Advanced QFT: Problem Set 3

February 22, 2019

Problem 1 *Non-Abelian gauge theory*

Consider N^2-1 real fields $\Phi^a(x)$ that transform the adjoint representation of an $SU(N)$ gauge symmetry, with covariant derivative

$$D_\mu \Phi^a(x) = \partial_\mu \Phi^a(x) - igA^c_\mu(x)\,[T^c_A]^{ab}\,\Phi^b(x)\,.$$

In this representation, and introducing the matrix notation

$$A_\mu \equiv A^a_\mu T^a\,, \qquad \Phi \equiv \Phi^a T^a\,, \tag{1}$$

where T^a are the generators in the fundamental representation, the covariant derivative becomes

$$D_\mu \Phi(x) = \partial_\mu \Phi(x) - ig[A_\mu(x),\, \Phi(x)]\,.$$

(a) Verify the covariance of this derivative under the gauge transformation written appropriately for Φ transforming in the adjoint representation

$$\Phi' = U(x)\Phi U^\dagger(x)\,, \qquad A'_\mu = U(x)A_\mu U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x)\,, \tag{2}$$

i.e. show that $D'_\mu\Phi' = U(D_\mu\Phi)U^\dagger$ (you may find it useful to use $\partial_\mu(UU^\dagger)=0$).

(b) Show that $[D_\mu, D_\nu]\Phi = -ig[F_{\mu\nu}, \Phi]$.

The non-abelian gauge field strength $F^a_{\mu\nu}(x)$ transforms in the adjoint representation of the gauge group. Therefore

$$D_\lambda F_{\mu\nu}(x) = \partial_\lambda F_{\mu\nu}(x) - ig[A_\lambda(x),\, F_{\mu\nu}(x)]\,.$$

(c) Prove the non-abelian Bianchi identity

$$D_\lambda F_{\mu\nu}(x) + D_\mu F_{\nu\lambda}(x) + D_\nu F_{\lambda\mu}(x) = 0\,.$$

Now, consider the classical Yang-Mills theory of the non-abelian gauge field $A^a_\mu(x)$ governed by the Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}\,F^{\mu\nu})\,.$$

(d) Show that the variation of the gauge field strength is $\delta F^{\mu\nu}(x) = D^\mu\,\delta A^\nu(x) - D^\nu\,\delta A^\mu(x)$, and use this observation to write the classical YM field equations of motion as $D^\mu F_{\mu\nu}=0$.

Next, let us add fermionic fields

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}\,F^{\mu\nu}) + \bar\Psi\,(i\gamma^\mu\partial_\mu + g\gamma^\mu A_\mu - m)\,\Psi\,.$$

(e) Show that in this case the classical YM field equations of motion become $(D^\mu F_{\mu\nu})^a = -gJ^a_\nu$ where the current $J^a_\mu = \bar{\Psi}\,\gamma_\mu\,T^a\,\Psi$

(f) Show that equation $D^\mu F_{\mu\nu} = -\,gJ_\nu$ requires the fermionic current J_ν to be covariantly conserved, $D^\mu J_\mu = 0$.

(g) Use Dirac equations for the fermionic fields to verify that the current $J^a_\mu = \bar{\Psi}\,\gamma_\mu\,T^a\,\Psi$ is covariantly conserved.

(h) Consider the second variantion of the Yang-Mills action expanded around some non-trivial solution $A^\mu(x)$ of the Yang-Mills equations and show that

$$\delta_2\left(\int d^4x\,\mathcal{L}_{YM}\right) = \int d^4x\,\left(\mathrm{Tr}(\delta A^\mu\,D^2\,\delta A_\mu) + \mathrm{Tr}((D_\mu\,\delta A^\mu)^2) + 4ig\mathrm{Tr}(F_{\mu\nu}\,\delta A^\mu\,\delta A^\nu)\right)\,.$$

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Problem 2 *The Axial Gauge*

(i) Determine the form of the ghost Lagrangrian for the case of a non–abelian gauge theory with the choice of gauge fixing function

$$f(A^a) = n^\mu A^a_\mu - \sigma^a(x)\,, \tag{3}$$

appropriate for the axial gauge. Determine the corresponding gauge fixing term.

(ii) Identify the form of the ghost–antighost–gluon vertex and determine the gluon propagator in this gauge.

(iii) The light cone gauge is a special case of the axial gauge, with $\xi = 0$ and $n^2 = 0$. In this gauge, the gluon propagator is given by

$$i\tilde{\Delta}^{\mu\nu}_{ab}(p) = \frac{i}{p^2 + i\epsilon}\left(g^{\mu\nu} - \frac{n^\mu p^\nu + n^\nu p^\mu}{(n\cdot p)}\right)\delta^{ab}\,. \tag{4}$$

Using the result from part (i), show that the contribution from ghost fields vanishes in this gauge.

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Problem 3 *QCD scattering process*

Consider the process $gg \to t\bar{t}$, where g is a gluon and t is the top quark.

(i) Draw the contributing Feynman diagrams at leading order and write down the associated colour factors with each diagram.

(ii) Calculate the amplitude squared for the diagram with an s–channel gluon exchange in terms of the centre–of–mass energy, $\sqrt{s}$, and the squared four-momentum transfer, t. In your calculation, average over initial colours and spins and sum over final colours and spins and do not neglect the top quark mass, m_t.

(iii) Consider the massless quark ($m_t = 0$) approximation. Writing the amplitude (including all diagrams, not just s–channel gluon exchange) in the form $\epsilon^\mu_1(k_1)\epsilon^\nu_2(k_2)\mathcal{M}_{\mu\nu}$, where $\epsilon_{1,2}$ are the gluon polarization vectors, consider the replacement $\epsilon_{1,2} \to k_{1,2}$. Show that the corresponding amplitude is of the form

$$k^\mu_1 k^\nu_2 \mathcal{M}_{\mu\nu} \propto ([T^a, T^b] - if^{abc}T^c)\,, \tag{5}$$

and determine the proportional factor in terms of Dirac spinors (it is not necessary to perform any explicit contraction).

(iv) What does this imply for $k^\mu_1 k^\nu_2 \mathcal{M}_{\mu\nu}$? How would this expression be different for QED? Briefly discuss the wider importance of this result.

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