MMathPhys Renormalisation Group
Homework 2: Real Space RG

February 6, 2019

1 Scaling variables and scaling operators at the 2D Ising fixed point

Recap: Recall from the lecture[1] that, close to the Ising fixed point $\mathbf{K}_*$, we can choose a basis of 'scaling variables' u_α that behave simply under coarse-graining:

$$u'_\alpha = b^{y_\alpha} u_\alpha + \mathcal{O}(u^2). \tag{1}$$

The relevant couplings have 'RG eigenvalue' $y > 0$ and grow under the RG. Irrelevant couplings, with $y < 0$, get smaller under the RG (so that perturbing the fixed point by an irrelevant coupling does not change the universal behaviour). Couplings with $y = 0$ are referred to as marginal. (When these occur we must examine higher order terms in Eq. 1.)

Recall that when we write the Hamiltonian in terms of the scaling variables,

$$\mathcal{H} = \mathcal{H}_* + \sum_\alpha u_\alpha \sum_{\vec{r}} O_\alpha(\vec{r}) \tag{2}$$

the local terms $O_\alpha(\vec{r})$ appearing are referred to as 'scaling operators'.[2] When

[1] See Cardy ch. 3

[2] In our *approximate* calculation on the triangular lattice, the lattice spin $S_{\vec{r}}$ behaved as a scaling operator (the corresponding scaling field being h). In reality a microscopic operator such as $S_{\vec{r}}$ is equal to a sum of scaling operators, because of the change of basis required to relate microscopic couplings to scaling variables. However, when we write $S_{\vec{r}}$ as a sum of ($\mathbb{Z}_2$ odd) scaling operators, a single term dominates the large-distance correlation functions. This is the operator in the sum with the smallest x. Since this operator has the largest y, it is also the most relevant term if we perturb the critical point with a magnetic field term $h \sum_{\vec{r}} S(\vec{r})$. We sometimes use a loose notation in which we use the same symbol to refer to both the microscopic operator and the leading scaling operator with the same symmetry.

we coarse-grain, replacing microscopic spins S_r with block spins S'_R, the microscopic and coarse-grained operators are related as

$$O_\alpha(\vec{r}) \sim b^{-x_\alpha} O'_\alpha(\vec{R}), \tag{3}$$

where R is the coordinate of the block containing the microscopic spin $\vec{r}$. The operator on the left is a function of the spins S (close to $\vec{r}$) and the operator on the right is a function of the blocked spins S'. Since we rescale coordinates when we do the rescaling transformation, we have

$$\vec{R} \sim \vec{r}/b. \tag{4}$$

Recall that this gives the following for the correlation function $G_\alpha(\vec{r}) = \langle O_\alpha(0) O_\alpha(\vec{r})\rangle_{K_*}$ at the fixed point:

$$G_\alpha(\vec{r}) = \langle O_\alpha(0) O_\alpha(\vec{r})\rangle_{K_*} = b^{-2x_\alpha} \left\langle O'_\alpha(0) O'_\alpha(\vec{r}/b)\right\rangle_{K_*} \tag{5}$$

Since the coarse-grained spins S' have the same couplings as the original ones S, the primes on the right hand side do not matter:

$$G_\alpha(\vec{r}) = b^{-2x_\alpha} G_\alpha(\vec{r}/b) \quad \longrightarrow \quad G_\alpha(\vec{r}) \sim |\vec{r}|^{-2x_\alpha}. \tag{6}$$

Finally, recall that x_α and y_α are related by

$$y_\alpha = d - x_\alpha. \tag{7}$$

We see this by using (3) to write (2) in terms of coarse-grained variables S':

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_* + \sum_\alpha u_\alpha \sum_{\vec{R}} \sum_{\vec{r}\in\vec{R}} b^{-x_\alpha} O'_\alpha(\vec{R}) && (8)\\ &= \mathcal{H}_* + \sum_\alpha \left(b^{d-x_\alpha} u_\alpha\right) \sum_{\vec{R}} O'_\alpha(\vec{R}), && (9) \end{aligned}$$

since b^d is the number of spins in the block. Identifying u'_α with $b^{d-x_\alpha} u_\alpha$ gives the relation between y and x.

a) Random coupling Instead of perturbing $\mathcal{H}_*$ with a uniform coupling $u(r)$, perturb it with a random coupling that depends on the site r, so that the $u(r)$ are i.i.d. random variables. Let the average vanish, $\overline{u(r)} = 0$, and let the rms value be $\Delta \equiv \sqrt{\overline{u(r)^2}}$. Following similar logic to that above, what is the RG eigenvalue of Δ? Is a random magnetic field a relevant or an irrelevant perturbation of the critical 2D Ising model?

b) Coupled copies of Ising Consider a 2D model where at each site we have m different Ising spins $S^{(1)}(\vec{r}), \ldots, S^{(m)}(\vec{r})$. Initially they do not interact with each other, and we have m decoupled copies of the Ising model:

$$\mathcal{H}[S^{(1)}, \ldots, S^{(m)}] = \sum_{c=1}^{m} \mathcal{H}[S^{(c)}]. \tag{10}$$

