1 A phase transition for Ising spins on a tree?

Consider the Ising model on the tree shown in Fig. 1, with ferromagnetic coupling $J > 0$ on every bond. We also add a small positive magnetic field $h \ll 1$ for the spins in layer $m = 1$ only (we refer to this lowest layer as the boundary of the tree):

$$Z = \sum_{\{S_i\}} e^{J \sum_{\langle ij \rangle} S_i S_j + h \sum_{i \in \text{boundary}} S_i}. \quad (1)$$
Generalising slightly, we allow the tree to have a branching factor $f$ (the figure shows $f = 2$; note that $f = 1$ is a 1D chain).

We will answer the question: does the small symmetry-breaking boundary field $h$ lead to a nonzero magnetization $\langle S_{\text{root}} \rangle$ in the limit $N \to \infty$?

**Definitions:** Let $Z_+^{(N)}$ be the partition function for a tree with $N$ generations of spins, and with the root spin fixed to be $+1$, and let $Z_-^{(N)}$ be the partition function with the root spin fixed to be $-1$. (Note that $Z_+^{(1)} = e^h$ and $Z_-^{(1)} = e^{-h}$.)

a) Derive recursion relations for $Z_+^{(N+1)}$ and $Z_-^{(N+1)}$ in terms of $Z_+^{(N)}$ and $Z_-^{(N)}$ for general $f$. (Note that if we fix the value of the root spin, the remaining degrees of freedom split up into sub-trees that do not interact with each other.) Turn these into a recursion relation for $r^{(N)} = Z_+^{(N)}/Z_-^{(N)}$ only.

b) Note that for small $h$, $Z_+/Z_-$ is initially close to 1, so write $r^{(N)} = 1 + \epsilon^{(N)}$. Write a linearized recursion relation for $\epsilon^{(N)}$.

By considering when $\epsilon^{(N)}$ tends to zero at large $N$, define a critical $J_c$ as a function of $f$. What is $\lim_{N \to \infty} \langle S_{\text{root}} \rangle$ when $J < J_c$?

c) Think of the recursion relation for $r^{(N)}$ as a map on the line $r \in [1, \infty]$. Is the fixed point at $r = \infty$ stable or unstable? What about the fixed point at $r = 1$, for the cases $J < J_c$ and $J > J_c$?

d) Making the simplest assumption about the flows in between these limits, draw “flow diagrams” for the recursion relation for the cases $J < J_c$ and $J > J_c$. Is $\lim_{h \to 0^+} \lim_{N \to \infty} \langle S_{\text{root}} \rangle$ zero or nonzero when $J > J_c$?

2 **Fractal dimension**

The figure shows the first steps in the construction of the Sierpinski triangle by iteratively removing down-pointing triangles from the shaded region.\(^1\)

a) After $k$ iterations, let the side length of the smallest remaining up-pointing triangles be $a$ and let the number of these triangles be $N$. Writing

\(^1\)The Sierpinski triangle is the subset of the plane that remains in the limit of an infinite number of iterations.
Figure 2: First steps in the construction of the Sierpinski triangle

\[ N = (L/a)^{d_f}, \text{ what is } d_f? \]

[So long as we look at features on scales greater than \( a \) and smaller than \( L \), the triangle is scale invariant: it looks the same if we zoom in (by a rescaling factor \( b = 2 \)). The quantity \( d_f \) is the ‘fractal dimension’ of this scale-invariant pattern. It characterises how the number of sub-blocks scales with the linear size of the total shape. The percolation clusters discussed in the lecture are random fractals that are statistically scale-invariant. They have \( d_f = 91/48 \simeq 1.89. \]

3 Ising correlations along a boundary (semi-infinite transfer matrix)

Review: Recall from the lecture that in the limit of high temperature (small \( \mu = \tanh J \)) the two-point function in the bulk Ising model can be
written as a sum over directed paths

$$\langle S_i S_j \rangle \simeq \sum_{\text{shortest paths } i \to j} \mu^\text{length}. \quad (2)$$

This led to the result that in 2D the correlator decayed exponentially with a power-law prefactor $R^{-1/2}$. On the other hand, in 1D, we know from transfer matrix calculations that correlators decay exponentially, without any power-law prefactor.

**Setup**: Now consider the 2D Ising model with boundary shown in Fig. 3. (The model is infinite in the $+y$ and $\pm x$ directions.) The coupling for the bonds on the boundary is different than that for the bonds in the bulk. We parameterise this as shown in the figure — the constant $\mu$ in the bulk is replaced by $\mu \times s$ on the boundary.

Using the coordinates in the figure, the boundary correlation function $\langle S(0,0)S(R,0) \rangle$ between the two points marked in the figure ($R$ is arbitrary but even) is now

$$\langle S(0,0)S(R,0) \rangle \simeq \mu^R Z_{\text{eff}}, \quad Z_{\text{eff}} = \sum_{\text{directed paths } (0,0) \to (R,0)} s^{\text{no. boundary steps}} \quad (3)$$

where $Z_{\text{eff}}$ is a sum over directed (rightgoing) paths, i.e. an effective partition function for a directed string of length $R$ that lives on the bonds of the lattice shown.

**a)** Write this correlation function in terms of a semi-infinite transfer matrix $T_{y,y'}$, where $y$ is the vertical coordinate shown.

The transfer matrix may have two types of eigenstates: states that look like plane waves for $y > 0$ with momentum $k$ (or rather superpositions of waves at $+k$ and $-k$), and bound states (evanescent waves) that decay exponentially in $y$ for $y > 0$.

**b)** From the eigenvalue equation $T_{yy'} \psi_{y'} = \lambda \psi_y$ for $y > 0$ show that any plane wave eigenstate has eigenvalue between -2 and 2 and any bound state has eigenvalue > 2.

**c)** Try a bound state wavefunction of the form $\psi \propto (1, \alpha, \alpha e^{-\beta}, \alpha e^{-2\beta}, \ldots)$. For what range of $s$ does a bound state exist?
d) In the regime where the bound state exists, what is the free energy per unit length for the directed path? (i.e. $\lim_{R \to \infty} F/R$, with $Z_{\text{eff}} = e^{-F}$.) What is the correlation length for $\langle S(0,0)S(R,0) \rangle$ in this regime? Do you expect there to be a power-law prefactor in the correlation function?

e) How does the size of the bound state behave, asymptotically, as it is about to disappear?

f) Consider the regime where there is no bound state. In this regime, how does the path’s typical $y$ value (when $x$ is, say, $\sim R/2$) scale with $R$? Draw a cartoon of a trajectory in each of the two ‘phases’\(^2\). Without appealing to the transfer matrix, what are the path’s free energy per unit length, and the correlation length, in the phase without a bound state?

[In the phase without a bound state the power law prefactor in the correlation function is $R^{-3/2}$, but you do not need to derive this. Right at the critical point it is $R^{-1/2}$.]

\(^2\)Note that the two phases of the string do not correspond to the ordered and disordered phases of the Ising model. We are considering small $\mu$, so the Ising model is always disordered.