AQFT: Problem Sheet 3

February 27, 2019

Problem 1

a) We have

$$D'_\mu \Phi' = \partial_\mu (U\Phi U^\dagger) - ig[UA_\mu U^\dagger + \frac{i}{g} U\partial_\mu U^\dagger, U\Phi U^\dagger] \,, \tag{1}$$

$$= U(\partial_\mu \Phi)U^\dagger - igU[A_\mu, \Phi]U^\dagger + (\partial_\mu U)\Phi U^\dagger + U(\partial_\mu U^\dagger)U\Phi U^\dagger \,, \tag{2}$$

$$= U(D_\mu \Phi)U^\dagger \,, \tag{3}$$

where in going to the last line we have used $\partial_\mu(U^\dagger U) \Rightarrow (\partial_\mu U)U^\dagger = -U\partial_\mu U^\dagger$.

b)

$$\begin{aligned} D_\mu D_\nu \Phi \;&=\; D_\mu(\partial_\nu \Phi \;-\; ig[A_\nu, \Phi]) \;=\; \partial_\mu\big(\partial_\nu \Phi \;-\; ig[A_\nu, \Phi]\big) \;-\; ig\big[A_\mu, (\partial_\nu \Phi \;-\; ig[A_\nu, \Phi])\big] \\ &=\; \partial_\mu \partial_\nu \Phi \;-\; ig[(\partial_\mu A_\nu), \Phi] \;-\; ig[A_\nu, \partial_\mu \Phi] \;-\; ig[A_\mu, \partial_\nu \Phi] \;-\; g^2[A_\mu, [A_\nu, \Phi]]. \end{aligned} \tag{4}$$

Similarly,

$$D_\nu D_\mu \Phi \;=\; \partial_\nu \partial_\mu \Phi \;-\; ig[(\partial_\nu A_\mu), \Phi] \;-\; ig[A_\mu, \partial_\nu \Phi] \;-\; ig[A_\nu, \partial_\mu \Phi] \;-\; g^2[A_\nu, [A_\mu, \Phi]]. \tag{5}$$

The difference between these two formulas cancels several terms living behind

$$\begin{aligned} [D_\mu, D_\nu]\Phi \;&=\; -ig[(\partial_\mu A_\nu), \Phi] \;+\; ig[(\partial_\nu A_\mu), \Phi] \;-\; g^2[A_\mu, [A_\nu, \Phi]] \;+\; g^2[A_\nu, [A_\mu, \Phi]] \\ &=\; -ig[(\partial_\mu A_\nu - \partial_\nu A_\mu), \Phi] \;-\; g^2[[A_\mu, A_\nu], \Phi] \\ &=\; -ig[F_{\mu\nu}, \Phi], \end{aligned} \tag{6}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ as usual.

c)

$$\begin{aligned} D_\lambda F_{\mu\nu} \;&=\; \partial_\lambda\big(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\big) \;-\; ig\big[A_\lambda, (\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])\big] \\ &=\; (\partial_\lambda \partial_\mu A_\nu - \partial_\lambda \partial_\nu A_\mu) \;-\; ig\big([A_\mu, \partial_\lambda A_\nu] - [A_\nu, \partial_\lambda A_\mu]\big) \\ &\quad -\; ig\big([A_\lambda, \partial_\mu A_\nu] - [A_\lambda, \partial_\nu A_\mu]\big) \;-\; g^2[A_\lambda, [A_\mu, A_\nu]]. \end{aligned} \tag{7}$$

For each group of terms here, summing over cyclic permutation of the Lorentz indices $\lambda \to \mu \to \nu \to \lambda$ produces a zero:

$$(\partial_\lambda \partial_\mu A_\nu - \partial_\lambda \partial_\nu A_\mu) \;+\; (\partial_\mu \partial_\nu A_\lambda - \partial_\mu \partial_\lambda A_\nu) \;+\; (\partial_\nu \partial_\lambda A_\mu - \partial_\nu \partial_\mu A_\lambda) \;=\; 0,$$

