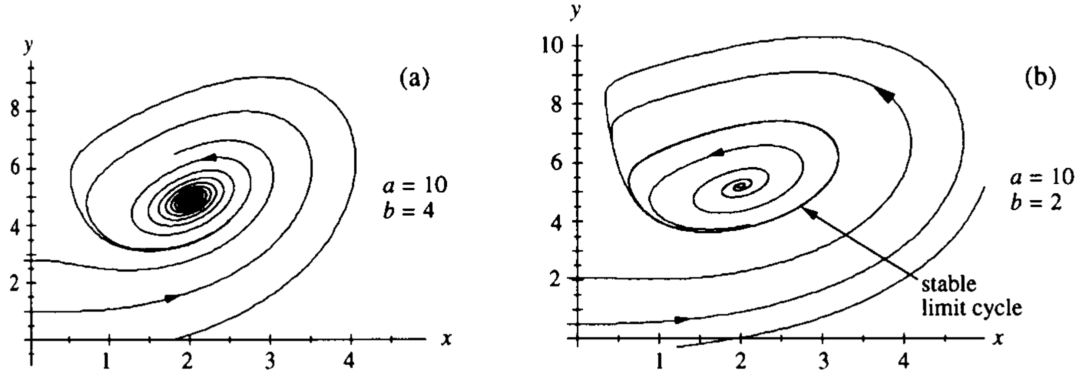
Lengyel et al. (1990) derived a simple model for another oscillating chemical reaction between ClO_2, iodine (I_2) and malonic acid (MA)

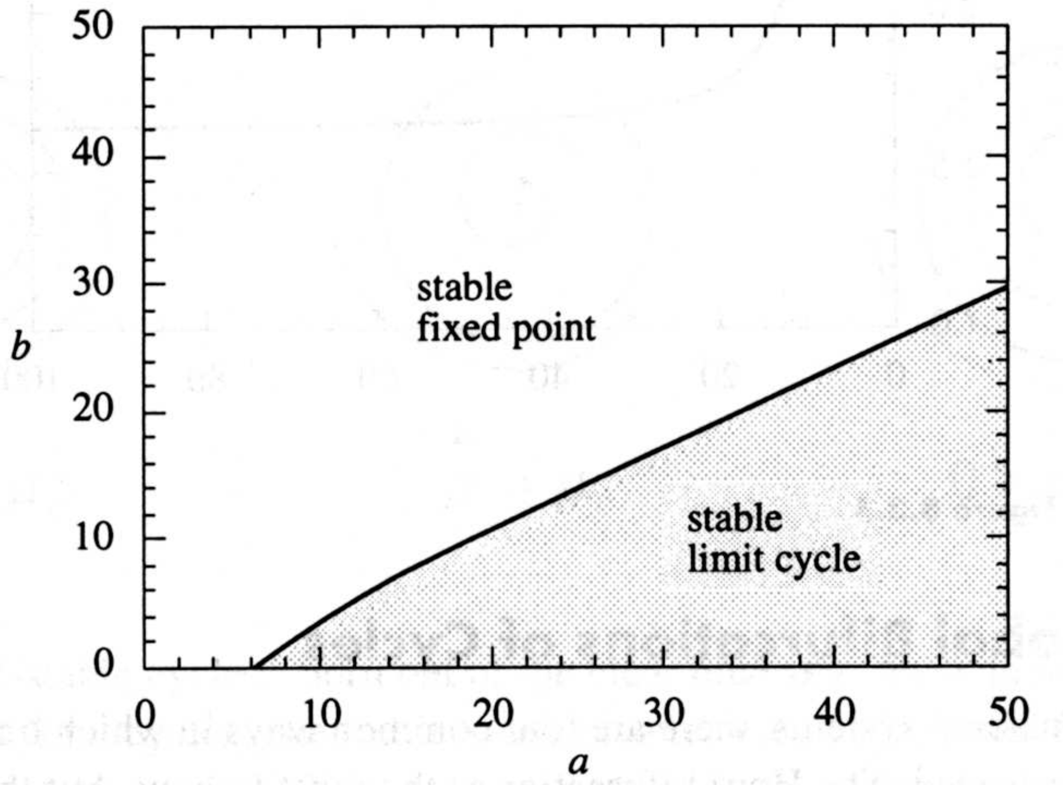
$$
\begin{aligned}
MA + I_2 &\rightarrow IMA + I^- + H^+ \\
ClO_2 + I^- &\rightarrow ClO_2^- + \frac{1}{2}I_2 \\
ClO_2 + 4I^- + 4H^+ &\rightarrow Cl^- + 2I_2 + 2H_2O
\end{aligned}
$$

from which equations for the rate of change of concentration of I_2, ClO_2 and MA can be derived which depend on products of the concentrations of the other reagents with rate constants k_i. After suitable nondimensionalisation (see Strogatz Ch. 8 for details), this reduces to the dynamical system

$$
\begin{aligned}
\dot{x} &= a - x - \frac{4xy}{1+x^2} \\
\dot{y} &= bx\left(1 - \frac{y}{1+x^2}\right)
\end{aligned}
$$

where a and b are constants. This turns out to be a 2-dimensional nonlinear autonomous dynamical system which can exhibit periodic oscillations....





