AQFT / Tutorial 1

(1.1)

PROBLEM 1: SYMMETRIES

→ three component scalar field:

$$\Phi(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{bmatrix}$$

signature: $(+,-,-,-)$

→ Lagrangian is:

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{3}(\partial_\mu \phi_a)(\partial^\mu \phi_a) - \frac{1}{2}\sum_{a=1}^{3} m^2 \phi_a^2$$

$$= \frac{1}{2}(\partial_\mu \Phi)^T(\partial^\mu \Phi) - \frac{1}{2} m^2 \Phi^T \Phi$$

(1A) equations of motion:

$$LHS = \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_a)}\right) = \partial_\mu(\partial^\mu \phi_a)$$

$$RHS = \frac{\delta \mathcal{L}}{\delta \phi_a} = -m^2 \phi_a$$

which gives:

$$\boxed{\partial_\mu \partial^\mu \phi_a + m^2 \phi_a = 0}$$

KLEIN-GORDON EQUATION
for each scalar field ϕ_a

(1B) symmetry

→ clear that $\mathcal{L}$ is invariant under $SO(3)$ transformation:

$$\Phi \rightarrow \Phi' = U(\theta)\,\Phi$$

where $U(\theta) \in SO(3)$

// $U^T U = \mathbb{1}$, $\det U = +1$ //

→ generators of $SO(3)$ Lie algebra:

$$J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{bmatrix} \qquad J_2 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{bmatrix} \qquad J_3 = \begin{bmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- in short: $(J_a)_{bc} = -i\,\epsilon_{abc}$
- under this transformation:

$$\phi_a' = U_{ab}\phi_b = (\mathbb{1} + i\vec{J}\cdot\vec{\Theta})\,\phi_b$$

three generators $\vec{J} = (J_1, J_2, J_3)$

parameters of the transformation $\vec{\Theta} = (\Theta_1, \Theta_2, \Theta_3)$

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further:

$$\phi_a' = \phi_a + \delta\phi_a$$
$$= \phi_a + i(J_\Theta)_{ab}\,\Theta_c\,\phi_b$$
$$= \phi_a + \epsilon_{cab}\,\Theta_c\,\phi_b$$

hence

$$\boxed{\delta\phi_a = \epsilon_{abc}\,\phi_b\,\Theta_c}$$

→ the Noether current is:

$$J_\Theta^\mu = \sum_a \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_a)}\cdot\delta\phi_a$$
$$= \sum_a (\partial^\mu\phi_a)\cdot\epsilon_{abc}\,\phi_b\,\Theta_c$$
$$= \Theta_c\cdot\left(-\sum_a \epsilon_{cba}\,\phi_b\,(\partial^\mu\phi_a)\right)$$
$$\vec{J}^\mu = \vec{\Theta}\cdot\left(-\vec{\Phi}\times\partial^\mu\vec{\Phi}\right)$$

⇒ three conserved currents

$$\vec{J}^\mu = \begin{pmatrix} J_1^\mu \\ J_2^\mu \\ J_3^\mu \end{pmatrix} = -\vec{\Phi}\times(\partial^\mu\vec{\Phi})$$

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PROBLEM 2: WARD-TAKAHASI IDENTITIES & SCALAR QED

$\boxed{Q2}$ SCALAR QED

→ fields: $\phi(x)$ - complex scalar field
$A_\mu(x)$ - (real) gauge field (vector)

→ lagrangian:

$$\mathcal{L}_{sQED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D_\mu\phi) - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2$$

field strength tensor
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

covariant derivative
$D_\mu = \partial_\mu - ieA_\mu$

(2A) Gauge transformation:

$$\phi(x) \rightarrow \phi(x)e^{ie\omega(x)}$$
$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\omega(x)$$

// $\omega(x)$ - space-dependent function! local symmetry! //

(i) $\phi^*\phi$ - manifestly invariant (since phases cancel)!