$$
\begin{aligned}
\big([A_\mu,\partial_\lambda A_\nu]-[A_\nu,\partial_\lambda A_\mu]\big)\;+\;\big([A_\nu,\partial_\mu A_\lambda]-[A_\lambda,\partial_\mu A_\nu]\big)\;+\;\big([A_\lambda,\partial_\nu A_\mu]-[A_\mu,\partial_\nu A_\lambda]\big) &= 0,\\
\big([A_\lambda,\partial_\mu A_\nu]-[A_\lambda,\partial_\nu A_\mu]\big)\;+\;\big([A_\mu,\partial_\nu A_\lambda]-[A_\mu,\partial_\lambda A_\nu]\big)\;+\;\big([A_\nu,\partial_\lambda A_\mu]-[A_\nu,\partial_\mu A_\lambda]\big) &= 0,\\
[A_\lambda,[A_\mu,A_\nu]]\;+\;[A_\mu,[A_\nu,A_\lambda]]\;+\;[A_\nu,[A_\lambda,A_\mu]] &= 0,
\end{aligned}
\tag{8}
$$

and consequently

$$
D_\lambda F_{\mu\nu}\;+\;D_\mu F_{\nu\lambda}\;+\;D_\nu F_{\lambda\mu}\;=\;0. \tag{9}
$$

d) Let $A^\mu(x)\to A^\mu(x)+\delta A^\mu(x)$; then the first variation of $F^{\mu\nu}(x)$ is

$$
\begin{aligned}
\delta_a\left(F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu-ig[A^\mu,A^\nu]\right) &= \partial^\mu\delta A^\nu\;-\;\partial^\nu\delta A^\mu\;-\;ig[A^\mu,\delta A^\nu]\;-\;ig[\delta A^\mu,A^\nu]\\
&= D^\mu\delta A^\nu\;-\;D^\nu\delta A^\mu
\end{aligned}
\tag{10}
$$

and consequently,

$$
\delta_1\mathcal{L}_{\rm YM}\;=\;-2\,\mathrm{Tr}\left(F_{\mu\nu}\,D^\mu\delta A^\nu\right). \tag{11}
$$

Next, consider a total derivative of general form

$$
\begin{aligned}
&\mathrm{Tr}\big((D^\mu B)C\big)\;+\;\mathrm{Tr}\big(B(D^\mu C)\big)\;=\;\mathrm{Tr}\big(D^\mu(BC)\big)\;\;,\\
&=\;\mathrm{Tr}\big(\partial^\mu(BC)-i[A^\mu,BC]\big)\;=\;\partial^\mu\big(\mathrm{Tr}(BC)\big)\;+\;0,
\end{aligned}
\tag{12}
$$

which allows us to integrate by parts traces involving gauge-covariant derivatives. Thus, eq. (11) can be written as

$$
\delta_1\mathcal{L}_{\rm YM}\;=\;2\,\mathrm{Tr}\left((D^\mu F_{\mu\nu})\delta A^\nu\right)\;+\;\partial^\mu(\cdots)\;=\;(D^\mu F_{\mu\nu})^a\,\delta A^{a\nu}\;+\;\partial^\mu(\cdots) \tag{13}
$$

and consequently

$$
\frac{\delta}{\delta A^{a\nu}(x)}\left[S_{YM}=\int\mathcal{L}_{\rm YM}d^4x\right]=(D^\mu F_{\mu\nu})^a, \tag{14}
$$

which immediately implies the classic field equation of motion

$$
D^\mu F_{\mu\nu}\;=\;0. \tag{15}
$$

e) Eq. (12) follows immediately from the previous result and writing the new Lagrangian (11) as

$$
\mathcal{L}\;=\;\mathcal{L}_{\rm YM}\;+\;\overline{\psi}(i\partial\!\!\!/-m)\psi\;+\;gA^{a\nu}J^a_\nu\,. \tag{16}
$$

f):

$$
D_\nu J^\nu\;\propto\;D_\nu(D_\mu F^{\mu\nu})\;=\;-\frac{1}{2}[D_\mu,D_\nu]F^{\mu\nu}\;=\;\frac{i}{2}[F_{\mu\nu},F^{\mu\nu}]\;=\;0. \tag{17}
$$

where the first equality follows from $F^{\mu\nu}=-F^{\nu\mu}$, the second from previous result (b), and the third from the fact that $F^{\mu\nu}$ commutes with itself.

