AQFT: Problem Sheet 4

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Problem 1: BRST Transformation

The action of δ_B on some field object $O(x)$ is defined by

$$\delta O(x) = \theta \delta_B O(x) \ . \tag{1}$$

Now, consider the action of δ_B on the product of two arbitrary fields Φ_i:

$$\begin{aligned} \delta(\Phi_1\Phi_2) &= (\delta\Phi_1)\Phi_2 + \Phi_1(\delta\Phi_2) \ , & (2)\\ &= \theta(\delta_B\Phi_1)\Phi_2 + \Phi_1\theta(\delta_B\Phi_2) \ , & (3)\\ &= \theta\left[(\delta_B\Phi_1)\Phi_2 \pm \Phi_1(\delta_B\Phi_2)\right] \ , & (4) \end{aligned}$$

where the '$\pm$' depends on whether the statistics of the field Φ_1 are fermionic (in which case we have a minus sign from anticommuting with the θ) or bosonic (in which case we do not). Thus we have

$$\delta_B(\Phi_1\Phi_2) = (\delta_B\Phi_1)\Phi_2 \pm \Phi_1(\delta_B\Phi_2) \ , \tag{5}$$

Turning to the case at hand, for the matter field ϕ we have

$$\begin{aligned} \delta_B^2\phi_i &= ig\left[(\delta_B c^a)(T^a)_{ij}\phi_j - c^a(T^a)_{ij}(\delta_B\phi_j)\right] \ , & (6)\\ &= ig^2\left[-\frac{1}{2}f^{bca}c^bc^c(T^a)_{ij}\phi_j - ic^ac^b(T^aT^b)_{ik}\phi_k\right] \ , & (7)\\ &= ig^2\left[-\frac{1}{2}f^{bca}c^bc^c(T^a)_{ij}\phi_j - \frac{i}{2}c^ac^b[T^a,T^b]_{ik}\phi_k\right] \ , & (8)\\ &= \frac{ig^2}{2}\left[-f^{bca}c^bc^c(T^a)_{ij}\phi_j + f^{abc}c^ac^b(T^c)_{ij}\phi_j\right] \ , & (9)\\ &= 0 \ , & (10) \end{aligned}$$

For the gauge field A_μ we have

$$\delta_B A_\mu^a(x) = D_\mu^{ab}c^b(x) \ , \tag{11}$$

but we have already shown in Section 17.2 of the notes that the right hand side of this is invariant under a BRST transformation, so that $\delta_B^2 A_\mu^a(x) = 0$ follows immediately from this. For the ghost field we have

$$\delta_B^2 c^c = \frac{1}{4}g^2 f^{abc}\left[(\delta_B c^a)c^b - c^a(\delta_B c^b)\right] \ , \tag{12}$$

$$= \frac{1}{4} g^2 f^{abc} \left[f^{dea} c^d c^e c^b - f^{deb} c^a c^d c^e \right] \ , \quad (13)$$

$$= \frac{1}{2} g^2 f^{abc} f^{dea} c^d c^e c^b \ , \quad (14)$$

which vanishes from the Jacobi identity. Finally, for the antighost field

$$\delta_B^2 \bar{c}^a = -\frac{1}{\xi} \partial^\mu (\delta_B A_\mu^a) = -\frac{1}{\xi} \partial^\mu D_\mu^{ab} c^b \ , \quad (15)$$

which vanished from the EOM of the antighost field (see the notes for further discussion). Having proved that the BRST operator is nilpotent when acting individually on the fields entering the QCD Lagrangian, we can generalise to any arbitrary product of such fields. Starting with the case of two fields, we have

$$\delta_B^2(\Phi_1 \Phi_2) = \delta_B \left[(\delta_B \Phi_1) \Phi_2 \pm \Phi_1 (\delta_B \Phi_2) \right] \ , \quad (16)$$

$$= (\delta_B^2 \Phi_1) \Phi_2 \mp (\delta_B \Phi_1)(\delta_B \Phi_2) \pm (\delta_B \Phi_1)(\delta_B \Phi_2) + \Phi_1 (\delta_B^2 \Phi_2) \ , \quad (17)$$

$$= (\delta_B^2 \Phi_1) \Phi_2 + \Phi_1 (\delta_B^2 \Phi_2) \ , \quad (18)$$

$$= 0 \ , \quad (19)$$

where the last line follows from the results above. This can then be generalised to any arbitrary product of fields:

$$\delta_B^2 (\Phi_1 \Phi_2 \cdots \Phi_n) = 0 \ . \quad (20)$$

Problem 2: Spontaneous Symmetry Breaking (1)