(ii) $F_{\mu\nu}$ - invariant (since derivatives commute: $\partial_\mu\partial_\nu\omega - \partial_\nu\partial_\mu\omega = 0$)

// NOTE: this is only true for U(1) symmetry (abelian group)!! for non-abelian groups, $F_{\mu\nu}$ is not gauge invariant object, $Tr(F_{\mu\nu}F^{\mu\nu})$, however, is! //

(iii) covariant derivative: term:

$$D_\mu\phi \equiv (\partial_\mu - ieA_\mu)\phi \longrightarrow D'_\mu\phi' = (\partial_\mu - ieA'_\mu)\phi'$$

$$\begin{aligned} D'_\mu\phi' &= (\partial_\mu - ieA_\mu - ie(\partial_\mu\omega))\phi e^{ie\omega} \\ &= e^{ie\omega}(\partial_\mu\phi + ie(\partial_\mu\omega)\phi) - e^{ie\omega}ieA_\mu\phi - e^{ie\omega}ie(\partial_\mu\omega)\phi \\ &= e^{ie\omega}\left[\partial_\mu + \cancel{ie(\partial_\mu\omega)} - ieA_\mu - \cancel{ie(\partial_\mu\omega)}\right]\phi \\ &= e^{ie\omega}\left[\partial_\mu - ieA_\mu\right]\phi \\ &= e^{ie\omega}D_\mu\phi \end{aligned}$$

$$D_\mu\phi \rightarrow e^{ie\omega}D_\mu\phi$$

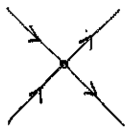
hence phases cancel out in $(D_\mu\phi)^*(D_\mu\phi)$, which is indeed invariant

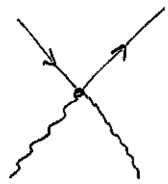
(2B) Lagrangian in R_ξ-gauge:

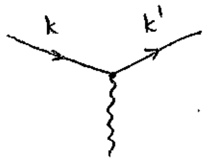
$$\mathcal{L}_{eff} = \mathcal{L}_{sQED} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

FEYNMAN RULES:

(i) interaction terms:

 $= -i\lambda$

 $= 2ie^2 g_{\mu\nu}$



$= +ie(k+k')_\mu$

(ii) propagators:

 $= \dfrac{i}{p^2 - m^2 + i\varepsilon}$

$= \dfrac{-i}{k^2 + i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\dfrac{k_\mu k_\nu}{k^2}\right] \equiv D_{\mu\nu}$

Consider the generating functional:

$$Z[K, K^*, J_\mu] = \mathcal{N}\int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu \exp\left\{ i\int d^4x \left(\mathcal{L}_{eff} + \phi K^* + K\phi^* + J_\mu A^\mu\right)\right\}$$

→ under the gauge transformation:

(a) measure: $\mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu = \mathcal{D}\phi'\mathcal{D}\phi'^*\mathcal{D}A'_\mu$

(b) integrand:

$$\mathrm{EXP} = \left[\mathcal{L}_{eff} + \phi K^* + K\phi^* + J_\mu A^\mu\right] +$$
$$+\left[-\frac{1}{\xi}(\partial_\mu A^\mu)\partial^2\omega + ie\omega\phi K^* - ie\omega K\phi^* + J_\mu\partial^\mu\omega\right]$$
$$+ \mathcal{O}(\omega^2)$$

→ after integrating by parts:

$$EXP = \int d^4x \left[\ldots\ldots \right]$$
$$+ \int d^4x \left[-\frac{1}{\xi} \partial^2(\partial_\mu A^\mu) + ie\phi K^* - ie K\phi^* - \partial_\mu J^\mu \right] \omega$$
$$+ \mathcal{O}(\omega^2)$$

hence, requiring the generating functional is invariant under gauge transformation we need:

$$\int d^4x \left[-\frac{1}{\xi} \partial^2 \partial_\mu A^\mu + ie\phi K^* - ie K\phi^* - \partial_\mu J^\mu \right] \omega(x) = 0$$

→ use:

$$A^\mu(x) = \frac{\delta W}{\delta J_\mu(x)} \qquad \phi = \frac{\delta W}{\delta K^*} \qquad \phi^* = \frac{\delta W}{\delta K}.$$

and get:

$$\boxed{-\frac{1}{\xi} \partial^2 \partial_\mu \frac{\delta W}{\delta J_\mu} + ie \frac{\delta W}{\delta K^*} K^* - ie \frac{\delta W}{\delta K} K - \partial_\mu J^\mu = 0} \quad (*)$$

