Advanced QFT: Problem Set 2

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Problem 1 *Dirac equation*

a) Demonstrate the Lorentz invariance of the Dirac equation.
b) Check whether $\psi^\dagger(x)\psi(x)$ and $\overline{\psi}(x)\psi(x)$ are Lorentz invariant.
c) Show that $\sum_r u_r(p)\overline{u}_r(p) = \not{p} + m$ and that $\sum_r \mathrm{v}_r(p)\overline{\mathrm{v}}_r(p) = \not{p} - m$.

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Problem 2 *Grassman variables*

Evaluate the following Grassman integrals

$$\begin{aligned}
&\textbf{a)} \int d^N\theta^*\, d^N\theta \,\times e^{-\vec{\theta}^\dagger\vec{\theta}} = 1 \\
&\textbf{b)} \frac{1}{\det(A)}\int d^N\theta'^*\, d^N\theta' \,\times e^{-\vec{\theta'}^\dagger A\vec{\theta'}} = 1 \\
&\textbf{c)} \int d^N\theta^*\, d^N\theta \,\times e^{-\vec{\theta}^\dagger A\vec{\theta}+\vec{K}^\dagger\vec{\theta}+\vec{\theta}^\dagger\vec{K}} = \det(A)\, e^{\vec{K}^\dagger A^{-1}\vec{K}}. \\
&\textbf{d)} \int d^N\theta^*\, d^N\theta \,\times \theta_k\,\theta_l^*\, e^{-\vec{\theta}^\dagger A\vec{\theta}} = \det(A)\left(A^{-1}\right)_{kl}.
\end{aligned} \tag{1}$$

where $\{\theta_i\}$ and $\{\theta_i^*\}$ are two independent sets of Grassmann numbers and A is a c-number skew-symmetric $N\times N$ matrix. Notice that the determinant is in the numerator, whereas for c-number Gaussian integrals

$$\int_{-\infty}^{-\infty} exp\left(-\frac{1}{2}x^T Ax\right) d^n x = \sqrt{\frac{(2\pi)^n}{det(A)}}$$

List of conventions

$$\begin{aligned}
\int d\theta\theta &= 1 \\
f(\theta+d\theta) &= f(\theta) + d\theta f'(\theta) \quad (left\ derivative) \\
d^N\theta &= d\theta_N d\theta_{N-1}...d\theta_1 \quad (descending\ order) \\
d^N\theta^* d^N\theta &= d\theta_N^* d\theta_N...d\theta_1^* d\theta_1 \quad (descending\ order) \\
f(\theta) &= f_0 + \theta_i f_1^i + ... + \theta_1\theta_2...\theta_N f_N; \quad (ascending\ order)
\end{aligned}$$

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Problem 3 *Gamma matrix manipulation*

Consider the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.
a) Show that γ^5 anticommutes with each of the γ^μ matrices, $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$.

b) Show that γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

c) Show that $\gamma^5 = (-i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$.
(Sign convention: $\epsilon^{0123} = +1$, $\epsilon_{0123} = -1$.)

d) Show that $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5$.

e) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\gamma^{[\mu}\gamma^{\nu]}$, $\gamma^5\gamma^\mu$ and γ^5.

f) Using $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ show that in d dimensions

$$
\begin{aligned}
\gamma^\mu\gamma_\mu &= d\,I_4,\\
\gamma^\mu\gamma^\nu\gamma_\mu &= -(d-2)\,\gamma^\nu,\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 2\,\{\gamma^\rho,\gamma^\nu\} + (d-4)\gamma^\nu\gamma^\rho = 4g^{\nu\rho}I_4 - (4-d)\gamma^\nu\gamma^\rho,\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma,\\
\not{p}^2 &= p^2.
\end{aligned}
$$

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Problem 4 *Anomalous magnetic moment of the electron*

a) Draw the Feynman diagram that represents the leading order contribution to the interaction between a photon and an electron. Write down the amplitude for this diagram.

b) Evaluating $\bar{u}(p',s')\,(\not{p}'\,\gamma^\mu + \gamma^\mu\not{p})\,u(p,s)$ in two different ways, prove the Gordon identity

$$
\bar{u}(p',s')\gamma^\mu u(p,s) = \frac{1}{2m}\bar{u}(p',s')\Big[(p+p')^\mu + i\sigma^{\mu\nu}(p'-p)_\nu\Big]u(p,s).
$$

c) Prove that the most general matrix element of the electromagnetic current allowed by Lorentz invariance and current conservation is

$$
\begin{aligned}
\langle p',s'\,|J^\mu(x)|\,p,s\rangle &= \bar{u}(p',s')\left[\gamma^\mu F_1\left(q^2\right) + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}F_2\left(q^2\right)\right]u(p,s)\\
&= \bar{u}(p',s')\left[\frac{(p'+p)^\mu}{2m}F_1\left(q^2\right) + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\Big(F_1\left(q^2\right)+F_2\left(q^2\right)\Big)\right]u(p,s), \qquad (-2)
\end{aligned}
$$

where $F_1\left(q^2\right)$ and $F_2\left(q^2\right)$ are two arbitrary functions of the momentum transfer, $q^\mu = p^\mu - p'^\mu$, called form factors. $|p,s>$ is a one-particle state of an electron with momentum p^μ and spin s. Notice that $< p',s'|J^\mu(x)|p,s >$ transforms as a vector under Lorentz transformations, so a good start is to write down all possible vector interactions. Using translation invariance

$$
\langle p',s'\,|J^\mu(x)|\,p,s\rangle = \langle p',s'\,|J^\mu(0)|\,p,s\rangle\, e^{i(p'-p)x},
$$

and the conservation of the electromagnetic current, $\partial_\mu J^\mu = 0$, show that

$$(p' - p)_\mu \langle p', s' | J^\mu(0) | p, s \rangle = 0$$

The coefficient of the first term in eq(**??**) is what we observe as an electric charge. Therefore it is natural to absorb $F_1(d^2)$ into the definition of the electric charge e, i.e. we set $F_1(0) = 1$ in the on-shell scheme.On the other hand, the second term is related to the magnetic moment of the electrn, which gets shifted from the tree-level value by $1 + F_2(0)$, which is now to be calculated to one-loop order.

d) Draw the Feynman diagrams that contribute to next-to-leading order to the interaction between a photon and an electron. Explain why only one of these diagrams contribtes to $F_2(q^2)$ and evaluate the diagram. In the calculation, concentrate on those terms that contribte to $F_2(q^2)$, which are finte and do not need any regulator, and drop the other contributions. Evaluate your result for $F_2(0)$ numerically, and compare it to the experimental value for $(g_e - 2)$.

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