3. (a) (i) We have

$$\exp(i\alpha\gamma_5) = \mathbb{1} + i\alpha\gamma_5 + \frac{(i\alpha)^2}{2!}(\gamma_5)^2 + \frac{(i\alpha)^3}{3!}(\gamma_5)^3 + \cdots \ , \quad (21)$$

$$= \mathbb{1} \left(1 - \frac{\alpha^2}{2!} - \cdots \right) + i\gamma_5 \left(\alpha - \frac{\alpha^3}{3!} + \cdots \right) \ , \quad (22)$$

$$= \mathbb{1} \cos\alpha + i\gamma_5 \sin\alpha \ , \quad (23)$$

as required.
(ii) The interaction term

$$g\overline{\psi}(\phi_1 + i\gamma_5 \phi_2)\psi \ , \quad (24)$$

is the only term in the Lagrangian to combine the Dirac and scalar fields, and therefore demanding invariance of this will dictate the symmetry obeyed by the scalar fields. We note that

$$\psi \to \exp(-\frac{1}{2} i\alpha\gamma_5)\psi \ , \quad (25)$$

implies

$$\overline{\psi} = \left(\exp\left(-\frac{1}{2} i\alpha\gamma_5 \right) \psi \right)^\dagger \gamma_0 = \overline{\psi} \exp\left(-\frac{1}{2} i\alpha\gamma_5 \right) \ , \quad (26)$$

where we have used that $\exp(i\frac{1}{2}\alpha\gamma_5)\gamma_0 = \gamma_0\exp(-i\frac{1}{2}\alpha\gamma_5)$, which follows from (23) and $\{\gamma_\mu,\gamma_5\}=0$. We have also used $\gamma_5^\dagger=\gamma_5$, and will use this in what follows as well. Thus we require

$$\exp\left(-i\frac{1}{2}\alpha\gamma_5\right)(\phi_1'+i\gamma_5\phi_2')\exp\left(-i\frac{1}{2}\alpha\gamma_5\right) = \exp\left(-i\alpha\gamma_5\right)(\phi_1'+i\gamma_5\phi_2') = \phi_1+i\gamma_5\phi_2\ , \tag{27}$$

where the second equality follows from (23). We therefore have

$$(\mathbb{1}\cos\alpha - i\gamma_5\sin\alpha)(\phi_1'+i\gamma_5\phi_2') = \phi_1'\cos\alpha+\phi_2'\sin\alpha+i\gamma_5(-\phi_1'\sin\alpha+\phi_2'\cos\alpha)\ . \tag{28}$$

Inverting the above gives the required transformation

$$\phi_1 \to \phi_1\cos\alpha - \phi_2\sin\alpha\ , \tag{29}$$
$$\phi_2 \to \phi_1\sin\alpha + \phi_2\cos\alpha\ , \tag{30}$$

which corresponds to a $SO(2)$ symmetry, $\phi=(\phi_1,\phi_2)\to O\phi$, where $O^T=O^{-1}$ and $\det(O)=1$.
(iii) To verify that the rest of the Lagrangian is invariant, we first note that all of the remaining scalar terms $\sim\phi^T\phi$ are manifestly invariant under $SO(2)$. Finally, we have

$$\overline{\psi}\gamma_\mu\psi \to \overline{\psi}\exp\left(-\frac{1}{2}i\alpha\gamma_5\right)\gamma_\mu\left(-\frac{1}{2}i\alpha\gamma_5\right)\psi = \overline{\psi}\gamma_\mu\psi\ , \tag{31}$$

from (23) and $\{\gamma_5,\gamma_\mu\}=0$. Thus the fermion kinetic term is invariant.
A Dirac mass term would be of the form

$$m\overline{\psi}\psi \to m\overline{\psi}\exp(-i\alpha\gamma_5)\psi\ , \tag{32}$$

and so this does not preserve the global symmetry.
(iv) We have

$$V(\phi_1,\phi_2) = -\frac{1}{2}\mu^2(\phi_1^2+\phi_2^2)+\frac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2\ . \tag{33}$$