(2c) derive the result (*) wrt. $K^*(y)$ and $K(z)$

$$-\frac{1}{\xi} \partial^2 \partial_\mu \frac{\delta^3 W}{\delta K_z \, \delta K_y^* \, \delta J_x^\mu} = -ie \frac{\delta^2 W}{\delta K_z \, \delta K_x^*} \delta(x-y) + ie \frac{\delta^2 W}{\delta K_y^* \, \delta K_x} \delta(x-z)$$

// use definition of connected correlation function: [P&S, eq. 11.86]

$$\frac{\delta^n W[J]}{\delta J(x_1) \ldots \delta J(x_n)} = (i)^{n+1} \langle \phi(x_1) \ldots \phi(x_n) \rangle_{connected}$$

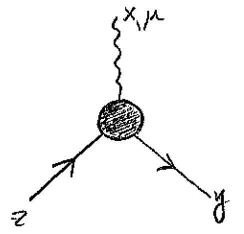
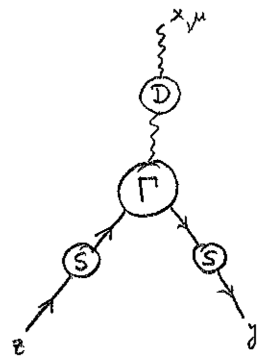
→ good reference: Peskin + Schroeder; chapter 11.5
P. Ramond, ch. 3.3 //

we obtain:

$$\boxed{-\frac{1}{\xi} \partial^2 \partial_\mu \langle T(\phi^*(z)\phi(y)A^\mu(x)) \rangle = -e \langle T(\phi^*(z)\phi(x)) \rangle \delta(x-y) + e \langle T(\phi(y)\phi^*(x)) \rangle \delta(x-z)}$$

(**)

(a) $\langle \Omega | T(\phi^*(z)\phi(y)A^\mu(x)) | \Omega \rangle =$

$=$  $=$ 

$$= \int d^4w \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} \int \frac{d^4k''}{(2\pi)^4} \left[S(k)\, e^{ik(z-w)} \right] \left[S(k')\, e^{ik'(w-y)} \right] \times$$

$$\times \left[D^{\mu\nu}(k'')\, e^{ik''(w-x)} \right] \left[ie\, \Gamma^\nu(k,k',k'') \right]$$

$$= \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} \int \frac{d^4k''}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(k''+k'-k)\, e^{ikz} \cdot e^{-ik'y} \cdot e^{-ik''x} \times$$

$$\times S(k) S(k') D^{\mu\nu}(k'') \left[ie\, \Gamma^\nu(k,k',k'') \right]$$

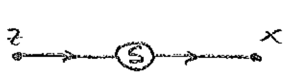
$$= \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} \cdot e^{ik(z-x)} \cdot e^{ik'(x-y)} \cdot S(k) \cdot S(k') \cdot D^{\mu\nu}(k-k') \cdot \left[ie\, \Gamma^\nu(k,k') \right]$$

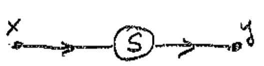
$\rightarrow$ then: (under Fourier Transform):

$$\frac{\partial}{\partial x^\mu} \langle \Omega | T(\phi_z^* \phi_y A_x^\mu) | \Omega \rangle = -i(k-k')_\mu \langle \Omega | T(\phi_z^* \phi_y A_x^\mu) | \Omega \rangle$$

$$\Box \frac{\partial}{\partial x^\mu} \langle \Omega | T(\phi_z^* \phi_y A_x^\mu) | \Omega \rangle = +i\,(k-k')^2 (k-k')_\mu \langle \Omega | T(\ldots) | \Omega \rangle$$

$\uparrow$
$\partial_\mu \partial^\mu$

(b)  $= \langle \Omega | T(\phi_z^* \phi_x) | \Omega \rangle = \int \frac{d^4k}{(2\pi)^4} S(k)\, e^{ik(z-x)}$