At the critical temperature, this Hamiltonian flows to a fixed point. This fixed point theory is simply m decoupled copies of the Ising fixed point theory. We now couple the copies with the small term:

$$\delta\mathcal{H} = u \sum_{\vec{r}} S^{(1)}(\vec{r}) S^{(2)}(\vec{r}) \ldots S^{(m)}(\vec{r}). \tag{11}$$

At the critical temperature this leads to a small perturbation of the fixed point of a similar form (with S replaced with the corresponding leading $\mathbb{Z}_2$ odd scaling operator at the Ising fixed point[3]). In the 2D Ising model the scaling dimension of S is $x_S = 1/8$.

i) What is the RG eigenvalue y_u of the coupling u above? What is x_u for the corresponding operator $O(\vec{r}) = S^{(1)}(\vec{r}) S^{(2)}(\vec{r}) \ldots S^{(m)}(\vec{r})$? For what range of m can you assert that a small perturbation of this form leaves the critical exponents of the original model unchanged?

ii) Recall that the *ordered* phase of a single copy has two symmetry-related ordered states. How many degenerate ordered states does the multi-copy model have when $u = 0$? What about when $u > 0$? (The other ones have an extensively larger free energy, so do not occur.)

c) Power-law interactions Consider a 2D Ising model to which we add a *long-range* coupling of the following form:

$$\delta\mathcal{H} = u \sum_{r} \sum_{r'} \frac{S_{\vec{r}} S_{\vec{r}'}}{|\vec{r} - \vec{r}'|^a}. \tag{12}$$

Assuming that the important terms are the ones where the separation $|r - r'|$ is much larger than the lattice spacing,[4] use the logic outlined at the end

[3] See previous footnote

[4] The short-distance terms only renormalize the short-range coupling J that we already have. This affects the location of the critical point J_c but is not important for the universality class of the transition. Also, note that when r and r' are well-separated they live in different blocks, so the operators S_r and $S_{r'}$ are renormalized independently (this would not be true if r and r' were adjacent sites).

of the recap to compute the RG eigenvalue of the coupling u, in terms of x_S and a. (Do not forget the rescaling of the spatial coordinate in Eq. 4!) For what range of a does the universality class of the transition remain the same as in the model with only nearest-neighbour couplings?

2 RSRG for a 1D random walk problem and an example of quenched disorder

A classical particle travels along the line. It travels at constant velocity until it encounters one of the junctions that are placed at unit intervals along the line. When it encounters a junction it is randomly either 'transmitted' (continues past the junction with unchanged velocity) or 'reflected' (velocity reversed).

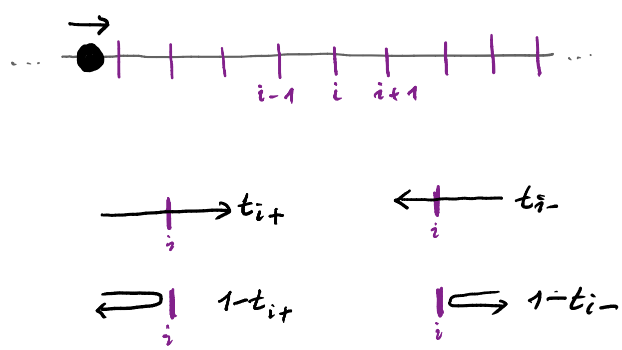

Figure 1: Transmission and reflection probabilities for particle

Each encounter with a junction is an independently random event with a certain probability for transmission (see Figure). This can depend on whether the junction is approached from the left or the right (we then denote these probabilities t_+ and t_- respectively) and may also be different for different junctions (in which case the transmission probabilities for the ith junction are denoted $t_{i\pm}$).

a) First consider the case where the junctions are symmetric and identical, so $t_{i+} = t_{i-} = t$.

i) Perform a real space RG transformation with rescaling factor $b = 2$ by grouping a pair of junctions into one coarse-grained junction, with transmission and reflection probabilities t' and $1 - t'$. (To obtain t' in terms of

t, you can sum up a series whose terms correspond to different numbers of internal reflections inside the junction.) Simplify the formula for t.

ii) Show that if we rewrite the RG equation in terms of the variable $u = (1-t)/t$ it takes a simple linear form. Draw a flow diagram.

iii) Let the system consist of $L = 2^k$ junctions. Compute the probability $t(L)$ that a particle, incident from the left, finally leaves the system to the right. You should find that $t(L)$ scales as const. $\times L^{-1}$ at large L.

Note: For $t = 1/2$, the above walk reduces to a simple random walk: the probabilities to hop from junction i to junctions $i+1$ and $i-1$ are $1/2$ each. For $1/2 < t < 1$ we have a persistent random walker who prefers to carry on in the same direction for a while. However the universal properties of the motion are the same for any $t \in (0, 1)$.[5]

Adding disorder: This random walk problem is one where we can investigate exactly the effect of disorder in the couplings of a statistical problem. That is, we will make the transmission probabilities $t_{i\pm}$ into random variables. We refer to these as 'quenched' random variables, meaning that they do not change during the stochastic motion of the particle. The stochastic motion takes place within a frozen random landscape defined by a realization of the $t_{i\pm}$s.

