# MMathPhys Renormalisation Group: Homework 1

January 23, 2019

## 1 A phase transition for Ising spins on a tree?

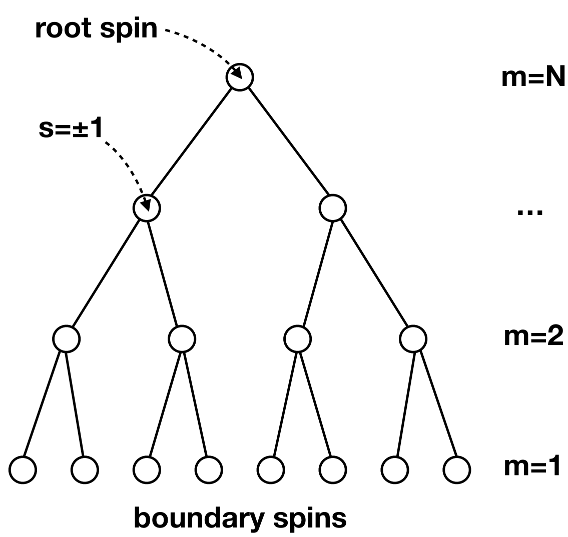


Figure 1: Ising model on a tree with branching factor $f = 2$

Consider the Ising model on the tree shown in Fig. 1, with ferromagnetic coupling $J > 0$ on every bond. We also add a small positive magnetic field $h \ll 1$ *for the spins in layer* $m = 1$ *only* (we refer to this lowest layer as the boundary of the tree):

$$Z = \sum_{\{S_i\}} e^{J \sum_{\langle ij \rangle} S_i S_j + h \sum_{i \in \text{boundary}} S_i}. \tag{1}$$

Generalising slightly, we allow the tree to have a branching factor $f$ (the figure shows $f = 2$; note that $f = 1$ is a 1D chain).

We will answer the question: does the small symmetry-breaking boundary field $h$ lead to a nonzero magnetization $\langle S_{\text{root}} \rangle$ in the limit $N \to \infty$?

**Definitions:** Let $Z_+^{(N)}$ be the partition function for a tree with $N$ generations of spins, and with the root spin fixed to be $+1$, and let $Z_-^{(N)}$ be the partition function with the root spin fixed to be $-1$. (Note that $Z_+^{(1)} = e^h$ and $Z_-^{(1)} = e^{-h}$.)

**a)** Derive recursion relations for $Z_+^{(N+1)}$ and $Z_-^{(N+1)}$ in terms of $Z_+^{(N)}$ and $Z_-^{(N)}$ for general $f$. (Note that if we fix the value of the root spin, the remaining degrees of freedom split up into sub-trees that do not interact with each other.) Turn these into a recursion relation for $r^{(N)} \equiv Z_+^{(N)}/Z_-^{(N)}$ only.

**b)** Note that for small $h$, $Z_+/Z_-$ is initially close to 1, so write $r^{(N)} = 1 + \epsilon^{(N)}$. Write a linearized recursion relation for $\epsilon^{(N)}$.

By considering when $\epsilon^{(N)}$ tends to zero at large $N$, define a critical $J_c$ as a function of $f$. What is $\lim_{N\to\infty} \langle S_{\text{root}} \rangle$ when $J < J_c$?

**c)** Think of the recursion relation for $r^{(N)}$ as a map on the line $r \in [1, \infty]$. Is the fixed point at $r = \infty$ stable or unstable? What about the fixed point at $r = 1$, for the cases $J < J_c$ and $J > J_c$?

**d)** Making the simplest assumption about the flows in between these limits, draw "flow diagrams" for the recursion relation for the cases $J < J_c$ and $J > J_c$. Is $\lim_{h\to 0_+} \lim_{N\to\infty} \langle S_{\text{root}} \rangle$ zero or nonzero when $J > J_c$?

## 2 Fractal dimension

The figure shows the first steps in the construction of the Sierpinski triangle by iteratively removing down-pointing triangles from the shaded region.[1]

**a)** After $k$ iterations, let the side length of the smallest remaining up-pointing triangles be $a$ and let the number of these triangles be $N$. Writing

[1] The Sierpinski triangle is the subset of the plane that remains in the limit of an infinite number of iterations.

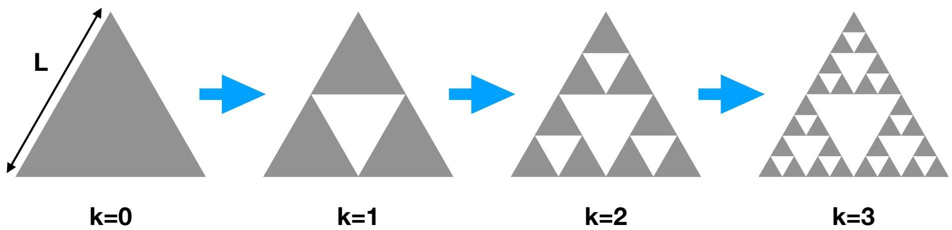


Figure 2: First steps in the construction of the Sierpinski triangle

$N = (L/a)^{d_f}$, what is $d_f$?