 $= \langle \Omega | T(\phi_x^* \phi_y) | \Omega \rangle = \int \frac{d^4k'}{(2\pi)^4} e^{ik'(x-y)} \cdot S(k')$

(c) $\delta(x-y) = \int \frac{d^4k'}{(2\pi)^4} e^{ik'(x-y)}$

$\delta(x-z) = \int \frac{d^4k}{(2\pi)^4} e^{ik(z-x)}$

Finally (**) leads to:

$$-\frac{i}{\xi}(k-k')^2(k-k')_\mu \; S(k)\,S(k')\,D^{\mu\nu}(k-k')\left[ie\,\Gamma^\nu(k,k')\right] =$$

$$= -e\,S(k) + e\,S(k')$$

// now, note that:

$$D^{\mu\nu}(p) = \frac{-i}{p^2}\left[g^{\mu\nu} - (1-\xi)p^\mu p^\nu/p^2\right]$$

and hence: $p^2 p_\mu D^{\mu\nu}(p) = -i\left[p^\nu - (1-\xi)p^\nu\right] = -i\xi p^\nu$ //

$$\Rightarrow \quad -(k-k')_\nu\, S(k)\,S(k')\left[ie\,\Gamma^\nu(k,k')\right] = e\,S(k') - e\,S(k)$$

$$\boxed{(k-k')_\nu\,\Gamma^\nu(k,k') = i\left[\frac{1}{S(k)} - \frac{1}{S(k')}\right]}$$

(2D) at leading order:

$$ie\,\Gamma^\nu(k,k') = +ie\,(k+k')^\nu$$

$$S(k) = \frac{i}{k^2-m^2}$$

$$\text{LHS} = (k-k')_\nu\,(+k+k')^\nu = k^2 - k'^2$$

$$\text{RHS} = i\left[\frac{k^2-m^2}{i} - \frac{k'^2-m^2}{i}\right] = k^2 - k'^2$$

$$\text{LHS} = \text{RHS} \quad \text{(ok)}$$

PROBLEM 3: SCALAR QED $\phi^+\phi^- \to \gamma\gamma$

Matrix elements:

$$i\mathcal{M}_t = (-ie)^2 \frac{i}{(p_1-p_3)^2-m^2} \left(p_1 + (p_1-p_3)\right)^\mu \Big(\underbrace{(p_1-p_3)}_{=p_1-p_2} - p_2 \Big)^\nu \; \varepsilon^*_{3\mu}\, \varepsilon^*_{4\nu}$$

$$= \frac{-ie^2}{(p_1-p_3)^2-m^2} \left(2p_1^\mu\right)\left(-2p_2^\mu\right) \varepsilon^*_{3\mu}\, \varepsilon^*_{4\nu}$$

$\boxed{k^\mu \varepsilon_\mu(k) = 0}$

$$= \frac{4ie^2}{t-m^2}\, p_1^\mu p_2^\nu \; \varepsilon^*_{3\mu}\, \varepsilon^*_{4\nu}$$

$$i\mathcal{M}_u = (-ie)^2 \frac{i}{(p_1-p_4)^2-m^2} \left(p_1 + (p_1-p_4)\right)^\nu \Big(\underbrace{(p_1-p_4)}_{=(p_3-p_2)} - p_2 \Big)^\mu \; \varepsilon^*_{3\mu}\, \varepsilon^*_{4\nu}$$

$$= \frac{4ie^2}{u-m^2}\, p_1^\nu p_2^\mu \; \varepsilon^*_{3\mu}\, \varepsilon^*_{4\nu}$$

$$i\mathcal{M}_4 = 2ie^2 g^{\mu\nu}\, \varepsilon^*_{3\mu}\, \varepsilon^*_{4\nu}$$

Full amplitude:

$$i\mathcal{M} = 2ie^2 \left[g^{\mu\nu} + \frac{2p_1^\mu p_2^\nu}{t-m^2} + \frac{2p_1^\nu p_2^\mu}{u-m^2} \right] \varepsilon^*_{3\mu}\, \varepsilon^*_{4\nu}$$

