1 Scaling variables and scaling operators at the 2D Ising fixed point

Recap: Recall from the lecture\(^1\) that, close to the Ising fixed point \( K_* \), we can choose a basis of ‘scaling variables’ \( u_\alpha \) that behave simply under coarse-graining:
\[
  u'_\alpha = b_{\alpha}^{\mu} u_\alpha + O(u^2).
\] (1)
The relevant couplings have ‘RG eigenvalue’ \( y > 0 \) and grow under the RG. Irrelevant couplings, with \( y < 0 \), get smaller under the RG (so that perturbing the fixed point by an irrelevant coupling does not change the universal behaviour). Couplings with \( y = 0 \) are referred to as marginal. (When these occur we must examine higher order terms in Eq. 1.)

Recall that when we write the Hamiltonian in terms of the scaling variables,
\[
  \mathcal{H} = \mathcal{H}_* + \sum_\alpha u_\alpha \sum_{\vec{r}} O_\alpha(\vec{r})
\] (2)
the local terms \( O_\alpha(\vec{r}) \) appearing are referred to as ‘scaling operators’.\(^2\) When

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\(^1\)See Cardy ch. 3

\(^2\)In our approximate calculation on the triangular lattice, the lattice spin \( S_r \) behaved as a scaling operator (the corresponding scaling field being \( h \)). In reality a microscopic operator such as \( S_r \) is equal to a sum of scaling operators, because of the change of basis required to relate microscopic couplings to scaling variables. However, when we write \( S_r \) as a sum of \( \mathbb{Z}_2 \) odd scaling operators, a single term dominates the large-distance correlation functions. This is the operator in the sum with the smallest \( x \). Since this operator has the largest \( y \), it is also the most relevant term if we perturb the critical point with a magnetic field term \( h \sum_{\vec{r}} S(\vec{r}) \). We sometimes use a loose notation in which we use the same symbol to refer to both the microscopic operator and the leading scaling operator with the same symmetry.
we coarse-grain, replacing microscopic spins $S_r$ with block spins $S'_{\vec R}$, the microscopic and coarse-grained operators are related as

$$O_\alpha(\vec r) \sim b^{-x_\alpha} O'_\alpha(\vec R),$$

(3)

where $R$ is the coordinate of the block containing the microscopic spin $\vec r$. The operator on the left is a function of the spins $S$ (close to $\vec r$) and the operator on the right is a function of the blocked spins $S'$. Since we rescale coordinates when we do the rescaling transformation, we have

$$\vec R \sim \vec r/b.$$  

(4)

Recall that this gives the following for the correlation function $G_\alpha(\vec r) = \langle O_\alpha(0) O_\alpha(\vec r) \rangle_{K_*}$ at the fixed point:

$$G_\alpha(\vec r) = \langle O_\alpha(0) O_\alpha(\vec r) \rangle_{K_*} = b^{-2x_\alpha} \langle O'_\alpha(0) O'_\alpha(\vec r/b) \rangle_{K_*}$$

(5)

Since the coarse-grained spins $S'$ have the same couplings as the original ones $S$, the primes on the right hand side do not matter:

$$G_\alpha(\vec r) = b^{-2x_\alpha} G_\alpha(\vec r/b) \quad \rightarrow \quad G_\alpha(\vec r) \sim |\vec r|^{-2x_\alpha}.$$  

(6)

Finally, recall that $x_\alpha$ and $y_\alpha$ are related by

$$y_\alpha = d - x_\alpha.$$  

(7)

We see this by using (3) to write (2) in terms of coarse-grained variables $S'$:

$$\mathcal{H} = \mathcal{H}_* + \sum_\alpha u_\alpha \sum_{\vec R} \sum_{\vec r \in \vec R} b^{-x_\alpha} O'_\alpha(\vec R)$$

$$= \mathcal{H}_* + \sum_\alpha \left( b^{d-x_\alpha} u_\alpha \right) \sum_{\vec R} O'_\alpha(\vec R),$$

(8)

(9)

since $b^d$ is the number of spins in the block. Identifying $u'_\alpha$ with $b^{d-x_\alpha} u_\alpha$ gives the relation between $y$ and $x$.

**a) Random coupling** Instead of perturbing $\mathcal{H}_*$ with a uniform coupling $u(r)$, perturb it with a random coupling that depends on the site $r$, so that the $u(r)$ are i.i.d. random variables. Let the average vanish, $\overline{u(r)} = 0$, and let the rms value be $\Delta \equiv \sqrt{\overline{(u(r))^2}}$. Following similar logic to that above, what is the RG eigenvalue of $\Delta$? Is a random magnetic field a relevant or an irrelevant perturbation of the critical 2D Ising model?
b) Coupled copies of Ising Consider a 2D model where at each site we have $m$ different Ising spins $S^{(1)}(\vec{r}), \ldots, S^{(m)}(\vec{r})$. Initially they do not interact with each other, and we have $m$ decoupled copies of the Ising model:

$$\mathcal{H}[S^{(1)}, \ldots, S^{(m)}] = \sum_{c=1}^{m} \mathcal{H}[S^{(c)}]. \quad (10)$$