g) The Dirac equations following from the Lagrangian (11) are

$$
(iD\!\!\!\!/-m)\psi\;=\;0,\qquad\overline{\psi}(-i\overleftarrow{D\!\!\!\!/}-m)\;=\;0, \tag{18}
$$

or

$$\begin{aligned} i\gamma^\mu\partial_\mu\psi \ +\ gA^a_\mu\gamma^\mu T^a\psi \ -\ m\psi \ &=\ 0,\\ -i\partial_\mu\bar\psi\gamma^\mu \ +\ gA^a_\mu\bar\psi\gamma^\mu T^a \ -\ m\bar\psi \ &=\ 0. \end{aligned} \tag{19}$$

Consequently, the fermionic current

$$J^{a\nu} \ =\ \bar\psi\gamma^\nu T^a\psi \tag{20}$$

(for fermions in the fundamental representation of the gauge group $T^a = \frac{1}{2}\lambda^a$) satisfies

$$\begin{aligned} \partial_\nu J^{a\nu} \ &=\ (\partial_\nu\bar\psi\gamma^\nu)T^a\psi \ +\ \bar\psi T^a(\gamma^\nu\partial_\nu\psi)\\ &=\ \bar\psi(im - ig\gamma^\nu T^b A^b_\nu)\,T^a\psi \ +\ \bar\psi T^a(-im + ig\gamma^\nu T^b A^b_\nu)\psi\\ &=\ -igA^b_\nu\times\bar\psi\gamma^\nu[T^b,T^a]\psi \ =\ -gf^{abc}A^b_\nu\bar\psi\gamma^\nu T^c\psi\\ &\equiv\ -gf^{abc}A^b_\nu J^{c\nu}, \end{aligned} \tag{21}$$

and therefore

$$D_\nu J^{a\nu} \ =\ \partial_\nu J^{a\nu} \ + gf^{abc}A^b_\nu J^{c\nu} \ =\ 0. \tag{22}$$

h) Finally, consider the second variation of the YM action. The first variation of the YM tension $F^{\mu\nu}$ is given by eq. (10) while the second variation is simply

$$\delta_2 F^{\mu\nu} \ =\ -ig[\delta A^\mu, \delta A^\nu]. \tag{23}$$

Therefore,

$$\begin{aligned} \delta_2\left[\frac{1}{2}\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu})\right] \ &=\ \frac{1}{2}\mathrm{Tr}\,(\delta_1 F_{\mu\nu}\,\delta_1 F^{\mu\nu}) \ +\ \mathrm{Tr}\,(F_{\mu\nu}\,\delta_2 F^{\mu\nu})\\ &=\ \frac{1}{2}\mathrm{Tr}\left((D^\mu\delta A^\nu - D^\nu\delta A^\mu)^2\right) \ -\ ig\mathrm{Tr}\,(F_{\mu\nu}\,[\delta A^\mu,\delta A^\nu])\\ &=\ \mathrm{Tr}\,((D_\mu\delta A_\nu)(D^\mu\delta A^\nu)) \ -\ \mathrm{Tr}\,((D_\mu\delta A_\nu)(D^\nu\delta A^\mu))\\ &\qquad -\ 2ig\mathrm{Tr}\,(F_{\mu\nu}\,\delta A^\mu\delta A^\nu)\,, \end{aligned} \tag{24}$$

which may be further simplified by discarding total derivatives *à la* eq. (12) in anticipation of integration $\int d^4x$. Indeed, the first term on the last line of eq. (24) becomes

$$\mathrm{Tr}\,((D_\mu\delta A_\nu)(D^\mu\delta A^\nu)) \ =\ \partial_\mu(\cdots) \ -\ \mathrm{Tr}\left(\delta A_\nu\, D^2\delta A^\nu\right), \tag{25}$$

while for the second term we have

$$\begin{aligned} -\mathrm{Tr}\,((D_\mu\delta A_\nu)(D^\nu\delta A^\mu)) \ &=\ \partial_\mu(\cdots) \ +\ \mathrm{Tr}\,(\delta A_\nu\, D_\mu D^\nu\delta A^\mu)\\ &=\ \partial_\mu(\cdots) \ +\ \mathrm{Tr}\,(\delta A_\nu\, D^\nu D_\mu\delta A^\mu) \ +\ \mathrm{Tr}\,(\delta A_\nu\,[D^\mu,D^\nu]\delta A_\mu)\\ &=\ \partial_\mu(\cdots) \ -\ \mathrm{Tr}\,((D^\nu\delta A_\nu)\,(D_\mu\delta A^\mu)) \ -\ ig\mathrm{Tr}\,(\delta A_\nu\,[F^{\mu\nu},\delta A_\mu])\\ &=\ \partial_\mu(\cdots) \ -\ \mathrm{Tr}\left((D^\nu\delta A_\nu)^2\right) \ -\ 2ig\mathrm{Tr}\,(F^{\mu\nu}\,\delta A_\mu\delta A_\nu)\,. \end{aligned} \tag{26}$$