Use polarisation sum:

$$\sum_{\lambda=\pm} \varepsilon^*_\mu(\lambda,k)\, \varepsilon_\nu(\lambda,k) = -g_{\mu\nu}$$

which gives the squared matrix element:

$$|\mathcal{M}|^2 = 16e^4 \left[1 - \frac{sm^2}{(t-m^2)(u-m^2)} + \left(\frac{sm^2}{(t-m^2)(u-m^2)} \right)^2 \right]$$

$$= 16(4\pi\alpha)^2 \left[1 - \;\dots\dots \right]$$

(3B) differential cross-section:

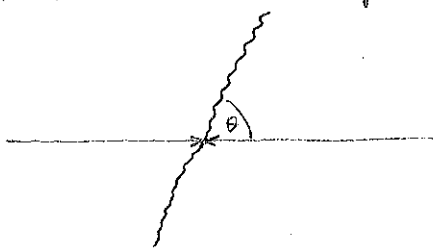
this only after integration over phase space

$$\frac{d\sigma}{d\Omega_{cm}} = \underbrace{\frac{1}{64\pi^2 s}\,\frac{|\vec{p}_3|}{|\vec{p}_1|}\,|\mathcal{M}|^2}_{\text{equation 11.31 of Srednicki}} \cdot \underbrace{\frac{1}{S}}_{\text{equation 11.36 of Srednicki}}$$

→ Momenta:

$p_1 = (E, 0, 0, +p)$ $\qquad$ $p_3 = (E, 0, +E\sin\theta, +E\cos\theta)$

$p_2 = (E, 0, 0, -p)$ $\qquad$ $p_4 = (E, 0, -E\sin\theta, -E\cos\theta)$



$|\vec{p}_3| = E \qquad |\vec{p}_1| = p$

→ symmetry factor: $\quad S = 2$ $\quad$ (two identical photons $\gamma\gamma$) (scalars are charged so distinguishable: $\phi^+\phi^-$)

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{64\pi^2 s}\left(\frac{E}{p}\right) 16\cdot 4\cdot 4\pi^2\alpha^2 \left[1 - \ldots\right]\cdot \frac{1}{2}$$

only after PS-integration

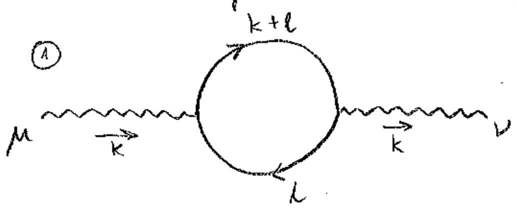
$$= \frac{4\alpha^2}{s}\left(\frac{E}{p}\right)\left[1 - \frac{m^2 s}{(t-m^2)(u-m^2)} + \left(\frac{m^2 s}{(t-m^2)(u-m^2)}\right)^2\right]$$

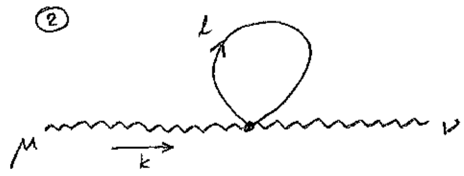
→ use FORM for evaluation of matrix element: performs algebra

PROBLEM 4: ELECTRIC CHARGE RENORMALISATION

4.1

(2A) At 1-loop we have two diagrams:





$$M_1 = (ie)^2 \int \frac{d^4 l}{(2\pi)^4} \left[D^{\mu\alpha}(k)\right]\left[(2l+k)^\alpha\right]\left[\frac{i}{(k+l)^2 - m^2}\right]\left[(2l+k)^\beta\right]\left[\frac{i}{l^2 - m^2}\right]\left[D^{\beta\nu}(k)\right]$$

$$M_2 = (2ie^2) \int \frac{d^4 l}{(2\pi)^4} \left[D^{\mu\alpha}(k)\right]\left[\frac{i g^{\alpha\beta}}{l^2 - m^2}\right]\left[D^{\beta\nu}(k)\right]$$

→ and together we get:

$$M = M_1 + M_2 \equiv D^{\mu\alpha}(k)\, \Sigma^{\alpha\beta}(k)\, D^{\beta\nu}(k)$$

$$\Sigma^{\alpha\beta}(k) = e^2 \int \frac{d^4 l}{(2\pi)^4} \frac{(2l+k)^\alpha (2l+k)^\beta - 2g^{\alpha\beta}\left((l+k)^2 - m^2\right)}{\left[(l+k)^2 - m^2\right]\left[l^2 - m^2\right]}$$

