Advanced QFT: Problem Set 4

Problem 1 *BRST transformation*

Writing the BRST transformations as $\phi_i \to \phi_i + \theta\delta_B\phi_i$, $A^a_\mu \to A^a_\mu + \theta\delta_B A^a_\mu$ etc, we have

$$\delta_B A^a_\mu(x) = D^{ab}_\mu c^b(x) = \partial_\mu c^a(x) - g f^{abc} A^c_\mu(x) c^b(x) \,, \tag{1}$$
$$\delta_B \phi_i(x) = i g c^a(x) (T^a_R)_{ij} \phi_j(x) \,, \tag{2}$$
$$\delta_B c^c(x) = -\frac{1}{2} g f^{abc} c^a(x) c^b(x) \,, \tag{3}$$
$$\delta_B \overline{c}^a(x) = -\frac{1}{\xi} \partial^\mu A^a_\mu \,. \tag{4}$$

Show that this operator has the property that $\delta_B^2 = 0$, when acting on an arbitrary operator $O(x)$. To do this, first demonstrate this is the case when acting on the gauge, ghost and matter fields, and then consider a product of arbitrary fields at arbitrary spacetime points to finish the proof.

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Problem 2 *Spontaneous Symmetry Breaking (1)*

Consider the following Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right) + \frac{1}{2}\mu^2(\phi_1^2+\phi_2^2) - \frac{1}{4}\lambda(\phi_1^2+\phi_2^2)^2 + i\overline{\psi}\partial\!\!\!/\psi - g\overline{\psi}(\mathbb{1}\phi_1 + i\gamma_5\phi_2)\psi \,, \tag{5}$$

where $\phi_{1,2}$ are real scalar fields and ψ is a Dirac fermion, while the parameters $\mu^2, \lambda > 0$ and $\mathbb{1}$ is a unit matrix in spinor space.

(i) Show that

$$\exp(i\alpha\gamma_5) = \mathbb{1}\cos\alpha + i\gamma_5\sin\alpha \,. \tag{6}$$

(ii) We require that this Lagrangian be invariant under the global symmetry

$$\psi \to \exp\left(-i\frac{1}{2}\alpha\gamma_5\right)\psi \,, \tag{7}$$

where α is a real parameter. Show that this implies the scalar doublet $\phi = (\phi_1, \phi_2)$ must obey a global $SO(2)$ symmetry with rotation angle α. [You may find the result from part (i) useful].

(iii) Verify that the entire Lagrangian is indeed invariant under this combined symmetry of the ϕ and ψ fields. Would a mass term for the Dirac field preserve this symmetry?

(iv) Show that the ground state, determined from the minimum of the potential $V(\phi_1, \phi_2)$, breaks the above symmetry spontaneously. Expanding around a conveniently chosen vacuum–expectation value v, show that the fermion acquires a mass, and determine its value in terms of the coupling g and v.

(v) Determine the masses of the remaining scalar bosons in the theory, and any new interaction vertices that been introduced by this symmetry breaking.

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Problem 3 *Spontaneous symmetry breaking (2)*

Consider a $SU(2) \times U(1)_Y$ gauge theory with a $Y = 0$, $SU(2)$ triplet of real scalar fields, $(\Phi)_i = \phi_i$, where Y is the $U(1)$ charge operator. The scalar potential is given by

$$V(\Phi) = -\frac{1}{2}m^2\Phi^T\Phi + \lambda(\Phi^T\Phi)^2 \ , \tag{8}$$

with $m^2, \lambda > 0$. After SSB, the electrically neutral ($Q = 0$) member of the scalar triplet acquires a vacuum expectation values (where $Q = T^3 + Y$).

i) Identify the subgroup that remains unbroken, and calculate the Higgs boson mass in this model.

ii) Calculate the vector boson masses and deduce the Feynman rule for the three–point vertex between the Higgs and the vector bosons.

[*The generators of* $SU(2)$ *for the triplet (adjoint) representation are given by* $(T^a)_{bc} = -i\epsilon_{abc}$.]

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