At the critical temperature, this Hamiltonian flows to a fixed point. This fixed point theory is simply $m$ decoupled copies of the Ising fixed point theory. We now couple the copies with the small term:

$$\delta \mathcal{H} = u \sum_{\vec{r}} S^{(1)}(\vec{r}) S^{(2)}(\vec{r}) \ldots S^{(m)}(\vec{r}). \quad (11)$$

At the critical temperature this leads to a small perturbation of the fixed point of a similar form (with $S$ replaced with the corresponding leading $\mathbb{Z}_2$ odd scaling operator at the Ising fixed point$^3$). In the 2D Ising model the scaling dimension of $S$ is $x_S = 1/8$.

i) What is the RG eigenvalue $y_u$ of the coupling $u$ above? What is $x_u$ for the corresponding operator $O(\vec{r}) = S^{(1)}(\vec{r}) S^{(2)}(\vec{r}) \ldots S^{(m)}(\vec{r})$? For what range of $m$ can you assert that a small perturbation of this form leaves the critical exponents of the original model unchanged?

ii) Recall that the ordered phase of a single copy has two symmetry-related ordered states. How many degenerate ordered states does the multi-copy model have when $u = 0$? What about when $u > 0$? (The other ones have an extensively larger free energy, so do not occur.)

$c)$ Power-law interactions Consider a 2D Ising model to which we add a long-range coupling of the following form:

$$\delta \mathcal{H} = u \sum_{\vec{r}} \sum_{\vec{r}'} S_{\vec{r}} S_{\vec{r}'} |\vec{r} - \vec{r}'|^a. \quad (12)$$

Assuming that the important terms are the ones where the separation $|r - r'|$ is much larger than the lattice spacing,$^4$ use the logic outlined at the end

$^3$See previous footnote

$^4$The short-distance terms only renormalize the short-range coupling $J$ that we already have. This affects the location of the critical point $J_c$ but is not important for the universality class of the transition. Also, note that when $r$ and $r'$ are well-separated they live in different blocks, so the operators $S_r$ and $S_{r'}$ are renormalized independently (this would not be true if $r$ and $r'$ were adjacent sites).
of the recap to compute the RG eigenvalue of the coupling $u$, in terms of $x_S$ and $a$. (Do not forget the rescaling of the spatial coordinate in Eq. 4!) For what range of $a$ does the universality class of the transition remain the same as in the model with only nearest-neighbour couplings?

2 RSRG for a 1D random walk problem and an example of quenched disorder

A classical particle travels along the line. It travels at constant velocity until it encounters one of the junctions that are placed at unit intervals along the line. When it encounters a junction it is randomly either ‘transmitted’ (continues past the junction with unchanged velocity) or ‘reflected’ (velocity reversed).

![Diagram of a 1D random walk](image)

Figure 1: Transmission and reflection probabilities for particle

Each encounter with a junction is an independently random event with a certain probability for transmission (see Figure). This can depend on whether the junction is approached from the left or the right (we then denote these probabilities $t_+$ and $t_−$ respectively) and may also be different for different junctions (in which case the transmission probabilities for the $i$th junction are denoted $t_i^\pm$).

a) First consider the case where the junctions are symmetric and identical, so $t_+ = t_− = t$.

i) Perform a real space RG transformation with rescaling factor $b = 2$ by grouping a pair of junctions into one coarse-grained junction, with transmission and reflection probabilities $t'$ and $1 − t'$. (To obtain $t'$ in terms of
You can sum up a series whose terms correspond to different numbers of internal reflections inside the junction. Simplify the formula for $t$.

**ii)** Show that if we rewrite the RG equation in terms of the variable $u = (1 - t)/t$ it takes a simple linear form. Draw a flow diagram.

**iii)** Let the system consist of $L = 2^k$ junctions. Compute the probability $t(L)$ that a particle, incident from the left, finally leaves the system to the right. You should find that $t(L)$ scales as $\text{const.} \times L^{-1}$ at large $L$.

**Note:** For $t = 1/2$, the above walk reduces to a simple random walk: the probabilities to hop from junction $i$ to junctions $i+1$ and $i-1$ are 1/2 each. For $1/2 < t < 1$ we have a persistent random walker who prefers to carry on in the same direction for a while. However the universal properties of the motion are the same for any $t \in (0, 1)$.\(^5\)

**Adding disorder:** This random walk problem is one where we can investigate exactly the effect of disorder in the couplings of a statistical problem. That is, we will make the transmission probabilities $t_{i\pm}$ into random variables. We refer to these as ‘quenched’ random variables, meaning that they do not change during the stochastic motion of the particle. The stochastic motion takes place within a frozen random landscape defined by a realization of the $t_{i\pm}$s.

In (b) and (c) we consider two types of quenched disorder with progressively more severe effect on the walker.