→ Feynman parametrisation:

$$\frac{1}{\left[(l+k)^2 - m^2\right]\left[l^2 - m^2\right]} = \int_0^1 dx \frac{1}{(l^2 + 2x\, l\cdot k + x k^2 - m^2)^2}$$

$$= \int_0^1 dx \frac{1}{(q^2 - \Delta^2)^2}$$

with: $q^\mu = l^\mu + x k^\mu$ $\qquad d^4 l = d^4 q$

$$\Delta^2 = m^2 - x(1-x)k^2$$

with this notation (after shift), we get:

$$\Sigma^{\alpha\beta}(k) = e^2 \int \frac{d^4 q}{(2\pi)^4} \int_0^1 dx \frac{(NUM)^{\alpha\beta}}{(q^2 - \Delta^2)^2}$$

where:

$$(NUM)^{\alpha\beta} = 4q^\alpha q^\beta + (1-2x)^2 k^\alpha k^\beta + 2(1-2x)\underbrace{\cancel{[q^\alpha k^\beta + q^\beta k^\alpha]}}_{\to 0}$$

$$-2g^{\alpha\beta}\left[q^2 + (1-x)^2k^2 + \underbrace{\cancel{2(1-x)q\cdot k}}_{\to 0} - m^2\right]$$

// note that $\int d^D q\, q^\alpha/(\ldots) = 0$ /

→ how to deal with tensor integrals?

$$\int \frac{d^D q}{(2\pi)^D}\frac{q^\alpha q^\beta}{(DEN)} = g^{\alpha\beta}\int\frac{d^D q}{(2\pi)^D}\frac{q^2}{(DEN)}\cdot\frac{1}{D} \qquad \left[\text{so: } q^\alpha q^\beta = g^{\alpha\beta}\frac{q^2}{D} \text{ under integral sign}\right]$$

→ which leads to:

$$(NUM)^{\alpha\beta} = g^{\alpha\beta}\left[4q^2/D - 2q^2 - 2(1-x)^2k^2 + 2m^2\right]$$

$$+ k^\alpha k^\beta\left[(1-2x)^2\right]$$

$$= q^2\left[g^{\alpha\beta}\left(\tfrac{4}{D}-2\right)\right] + \left[\left(-2(1-x)^2k^2 + 2m^2\right)g^{\alpha\beta} + k^\alpha k^\beta(1-2x)^2\right]$$

→ so that:

$$\Sigma^{\alpha\beta}(k) = e^2\int_0^1 dx \quad g^{\alpha\beta}\left(\tfrac{4}{D}-2\right)I_1(\Delta) + \left[\left(-2(1-x)^2k^2+2m^2\right)g^{\alpha\beta} + k^\alpha k^\beta(1-2x)^2\right]I_2(\Delta)$$

with:

$$I_1(\Delta) = \int\frac{d^D q}{(2\pi)^4}\frac{q^2}{(q^2-\Delta^2)^2} = \frac{(-1)\,i}{(4\pi)^{D/2}}\frac{D}{2}\frac{\Gamma(1-D/2)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{1-D/2}$$

$$I_2(\Delta) = \int\frac{d^D q}{(2\pi)^4}\frac{1}{(q^2-\Delta^2)^2} = \frac{(+1)\,i}{(4\pi)^{D/2}}\frac{\Gamma(2-D/2)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-D/2}$$

$$\boxed{D = 4-2\epsilon}$$

$$\cancel{\Sigma^{\alpha\beta}(k) = \frac{ie^2}{(4\pi)^{D/2}}\int_0^1 dx\left\{-g^{\alpha\beta}\frac{D}{2}\frac{\Gamma(1-D/2)}{1}\left(\frac{1}{\Delta}\right)^{1-D/2}\right.}$$

note that:

$$\left(\frac{4}{D}-2\right)I_1(\Delta) = 2\left(\frac{2}{D}-1\right)\frac{-i}{(4\pi)^{D/2}}\,\frac{D}{2}\,\frac{\Gamma(1-D/2)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{1-D/2}$$

