Homework IV Part II  
MMathPhys Renormalisation Group

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1 Sine-Gordon RG

In the lecture we obtained the RG equations

\[ \frac{dK}{d\tau} = 2\pi^3 g^2 + O(g^4), \quad \frac{dg}{d\tau} = \left(2 - \frac{\pi}{K}\right) g + O(g^3) \] (1)

for the Sine-Gordon model

\[ \mathcal{H} = \int d^2r \left( \frac{K}{2} (\nabla h)^2 + g \cos 2\pi h \right). \] (2)

a) (adapted from Wen): Consider a more general potential which we write as

\[ \mathcal{H} = \int d^2r \left( \frac{K}{2} (\nabla h)^2 + V(h) \right), \quad V(h) = \int d\ell \tilde{V}(\ell) e^{2\pi ih}. \] (3)

By generalising from the case we discussed, write the RG equation for \( \tilde{V}(\ell) \) to linear order in \( \tilde{V}(\ell) \). Reexpress this as a partial differential equation for \( V(h) \), in terms of the parameters \( \tau \) and \( h \). What would the initial periodic potential shown in Fig. 1 look like when \( \tau \gg 1 \)? (But while \( V \) is still small, so the linear approximation is valid. Assume also that the average is zero, \( \int dhV(h) = 0 \), since we can always subtract a constant from \( \mathcal{H} \).)

*** All the remaining questions on this set are optional!! ***

b) Consider the initial non-periodic potential \( V(h) = Ae^{-h^2/2\Delta} \). This is

\[ ^{1}\text{In many physical applications \( V(h) \) would be periodic, so \( \ell \) would be restricted to integers. Here we allow a non-periodic \( V \).} \]
a barrier disfavouring $h \sim 0$. What happens to $V(h)$ as the RG time increases (still within the linear approximation)?

c) Let us say we have a system with the potential in (b) that is defined on a spatial disc of radius $L$. Let us enforce the condition $h = 0$ on the boundary of the disc. Do you expect this system to exhibit spontaneous symmetry breaking in the limit of large $L$?

2 Spin-glass

Consider an Ising model in $d$ dimensions

$$\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij} S_i S_j \quad (4)$$

$J_{ij}$ is a random coupling associated with the bond $\langle ij \rangle$. It is equal to $+J$ with probability $1/2$ and to $-J$ with probability $1/2$. The $J_{ij}$s are independent, quenched random variables, representing non-fluctuating disorder in our system. Note that we must distinguish the thermal average, which we do in the background of a fixed random choice of $J_{ij}$,

$$\langle \ldots \rangle_{\{J_{ij}\}} = \frac{\sum_{\{S\}} e^{-\mathcal{H}(\{S\};\{J_{ij}\})}}{Z[\{J_{ij}\}]} \quad (5)$$

from the disorder average,

$$\langle \ldots \rangle = \sum_{\{J_{ij}\}} \frac{1}{\text{no. bonds}} \langle \ldots \rangle. \quad (6)$$
Fully averaged expectation values

\[ \langle S_i \ldots \rangle_J \]  

are hard to compute because the random variables appear both in the numerator and denominator of (5).

\textbf{a)} Draw the phase diagram of the model in one dimension as a function of \( J \). Exhibit an explicit equivalence to a model we already know in 1D.

\textbf{b)} In the one-dimensional \textbf{ferromagnet}, where \( J_{ij} = J \), the (unstable) zero-temperature fixed point can be distinguished from the disordered phase by the long distance behaviour of correlation functions: \( \langle S_i S_j \rangle = 1 \) at zero temperature, while the correlator decays exponentially with distance when the temperature is nonzero. Can we use the long distance behaviour of the analogous correlator,

\[ \langle S_i S_j \rangle_J \]  

in the spin-glass model (4) to distinguish the zero-temperature fixed point from the \( T > 0 \) phase? If not, why not?

\textbf{c)} Can you find an alternative correlation function \( G(i, j) \) which decays to zero at long distance for \( T > 0 \), but is nonzero at long distances at \( T = 0 \)? We are still in 1D here. Note: while \( G \) should be a disorder-averaged quantity, it is useful to note that we do not always have to perform the disorder average and the thermal average at the same time.

\textbf{d)} Assuming that in high enough dimensions \( G(i, j) \) can be nonvanishing at long distances even at nonzero temperature, draw a phase diagram with two phases. Think about the properties of the new phase.