**b)** We now allow the transmission probabilities to differ for different junctions. However each junction is still left-right symmetric, so we write $t_{i+} = t_{i-} = t_i$.

**i)** Derive $t'$ for a coarse-grained junction in terms of $t_1$, $t_2$ for the two constituent junctions (noting that the junction remains symmetric after coarse-graining), and rewrite in terms of the $u$ variable.

Now assume that the $u_i$s of the microscopic junctions are independent, identically distributed random variables with mean $\langle u_i \rangle = \overline{u}$ and standard devi-

\(^5\)The $1/L$ scaling you found above also applies for the probability that a Brownian particle, which undergoes Brownian motion on the interval $[0, L]$ with $L \gg 1$, and which starts off at a point $x > 0$ with $x = O(1)$, exits the interval via the right rather than the left boundary.
\[ \sqrt{\langle (u_i - \bar{u})^2 \rangle} = \Delta. \]

At a given stage \( k \) of the RG, denoting the coarse-grained parameter for a block by \( u^{(k)} \), let us write the renormalized mean and fluctuations as

\[ \bar{u}^{(k)} = \langle u^{(k)} \rangle, \quad r^{(k)} = \sqrt{\frac{\langle [u^{(k)} - \bar{u}^{(k)}]^2 \rangle}{\bar{u}^{(k)}}}, \]  

(13)

where \( r^{(k)} \) measures the relative strength of fluctuations at a given length-scale \( L = 2^k \) (note \( r^{(0)} = \Delta/\bar{u} \)).

ii) Write RG equations for \( \bar{u}^{(k)} \) and \( r^{(k)} \) (by noting that when you combine two junctions into one, the two constituents have statistically independent parameters). Draw a rough RG flow diagram in the \((1/u, r)\) plane.

**Note** that the fluctuations become negligible, relative to the mean, under the RG flow.

iii) Using the above observation, what is \( \langle t(L) \rangle \equiv \langle t^{(k)} \rangle \) for a junction of length \( L = 2^k \), to leading order at large \( L \)?

We see that in this case disorder has not changed the asymptotic scaling.

**Note:** in this 1D example the use of the RG is mostly illustrative, as we can solve the problem directly:

iv) Write the coarse-grained \( u(L) \equiv u^{(k)} \) for a chain of \( L = 2^k \) junctions in terms of the microscopic \( u_1, u_2, \ldots, u_L \). What is the form of the probability distribution of \( u(L) \)?

c) Now let us allow each junction to be asymmetric, so that \( t_{i+} \neq t_{i-} \). We take the \( t_{i\pm} \) to be independent, identically distributed random variables. While the junctions are not left-right symmetric, they are left-right symmetric ‘on average’, since the probability distribution of \( t_{i+} \) is the same as that of \( t_{i-} \).

i) Write \( t'_+ \) and \( t'_- \) in terms of \( t_{1\pm} \) and \( t_{2\pm} \) (for the left and right junctions respectively). Parametrize the \( t \)s as

\[ t_+ = \frac{1}{r\alpha}, \quad t_- = \frac{\alpha}{r}, \]  

(14)

\[ t'_+ = \frac{1}{r\alpha}, \quad t'_- = \frac{\alpha}{r}, \]  

(14)
so that $\alpha = \sqrt{t_-/t_+}$ is a measure of how asymmetric the junction is and $r = 1/\sqrt{t_+t_-}$. Write the RG equations for $r'$ and $\alpha'$ in terms of $r_1, r_2, \alpha_1, \alpha_2$. (You should find that the $\alpha$s form a very simple closed system.)

For simplicity let us now consider $\alpha$ only. In fact it is helpful to consider $\ln \alpha$. Note that microscopically

$$\langle \ln \alpha_i \rangle = \frac{1}{2} \langle \ln t_{i-} \rangle - \frac{1}{2} \langle \ln t_{i+} \rangle = 0$$

(15)

by the left-right symmetry ‘on average’. We define $\kappa = \sqrt{\langle (\ln \alpha_i)^2 \rangle}$. Note that $\kappa = 0$ if the microscopic junctions are perfectly symmetric.

**ii)** What is probability distribution of the coarse-grained quantity, $\alpha(L)$, after a large number $k$ of RG steps with $L = 2^k$?

**iii)** By writing the variance of $\log \alpha(L)$ (which you now know) in terms of $t_+(L)$ and $t_-(L)$, and using left-right symmetry on average, as well as the fact that $\log t \leq 0$, derive a bound of the form

$$\left\langle (\log t_+(L))^2 \right\rangle \geq \text{const} \times L.$$ 

(16)

Note: In part (b), the transmission probability of a long chain of junctions (large $L$) was approximately deterministic and was power-law small in $L$. By contrast, here the typical transmission probability is exponentially small in $L$, as suggested by the previous formula. The distribution of $\alpha$ also shows that long chains are extremely asymmetric (with a random preferred direction) even if the asymmetry of the microscopic junctions (measured by $\kappa$) is small.