$$=(-2\Delta)\,\frac{i}{(4\pi)^{D/2}}\left(1-\frac{D}{2}\right)\frac{\Gamma(1-D/2)}{\Gamma(2)}\cdot\left(\frac{1}{\Delta}\right)^{2-D/2}$$

$$=(-2\Delta)\,I_2(\Delta)$$

using: $z\cdot\Gamma(z)=\Gamma(z+1)$

$$\Sigma^{\alpha\beta}(k) = e^2\int_0^1 dx\,\Big[-2(m^2-x(1-x)k^2)g^{\alpha\beta} + (2m^2-2(1-x)^2k^2)g^{\alpha\beta} + k^\alpha k^\beta(1-2x)^2\Big]I_2(\Delta)$$

$$= e^2\int_0^1 dx\,\Big[-2(1-x)(1-2x)k^2 g^{\alpha\beta} + k^\alpha k^\beta (1-2x)^2\Big]I_2(\Delta)$$

// note:

$$\int_0^1 dx\,[-2(1-x)(1-2x)] = -\int_0^1 dx\Big[(1-2x)^2+(1-2x)\Big]$$

$$= -\int_0^1 dx\,(1-2x)^2 + 0$$ //

$$= e^2\int_0^1 dx\,(1-2x)^2\Big[-k^2 g^{\alpha\beta}+k^\alpha k^\beta\Big]I_2(\Delta)$$

$$\cancel{= \frac{ie^2}{16\pi^2}\,(4\pi)^\epsilon\Big[\frac{1}{\epsilon}-\gamma_E+O(\epsilon)\Big]\int_0^1 dx\,(1-2x)^2}$$

$$= (k^2 g^{\alpha\beta}-k^\alpha k^\beta)\underbrace{\Big[-e^2\int_0^1 dx\,(1-2x)^2 I_2(\Delta)\Big]}_{\Pi_{1\text{-loop}}(k^2)}$$

→ evaluation of $\Pi_{1loop}(k^2)$:

$$\Pi_{1loop}(k^2) = \frac{-ie^2\,(4\pi)^\epsilon}{16\pi^2}\Big[\frac{1}{\epsilon}-\gamma_E+O(\epsilon)\Big]\int_0^1 dx\,\frac{(1-2x)^2}{(m^2-x(1-x)k^2)^\epsilon}$$

not nice!

(MASS-SQUARED)$^\epsilon$

4.4

→ remember that we are no longer in 4-dim, coupling has dimension!

$$[e^2] = 4 - \text{dim} = +2\epsilon$$

$$e^2 \to \tilde{e}^2 \mu^{2\epsilon}$$

$$\Pi_{1loop}(k^2) = \frac{-i\tilde{e}^2}{16\pi^2}\left[\frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon^1)\right]\int_0^1 dx \frac{(1-2x)^2}{(1-x(1-x)\xi)^\epsilon} \cdot \left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon$$

$\xi \equiv \frac{k^2}{m^2}$

factor of 1/3, because $\int_0^1 dx\,(1-2x)^2 = \frac{1}{3}$

$$i\Pi_{1loop}(k^2) = \frac{-1}{12\pi}\left(\frac{\tilde{e}^2}{4\pi}\right)\left[\frac{1}{\epsilon} - 3\int_0^1 dx\,(1-2x)^2 \log(1-x(1-x)\xi) + \log\left(\frac{\mu^2}{m^2}\right) + \mathcal{O}(\epsilon^2) - \gamma_E + \log(4\pi)\right]$$

$\cancel{\frac{\alpha}{12\pi}}$

~~counter-term~~ ~~($\overline{MS}$)~~: $\cancel{\delta_3 = -\frac{\alpha}{12\pi}\left[\frac{1}{\epsilon} - \gamma_E + \log(4\pi)\right]}$

$$\boxed{I(\xi) = -3\int_0^1 dx\,(1-2x)^2\log(1-x(1-x)\xi)}$$

$\cancel{i\Pi_{1loop}(k^2)_{REN} = i\Pi_{1loop}(k^2) - \delta_3}$

Renormalization condition: $\Pi(0) = 0$

pole at $k^2=0$, residue $= 1$

then: $$\Pi_{REN}(k^2) = \Pi(k^2) - \Delta_3$$

$$\Rightarrow \Delta_3 = \Pi(0)$$
$$= -\frac{\alpha}{12\pi}\left[\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu^2}{m^2}\right) + \underbrace{I(0)}_{=0}\right]$$