Consequently,

$$\delta_2\left[\frac{1}{2}\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu})\right] \ =\ -\mathrm{Tr}\left(\delta A_\nu\, D^2\delta A^\nu\right) - \mathrm{Tr}\left((D^\nu\delta A_\nu)^2\right) - 4ig\mathrm{Tr}\,(F^{\mu\nu}\,\delta A_\mu\delta A_\nu) + \partial_\mu(\cdots) \tag{27}$$

and hence

$$\delta_2\left[\int d^4x\mathcal{L}_{\rm YM}\right] \;=\; \frac{1}{g^2}\int d^4x\left[{\rm Tr}\left(\delta A^\mu D^2\delta A_\mu\right) \;+\; {\rm Tr}\left((D_\mu\delta A^\mu)^2\right) \;+\; 4ig{\rm Tr}\left(F_{\mu\nu}\delta A^\mu\delta A^\nu\right)\right]. \tag{28}$$

Problem 2

(i) The gauge fixing function is given by

$$f(A^a) = n^\mu A^a_\mu - \sigma^a \;, \tag{29}$$

and the path integral is

$$Z = \int \mathcal{D}Ae^{iS[A]}\Delta(A)\delta(f(A)) \;, \tag{30}$$

where $\Delta(A)$ is Faddeev–Popov determinant, with

$$[\Delta(A)]^{-1} = \int \mathcal{D}\theta\delta(f(A_\theta)) \;. \tag{31}$$

The field A^a_μ transforms under a gauge transformation as

$$A^a_\mu \to A^a_\mu - (\delta^{ac}\partial_\mu\theta^c - gf^{abc}\theta^b A^c_\mu) \;. \tag{32}$$

For the axial gauge fixing function we have

$$f(A_\theta) = n^\mu A^a_\mu - \sigma^a - n^\mu(\partial_\mu\theta^a - gf^{abc}\theta^b A^c_\mu) \;, \tag{33}$$

$$\to n^\mu(\partial_\mu\theta^a - gf^{abc}\theta^b A^c_\mu) \;, \tag{34}$$

where in the second line we drop the $f(A)$ term as usual. We are then interested in the operator

$$K^{ab}(x,y) = n^\mu(gf^{abc}A^c_\mu - \delta^{ab}\partial_\mu)\delta^4(x-y) \;, \tag{35}$$

so that

$$\Delta(A) = \det(K) = \int \mathcal{D}c\mathcal{D}\overline{c}e^{iS_{\rm ghost}} \;, \tag{36}$$

with

$$S_{\rm ghost} = -\int {\rm d}^4x{\rm d}^4y\overline{c}^a(x)K^{ab}(x,y)c^b(y) \;, \tag{37}$$

and $c, \overline{c}$ are the usual ghost, antighost fields. We have

$$\mathcal{L}_{\rm ghost} = -\int {\rm d}^4y\overline{c}^a(x)n^\mu(gf^{abc}A^c_\mu - \delta^{ab}\partial_\mu)\delta^4(x-y)c^b(y) \;, \tag{38}$$

$$= -\overline{c}^a(x)n^\mu(gf^{abc}A^c_\mu - \delta^{ab}\partial_\mu)c^b(x) \;, \tag{39}$$

$$= \overline{c}^a(x)n^\mu\partial_\mu c^a(x) - gf^{abc}n^\mu A^c_\mu\overline{c}^a(x)c^b(x) \;. \tag{40}$$

Finally, the weighting function $\exp[-\frac{i}{2\xi}\int {\rm d}^4x\omega^a\omega^a]$ gives a gauge–fixing term

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(n^\mu A^a_\mu)^2 \;. \tag{41}$$