Closely related are waves of excitation in neural or cardiac tissue.

5.4 Poincaré Maps

These are useful for studying swirling flows, such as that near a periodic or quasi-periodic orbit or, as we shall see later, the flow in some chaotic systems.

Consider an n-dimensional system $\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X})$. Let S be an $n-1$ dimensional *surface of section*.

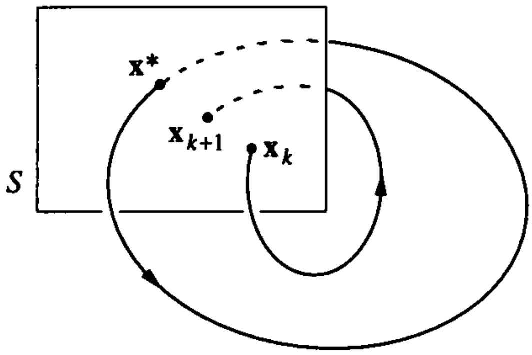


Fig. 5.4.1

S is required to be transverse to the flow i.e. all trajectories starting on S flow through it, not parallel to it.

The Poincaré map P is a mapping from S to itself, obtained by following trajectories from one intersection with S to the next. If $\mathbf{X}_k \subset S$ denotes the k^{th} intersection, then the Poincaré map is defined by

$$\mathbf{X}_{k+1} = P(\mathbf{X}_k)$$

Suppose that $\mathbf{X}^*$ is a *fixed point* of P i.e. $P(\mathbf{X}^*) = \mathbf{X}^*$. Then a trajectory starting at $\mathbf{X}^*$ returns to $\mathbf{X}^*$ after some time T and is therefore a closed orbit for the original system $\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X})$.

Hence the Poincaré map converts *problems about closed orbits* into *problems about fixed points of a mapping.*

The snag is that it is typically impossible to find an analytical formula for P!

Example of a rare exception $\begin{cases} \dot{r} = & r(1-r^2) \\ \dot{\theta} = & 1 \end{cases}$

Let S be the positive x-axis and r_0 is an initial condition on S. Since $\dot{\theta} = 1$, the first return to S occurs after a *time of flight* $t = 2\pi$. Then $r_1 = P(r_0)$, where r_1 satisfies

$$\int_{r_0}^{r_1} \frac{dr}{r(1-r^2)} = \int_0^{2\pi} dt = 2\pi \Rightarrow r_1$$

$$\text{Hence} \quad P(r) = [1 + e^{-4\pi}(r^{-2} - 1)]^{-1/2}$$

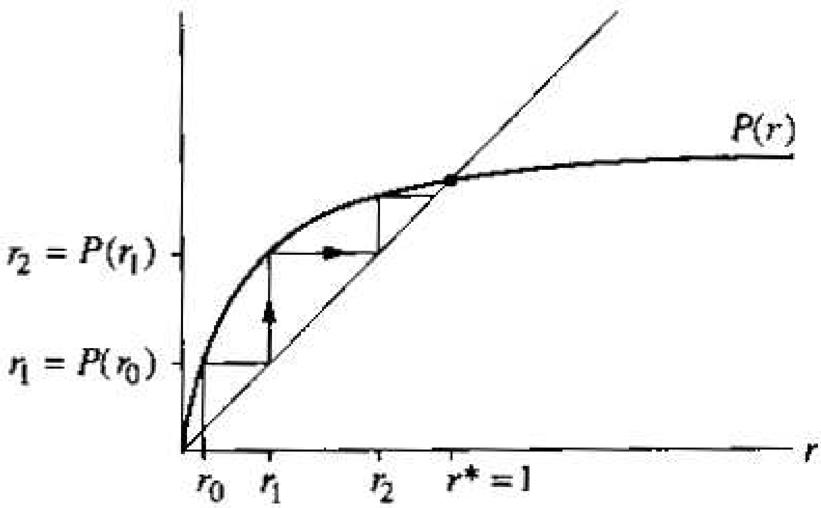


Fig. 5.4.2

"Cobweb" construction enables us to iterate the map graphically. The cobweb shows that $r^* = 1$ is stable and unique (as expected).