Correction is:

$$\Pi_{REN}(k^2) = +\frac{\alpha}{4\pi}\int_0^1 dx\,(1-2x)^2 \log(1-x(1-x)\xi)$$

$\xi = k^2/m^2$

(4C) → in the limit $|k^2| \gg m^2$ $(\xi \longrightarrow -\infty)$ we obtain:

$$\Pi_{REN}(k^2) = + \frac{\alpha}{4\pi} \int_0^1 dx \, (1-2x)^2 \log\left(x(1-x)(-\xi)\right)$$

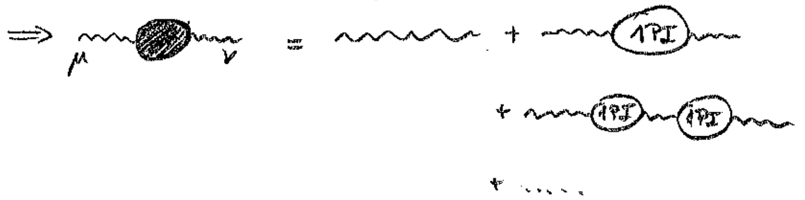
$$= \frac{\alpha}{4\pi} \left[\frac{1}{3} \log(-\xi) - \frac{8}{9} \right]$$

$$= \frac{\alpha}{12\pi} \left[\log\left(\frac{-k^2}{m^2}\right) - \frac{8}{3} \right]$$

→ effective coupling:

Ward identities tell us: $q_\mu \Pi^{\mu\nu}(q) = 0$

then we expect that: $\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$



$$= D^{\mu\nu}(q) + D^{\mu\rho}(q)\left[(q^2 g^{\rho\sigma} - q^\rho q^\sigma)\Pi(q^2)\right] D^{\sigma\nu}(q)$$
$$+ D^{\mu\rho}(q)\left[(q^2 g^{\rho\sigma} - q^\rho q^\sigma)\Pi(q^2)\right]\left[(q^2 g^{\sigma\kappa} - q^\sigma q^\kappa)\Pi(q^2)\right] D^{\kappa\nu}(q)$$
$$+ \dots$$

$$= \frac{D^{\mu\nu}(q)}{1 - \Pi(q^2)}$$

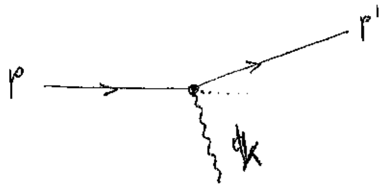
$$\alpha_0^{\,2} \longrightarrow \alpha^2_{eff}(q^2) = \left(\frac{\alpha_0}{1-\Pi(q^2)}\right)^2$$

where $\alpha_{eff}(q^2) = \dfrac{\alpha_0}{1-\Pi(q^2)}$

$$\Longrightarrow \frac{\alpha_0}{\alpha_{eff}(q^2)} = 1 - \frac{\alpha_0}{12\pi}\left[\log\left(\frac{-k^2}{m^2}\right) - \frac{8}{3}\right]$$

→ Comment on "$-k^2$"

- consider a scattering of a particle:



$q = p - p'$

- for simplicity work in high-energy limit and neglect masses:

$$p = (E, 0, 0, +E)$$
$$p' = (E', 0, E'\sin\theta, E'\cos\theta)$$

then:

$$k = (p - p') = (E - E', 0, -E'\sin\theta, E - E'\cos\theta)$$

$$k^2 = (E-E')^2 - (-E'\sin\theta)^2 - (E - E'\cos\theta)^2$$
$$= -2EE' + 2EE'\cos\theta$$
$$= 2EE'(\cos\theta - 1) \leq 0$$

momentum k is SPACE-LIKE: $k^2 < 0$