(ii) We identify the second term in (40) with the ghost–antighost–gluon vertex, and read off that the Feynman rule should be to assign a factor

$$\propto g f^{abc} n^\mu \ . \tag{42}$$

To calculate the gluon propagator, we identify the relevant terms in the Lagrangian

$$\mathcal{L} \ni \frac{1}{2} A^a_\mu \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu - \frac{1}{\xi} n^\mu n^\nu \right) A^a_\nu \ . \tag{43}$$

We are therefore interested in inverting

$$P^{\mu\nu}(p) = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} + \frac{1}{\xi} \frac{n^\mu n^\nu}{p^2} \ , \tag{44}$$

where the relative minus sign in the last term comes from the $\partial^2 \to -p^2$ ($\partial_\mu \partial_\nu \to -p_\mu p_\nu$) replacements in the first (second) terms, and the colour factor δ^{ab} inverts trivially, so we leave this out for now. To invert, we can write in the most general Lorentz covariant form

$$P^{-1}_{\mu\sigma} = A g_{\mu\sigma} + B n_\mu p_\sigma + C n_\sigma p_\mu + D p_\mu p_\sigma + E n_\mu n_\sigma \ , \tag{45}$$

and then we use $P^{\mu\nu} P^{-1}_{\mu\sigma} = g_\sigma{}^\mu$ to determine the unknown factors A–E. This gives

$$P^{-1}_{\mu\sigma} = g_{\mu\sigma} - \frac{n_\sigma p_\mu + n_\mu p_\sigma}{(p \cdot n)} + \frac{(n^2 + \xi p^2) p_\mu p_\sigma}{(p \cdot n)^2} \ , \tag{46}$$

which gives for the propagator

$$i \tilde{\Delta}^{\mu\nu}_{ab}(p) = \frac{i}{p^2 + i\epsilon} \left(g^{\mu\nu} - \frac{n^\mu p^\nu + n^\nu p^\mu}{(n \cdot p)} + \frac{(n^2 + \xi p^2) p^\mu p^\nu}{(p \cdot n)^2} \right) \delta^{ab} \ . \tag{47}$$

(iii) In any internal ghost loop this will always couple to a gluon propagator. In the light cone gauge we will then have a term

$$g f^{abc} n_\nu \left(g^{\mu\nu} - \frac{n^\mu p^\nu + n^\nu p^\mu}{(n \cdot p)} \right) \ , \tag{48}$$

$$\propto n^\mu - \frac{n^\mu (n \cdot p) + n^2 p^\mu}{n \cdot p} \ , \tag{49}$$

$$= 0 \ , \tag{50}$$

for $n^2 = 0$. Therefore ghost contributions will always vanish.

Problem 3: $gg \to t\bar{t}$

Production of heavy quarks plays an important role in phenomenology of hadronic collisions. In this problem we want to take a closer look at $gg \to t\bar{t}$ scattering. At tree level we have only three diagrams contributing - see Fig.1. We can use standard Feynman rules for QCD to describe each one of them. Squared matrix element reads:

$$|\mathcal{M}|^2 = (\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u)^* (\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u)$$

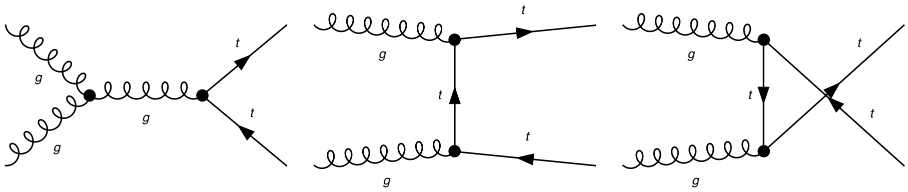

Figure 1. Diagrams contributing to $gg \to t\bar{t}$ at tree-level. They describe exchange of momenta in s, t and u channels: $\mathcal{M}_s$,$\mathcal{M}_t$ and $\mathcal{M}_u$

$$= |\mathcal{M}_s|^2 + |\mathcal{M}_t|^2 + |\mathcal{M}_u|^2 + 2\left(\mathcal{M}_s^*\mathcal{M}_t + \mathcal{M}_s^*\mathcal{M}_u + \mathcal{M}_t^*\mathcal{M}_u\right) \tag{51}$$