Requiring

$$\left.\frac{\partial V}{\partial\phi_1}\right|_{\phi_1=v_1,\phi_2=v_2} = \left.\frac{\partial V}{\partial\phi_2}\right|_{\phi_1=v_1,\phi_2=v_2} = 0\ , \tag{34}$$

implies that

$$[-\mu^2+\lambda(v_1^2+v_2^2)]v_1 = 0\ , \tag{35}$$
$$[-\mu^2+\lambda(v_1^2+v_2^2)]v_2 = 0\ . \tag{36}$$

This has two solutions

$$v_1=v_2=0 \Rightarrow V(v_1,v_2)=0\ , \tag{37}$$
$$v_1^2+v_2^2=\frac{\mu^2}{\lambda} \Rightarrow V(v_1,v_2) = -\frac{\mu^4}{4\lambda}\ . \tag{38}$$

As $\lambda>0$ it follows that the latter, with $V(v_1,v_2)<0$, is a minimum. The condition

$$\langle 0|\phi_1^2+\phi_2^2|0\rangle = v^2\ , \tag{39}$$

where $v^2 = \mu^2/\lambda$ is clearly not invariant under the rotational $SO(2)$ symmetry, as it picks out a particular point in the ϕ_1, ϕ_2 plane.
Without loss of generality we will assume that $\langle\phi_1\rangle = v$ and $\langle\phi_2\rangle = 0$, with $v = |\mu|/\sqrt{\lambda}$. We then expand around this, with

$$(\phi_1(x), \phi_2(x)) = (v + \sigma(x), \pi(x)) \ , \tag{40}$$

where we have defined new scalar fields $\sigma(x)$ and $\pi(x)$. Inserting into the original Lagrangian and eliminating μ^2 in favour of v, after some algebra we get

$$\mathcal{L} = \frac{1}{2}[\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\pi\partial^\mu\pi] - \lambda v^2\sigma^2 - \lambda v\sigma(\sigma^2+\pi^2) - \frac{1}{4}\lambda(\sigma^2+\pi^2)^2 + i\overline{\psi}\partial\!\!\!/\psi - g\overline{\psi}(v+\sigma+i\gamma_5\pi)\psi \ . \tag{41}$$

Identifying the term

$$\mathcal{L} \ni -gv\overline{\psi}\psi \ , \tag{42}$$

we see that fermion has acquired a mass

$$m_f = gv \ . \tag{43}$$

v) From the term

$$\mathcal{L} \ni -\lambda v^2\sigma^2 \ , \tag{44}$$

we can see that the scalar σ has mass

$$m_\sigma = \sqrt{2\lambda}v \ . \tag{45}$$

There is no mass term for the scalar π, which we therefore identify with the Goldstone boson associated with the symmetry breaking.
The original Lagrangian only contained quartic interaction terms between the scalar fields, however we can see from the term

$$\mathcal{L} \ni \lambda v\sigma(\sigma^2 + \pi^2) \ , \tag{46}$$

that cubic $\sigma\sigma\sigma$ and $\sigma\pi\pi$ interaction vertices, with $g \propto \lambda v$, are now present.