[So long as we look at features on scales greater than $a$ and smaller than $L$, the triangle is scale invariant: it looks the same if we zoom in (by a rescaling factor $b = 2$). The quantity $d_f$ is the 'fractal dimension' of this scale-invariant pattern. It characterises how the number of sub-blocks scales with the linear size of the total shape. The percolation clusters discussed in the lecture are random fractals that are statistically scale-invariant. They have $d_f = 91/48 \simeq 1.89$.]

# 3 Ising correlations along a boundary (semi-infinite transfer matrix)

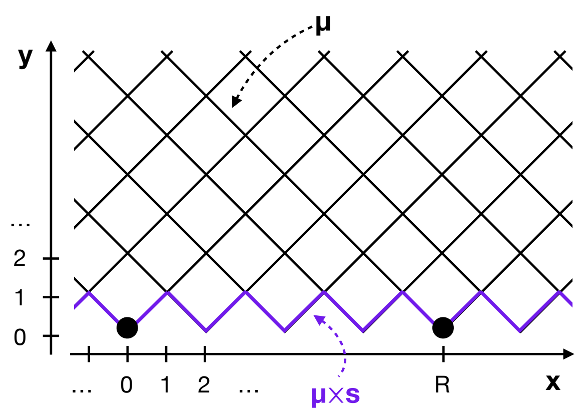


Figure 3: 2D Ising model with boundary

**Review:** Recall from the lecture that in the limit of high temperature (small $\mu = \tanh J$) the two-point function in the bulk Ising model can be written as a sum over directed paths

$$\langle S_i S_j \rangle \simeq \sum_{\text{shortest paths } i \to j} \mu^{\text{length}}. \tag{2}$$

This led to the result that in 2D the correlator decayed exponentially with a power-law prefactor $R^{-1/2}$. On the other hand, in 1D, we know from transfer matrix calculations that correlators decay exponentially, without any power-law prefactor.

**Setup:** Now consider the 2D Ising model with boundary shown in Fig. 3. (The model is infinite in the $+y$ and $\pm x$ directions.) The coupling for the bonds on the boundary is different than that for the bonds in the bulk. We parameterise this as shown in the figure — the constant $\mu$ in the bulk is replaced by $\mu \times s$ on the boundary.

Using the coordinates in the figure, the boundary correlation function $\langle S(0,0)S(R,0)\rangle$ between the two points marked in the figure ($R$ is arbitrary but even) is now

$$\langle S(0,0)S(R,0)\rangle \simeq \mu^R Z_{\text{eff}}, \qquad Z_{\text{eff}} = \sum_{\substack{\text{directed paths} \\ (0,0)\to(R,0)}} s^{\text{no. boundary steps}} \tag{3}$$

where $Z_{\text{eff}}$ is a sum over directed (rightgoing) paths, i.e. an effective partition function for a directed string of length $R$ that lives on the bonds of the lattice shown.

**a)** Write this correlation function in terms of a semi-infinite **transfer matrix** $T_{y,y'}$, where $y$ is the vertical coordinate shown.

The transfer matrix may have two types of eigenstates: states that look like plane waves for $y > 0$ with momentum $k$ (or rather superpositions of waves at $+k$ and $-k$), and bound states (evanescent waves) that decay exponentially in $y$ for $y > 0$.

**b)** From the eigenvalue equation $T_{yy'}\psi_{y'} = \lambda \psi_y$ for $y > 0$ show that any plane wave eigenstate has eigenvalue between -2 and 2 and any bound state has eigenvalue $> 2$.

**c)** Try a bound state wavefunction of the form $\psi \propto (1, \alpha, \alpha e^{-\beta}, \alpha e^{-2\beta}, \ldots)$. For what range of $s$ does a bound state exist?

**d)** In the regime where the bound state exists, what is the free energy per unit length for the directed path? (I.e. $\lim_{R\to\infty} F/R$, with $Z_{\text{eff}} = e^{-F}$.) What is the correlation length for $\langle S(0,0)S(R,0)\rangle$ in this regime? Do you expect there to be a power-law prefactor in the correlation function?

**e)** How does the size of the bound state behave, asymptotically, as it is about to disappear?

**f)** Consider the regime where there is no bound state. In this regime, how does the path's typical $y$ value (when $x$ is, say, $\sim R/2$) scale with $R$? Draw a cartoon of a trajectory in each of the two 'phases'[2]. Without appealing to the transfer matrix, what are the path's free energy per unit length, and the correlation length, in the phase without a bound state?

[In the phase without a bound state the power law prefactor in the correlation function is $R^{-3/2}$, but you do not need to derive this. Right at the critical point it is $R^{-1/2}$.]

[2]Note that the two phases of the string do *not* correspond to the ordered and disordered phases of the Ising model. We are considering small $\mu$, so the Ising model is always disordered.