i)
The colour factors of each combination of Fig. 1 are

$$\begin{aligned}
[ss] &= \left(-if^{abc}T^c_{ij}\right)\left(if^{abe}T^e_{ji}\right) = f^{abc}f^{abe}\mathrm{Tr}\left(T^cT^e\right) \\
&= \frac{1}{2}N_c\delta_{ce}\delta_{ce} = \frac{1}{2}N_c\left(N_c^2-1\right) = C_AC_FN_c \\
[tt] &= \left(T^aT^b\right)_{ij}\left(T^bT^a\right)_{ij} = \mathrm{Tr}\left(abba\right) = C_F^2N_c \\
[uu] &= C_F^2N_c \\
[st] &= \left(-if^{abc}T^C_{ij}\right)\left(T^bT^a\right)_{ji} = -if^{abc}\mathrm{Tr}\left(cba\right) \\
&= \frac{1}{2}f^{abc}f^{abe}\mathrm{Tr}\left(T^cT^e\right) = \frac{1}{4}N_c(N_c^2-1) = \frac{N_c}{2}C_AC_F \\
[su] &= \frac{N_c}{2}C_AC_F \\
[tu] &= \left(T^aT^b\right)_{ij}\left(T^aT^b\right)_{ij} = \mathrm{Tr}\left(abab\right) = \frac{1}{2}C_F
\end{aligned} \tag{52}$$

To average over the initial state gluons we need to divide the whole result by a factor:

$$\left(\frac{1}{2}\cdot\frac{1}{N_c^2-1}\right)^2 \tag{53}$$

where we have taken into account number of gluon polarizations $(+/-)$ and colours (for $SU(N_c)$ we have (N_c^2-1) gauge fields in adjoint representation).

ii) For the $g(k_1)g(k_2) \to t(p_1)\bar{t}(p_2)$ process, the amplitude with a s–channel gluon is

$$\mathcal{M}_s = \frac{g^2}{s}\epsilon^\mu(k_1)\epsilon^\nu(k_2)(if^{abc}T^c)\overline{u}(p_1)\gamma^\beta v(p_2)\tilde{C}_{\mu\nu\beta}\ , \tag{54}$$

with

$$\tilde{C}_{\mu\nu\beta} = g_{\mu\nu}(k_2-k_1)_\beta + 2g_{\beta\mu}k_{1\nu} - 2g_{\nu\beta}k_{2\mu}\ , \tag{55}$$

where $k\cdot\epsilon(k) = 0$ has been used.

For non-abelian gauge theory sum over polarizations of external gauge bosons can be written as:

$$\sum_{\sigma=+/-} \varepsilon^{\mu}(k,\sigma)\varepsilon^{\nu}(k,\sigma) = -g^{\mu\nu} + \frac{k^{\mu}n^{\nu} + k^{\nu}n^{\mu}}{k \cdot n} \tag{56}$$

where n^{μ} is a four vector satisfying $n \cdot \varepsilon(k,\sigma) = 0$, $n^2 = 0$ and $k \cdot n \neq 0$ (as it cannot be simply proportional to k^{μ}) If the calculation is done properly and the result is gauge invariant the η-dependence cancels out. In fact with the s–channel diagram written in the form (55) we have

$$k_1^{\mu} M_{\mu\nu} = k_2^{\nu} M_{\mu\nu} = 0 \;, \tag{57}$$

when summed over all three diagram contributions; hence, the terms $\sim k_{1,2}$ in the spin sum will cancel when summed over all three diagrams. We therefore drop these (i.e. keep only the $\sim g^{\mu\nu}$ contribution) to get

$$|\mathcal{M}_s|^2 = \frac{g^2}{4s^2}\left(\frac{N_c}{2(N_c^2-1)}\right)\tilde{C}_{\mu\nu\alpha}\tilde{C}^{\mu\nu\beta}\mathrm{Tr}[\gamma_\beta(\not{p}_2 - m_t)\gamma^\alpha(\not{p}_1 + m_t)] \;, \tag{58}$$

$$= \frac{4g^2}{s^2}\left(\frac{N_c}{2(N_c^2-1)}\right)(m_t^2 - t)(m_t^2 - u) \;, \tag{59}$$

where m_t is the top quark mass.