Problem 3: Spontaneous Symmetry Breaking (2)

(i) Minimising the potential, we have find a SSB vev for

$$v = \frac{m}{2\sqrt{\lambda}} \ . \tag{47}$$

The generators of SU(2) in the adjoint representation are given explicitly by

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad T^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad T^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{48}$$

Now, we have

$$Q\Phi = (Y + T_3)\Phi = T_3\Phi \ , \tag{49}$$

as we have $Y = 0$ for the field triplet. Thus the electrically neutral member of the triplet must be an eigenstate of T_3 with zero eigenvalue. In other words, we choose the vev to be

$$\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} , \tag{50}$$

and we have that two generators $T_{1,2}$ are broken while T_3 is unbroken. We therefore expect a $U(1)$ (or more precisely $U(1) \times U(1)_Y$) symmetry after SSB.
Now, we can expand our fields as

$$\Phi = \begin{pmatrix} 0 \\ 0 \\ v+h \end{pmatrix} , \tag{51}$$

where we take the unitary gauge, i.e. we set the Goldstone fields in the first and second rows to zero. Substituting this into the potential we get

$$V(\Phi) = -\frac{m^2}{2}(h+v)^2 + \lambda(h+v)^4 , \tag{52}$$

$$= h^2 \left(-\frac{m^2}{2} + 6\lambda v^2 \right) + O(h^3) + O(h^4) , \tag{53}$$

$$= m^2 h^2 + O(h^3) + O(h^4) , \tag{54}$$

and thus we have

$$m_h = \sqrt{2}m = 2v\sqrt{2\lambda} . \tag{55}$$

b) For the gauge boson masses we are interested in

$$g^2 \Phi_j (T^a T^b)_{jk} \Phi_k \to g^2 v_j (T^a T^b)_{jk} v_k \equiv (M^2)_{ab} . \tag{56}$$

We know that T_3 acting on $\mathbf{v}$ is zero, as is Y, so we are only interested in

$$T^1 \mathbf{v} = \begin{pmatrix} 0 \\ -iv \\ 0 \end{pmatrix} \qquad T^2 \mathbf{v} = \begin{pmatrix} iv \\ 0 \\ 0 \end{pmatrix} \qquad , \tag{57}$$

which gives

$$(M^2)_{ab} = \begin{pmatrix} g^2 v^2 & 0 & 0 & 0 \\ 0 & g^2 v^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{58}$$

and hence the masses of the gauge bosons (which we label $W^\pm$ according to their electric charge, as in the SM) is

$$M_W = gv . \tag{59}$$

The gauge boson interactions with the Higgs come from the kinetic term

$$\mathcal{L}_{\text{kin.}} = \frac{1}{2}(D_\mu \Phi)^T (D^\mu \Phi) , \tag{60}$$

where

$$(D_\mu \Phi)_i = \partial_\mu \phi_i + igT^a_{ij} W^a_\mu \phi_j \ , \tag{61}$$

where we have used that $Y = 0$ for the field triplet. Writing $g = e$ and $W^3_\mu = A_\mu$ we then have

$$D_\mu = \partial_\mu + \frac{ie}{\sqrt{2}} \left(T^+ W^+_\mu + T^- W^-_\mu \right) + ieQA_\mu \ , \tag{62}$$

where $T^\pm = T^1 \pm iT^2$ and we have used that $T_3 \Phi = Q\Phi$. Using the explicit form for the generators, when we expand as in (51) we find

$$D_\mu \Phi = \begin{pmatrix} -\frac{ie}{\sqrt{2}}(W^+_\mu - W^-_\mu)(v+h) \\ \frac{e}{\sqrt{2}}(W^+_\mu + W^-_\mu)(v+h) \\ \partial_\mu h \end{pmatrix} \ , \tag{63}$$

from which we get

$$\frac{1}{2}(D_\mu \Phi)^T (D^\mu \Phi) = \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{4}(v+h)^2 \left[(W^+_\mu + W^-_\mu)^2 - (W^+_\mu - W^-_\mu)^2 \right] \ , \tag{64}$$

$$= \frac{1}{2}(\partial_\mu h)^2 + e^2 (v^2 + 2vh + h^2) W^+_\mu W^{\mu -} \ , \tag{65}$$

$$= \frac{1}{2}(\partial_\mu h)^2 + (M^2_W + 2eM_W + e^2 h^2) W^+_\mu W^{\mu -} \ , \tag{66}$$

and hence we can read off a three–point vertex of $2ieM_W g^{\mu\nu}$.