In (b) and (c) we consider two types of quenched disorder with progressively more severe effect on the walker.

b) We now allow the transmission probabilities to differ for different junctions. However each junction is still left-right symmetric, so we write $t_{i+} = t_{i-} = t_i$.

i) Derive t' for a coarse-grained junction in terms of t_1, t_2 for the two constituent junctions (noting that the junction remains symmetric after coarse-graining), and rewrite in terms of the u variable.

Now assume that the u_is of the microscopic junctions are independent, identically distributed random variables with mean $\langle u_i \rangle = \overline{u}$ and standard devi-

[5] The $1/L$ scaling you found above also applies for the probability that a Brownian particle, which undergoes Brownian motion on the interval $[0, L]$ with $L \gg 1$, and which starts off at a point $x > 0$ with $x = O(1)$, exits the interval via the right rather than the left boundary.

ation $\sqrt{\langle (u_i - \bar{u})^2 \rangle} = \Delta$.

At a given stage k of the RG, denoting the coarse-grained parameter for a block by $u^{(k)}$, let us write the renormalized mean and fluctuations as

$$\overline{u}^{(k)} = \left\langle u^{(k)} \right\rangle, \qquad r^{(k)} = \frac{\sqrt{\left\langle \left[u^{(k)} - \overline{u}^{(k)} \right]^2 \right\rangle}}{\overline{u}^{(k)}}, \tag{13}$$

where $r^{(k)}$ measures the *relative* strength of fluctuations at a given length-scale $L = 2^k$ (note $r^{(0)} = \Delta / \overline{u}$).

ii) Write RG equations for $\overline{u}^{(k)}$ and $r^{(k)}$ (by noting that when you combine two junctions into one, the two constituents have statistically independent parameters). Draw a rough RG flow diagram in the $(1/u, r)$ plane.

Note that the fluctuations become negligible, relative to the mean, under the RG flow.

iii) Using the above observation, what is $\langle t(L) \rangle \equiv \left\langle t^{(k)} \right\rangle$ for a junction of length $L = 2^k$, to leading order at large L?

We see that in this case disorder has not changed the asymptotic scaling.

Note: in this 1D example the use of the RG is mostly illustrative, as we can solve the problem directly:

iv) Write the coarse-grained $u(L) \equiv u^{(k)}$ for a chain of $L = 2^k$ junctions in terms of the microscopic $u_1, u_2, \ldots, u_L$. What is the form of the probability distribution of $u(L)$?

c) Now let us allow each junction to be **asymmetric**, so that $t_{i+} \neq t_{i-}$. We take the $t_{i\pm}$ to be independent, identically distributed random variables. While the junctions are not left-right symmetric, they are left-right symmetric 'on average', since the probability distribution of t_{i+} is the same as that of t_{i-}.

i) Write t'_+ and t'_- in terms of $t_{1\pm}$ and $t_{2\pm}$ (for the left and right junctions respectively). Parametrize the ts as

$$t_+ = \frac{1}{r\alpha}, \qquad t_- = \frac{\alpha}{r}, \tag{14}$$

so that $\alpha = \sqrt{t_-/t_+}$ is a measure of how asymmetric the junction is and $r = 1/\sqrt{t_+ t_-}$. Write the RG equations for r' and α' in terms of $r_1, r_2, \alpha_1, \alpha_2$. (You should find that the αs form a very simple closed system.)

For simplicity let us now consider α only. In fact it is helpful to consider $\ln \alpha$. Note that microscopically

$$\langle \ln \alpha_i \rangle = \frac{1}{2} \langle \ln t_{i-} \rangle - \frac{1}{2} \langle \ln t_{i+} \rangle = 0 \tag{15}$$

by the left-right symmetry 'on average'. We define $\kappa = \sqrt{\langle (\ln \alpha_i)^2 \rangle}$. Note that $\kappa = 0$ if the microscopic junctions are perfectly symmetric.

ii) What is probability distribution of the coarse-grained quantity, $\alpha(L)$, after a large number k of RG steps with $L = 2^k$?

iii) By writing the variance of $\log \alpha(L)$ (which you now know) in terms of $t_+(L)$ and $t_-(L)$, and using left-right symmetry on average, as well as the fact that $\log t \leq 0$, derive a bound of the form

$$\left\langle (\log t_+(L))^2 \right\rangle \geq \text{const} \times L. \tag{16}$$

Note: In part (b), the transmission probability of a long chain of junctions (large L) was approximately deterministic and was power-law small in L. By contrast, here the typical transmission probability is **exponentially small in** L, as suggested by the previous formula. The distribution of α also shows that long chains are extremely asymmetric (with a random preferred direction) even if the asymmetry of the microscopic junctions (measured by κ) is small.