(iii) For the t–channel diagram we have

$$ik_{1\mu}k_{2\nu}\mathcal{M}_t^{\mu\nu} = -ig^2 T^a T^b \overline{u}(p_1)\frac{\not{k}_1(\not{p}_1 - \not{k}_1)\not{k}_2}{(p_1 - k_1)^2}v(p_2) \;, \tag{60}$$

$$= \frac{ig^2 T^a T^b}{2(p_1 \cdot k_1)}\overline{u}(p_1)\not{k}_1(\not{p}_1 - \not{k}_1)\not{k}_2 v(p_2) \;. \tag{61}$$

From quoted result, have $\not{k}_1\not{k}_1 = k_1^2 = 0$, thus

$$ik_{1\mu}k_{2\nu}\mathcal{M}_t^{\mu\nu} = \frac{ig^2 T^a T^b}{2(p_1 \cdot k_1)}\overline{u}(p_1)\not{k}_1\not{p}_1\not{k}_2 v(p_2) \;. \tag{62}$$

Anti–commute p_1 and k_1:

$$ik_{1\mu}k_{2\nu}\mathcal{M}_t^{\mu\nu} = \frac{ig^2 T^a T^b}{2(p_1 \cdot k_1)}\left(2(p_1 \cdot k_1)\overline{u}(p_1)\not{k}_2 v(p_2) - \overline{u}(p_1)\not{p}_1\not{k}_1\not{k}_2 v(p_2)\right) \;. \tag{63}$$

The second term vanishes due to the Dirac equation, $\overline{u}(p_1)\not{p}_1 = 0$, and thus we have

$$ik_{1\mu}k_{2\nu}\mathcal{M}_t^{\mu\nu} = ig^2 T^a T^b \overline{u}(p_1)\not{k}_2 v(p_2) \;. \tag{64}$$

For the u–channel diagram we have

$$ik_{1\mu}k_{2\nu}\mathcal{M}_u^{\mu\nu} = -ig^2 T^b T^a \overline{u}(p_1)\frac{\not{k}_2(\not{p}_1 - \not{k}_2)\not{k}_1}{(p_1 - k_2)^2}v(p_2) \;, \tag{65}$$

$$= \frac{ig^2 T^b T^a}{2(p_1 \cdot k_2)}\overline{u}(p_1)\not{k}_2(\not{p}_1 - \not{k}_2)\not{k}_1 v(p_2) \;, \tag{66}$$

$$= ig^2 T^b T^a \overline{u}(p_1) \not{k}_1 v(p_2) + 0 \ , \tag{67}$$

following same logic as above (alternatively can simply observe that we need to interchange $k_1 \leftrightarrow k_2$ and $a \leftrightarrow b$). Then need

$$\overline{u}(p_1)(\not{k}_1 + \not{k}_2)v(p_2) = \overline{u}(p_1)(\not{p}_1 + \not{p}_2)v(p_2) = 0 \ , \tag{68}$$

to give

$$ik_{1\mu}k_{2\nu}(\mathcal{M}_t^{\mu\nu} + \mathcal{M}_u^{\mu\nu}) = ig^2[T^a, T^b]\overline{u}(p_1)\not{k}_2 v(p_2) \ . \tag{69}$$

Finally calculate s–channel amplitude, with

$$\begin{aligned} ik_{1\mu}k_{2\nu}\mathcal{M}_s^{\mu\nu} &= g^2 \frac{f^{abc}T^c}{(k_1+k_2)^2}\overline{u}(p_1)\gamma_\sigma v(p_2) \\ &\cdot \left[(k_1-k_2)^\sigma (k_1\cdot k_2) + (2k_1+k_2)\cdot k_1 k_2^\sigma - (2k_2+k_1)\cdot k_2 k_1^\sigma\right] \ , \\ &= \frac{g^2 f^{abc}T^c}{(k_1+k_2)^2}\overline{u}(p_1)\gamma_\sigma v(p_2)(k_2-k_1)^\sigma (k_1\cdot k_2) \end{aligned} \tag{70}$$

Using (68) and rearranging we then get

$$ik_{1\mu}k_{2\nu}\mathcal{M}_s^{\mu\nu} = g^2 f^{abc}T^c \overline{u}(p_1)\not{k}_2 v(p_2) \ , \tag{71}$$

and therefore finally summing the three contributions we have

$$k_{1\mu}k_{2\nu}\mathcal{M}^{\mu\nu} = \left([T^a, T^b] - if^{abc}T^c\right) g^2 \overline{u}(p_1)\not{k}_2 v(p_2) \ , \tag{72}$$

as required.