Advanced QFT: Problem Set 1

February 13, 2019

Problem 1 *Symmetries*

The Lagrangian density of a real three-component scalar field is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^T(x)\partial^\mu \phi(x) - \frac{m^2}{2}\phi^T(x)\phi(x) \ , \tag{1}$$

where

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{pmatrix} . \tag{2}$$

Find the equations of motion for the scalar fields $\phi_i(x)$. Prove that the Lagrangian density is SO(3) invariant and show that the Noether currents are given by

$$J^\mu = -\phi \times \partial^\mu \phi \ . \tag{3}$$

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Problem 2 *Scalar QED: Ward–Takahashi identities*

The scalar QED Lagrangian is given by

$$\mathcal{L}_{\text{sQED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 \ , \tag{4}$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative.

(a) Show that the Lagragian is invariant under the gauge transformation

$$\begin{aligned} \phi(x) &\rightarrow e^{ie\alpha(x)}\phi(x) \ , & (5)\\ A_\mu(x) &\rightarrow A_\mu(x) + \partial_\mu\alpha(x) \ . & (6) \end{aligned}$$

(b) Consider the Lagrangian including the R_ξ gauge–fixing term $\mathcal{L}_{\text{sQED}} + \mathcal{L}_{gf}$, with

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \ . \tag{7}$$

By demanding the invariance of the action under the gauge transformation show that the generating functional for connected Green functions, $W = -i\log Z$, with

$$Z[\mathfrak{J}, \mathfrak{J}^*, \mathcal{J}^\mu] = \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu \, \exp\left[i\int \mathrm{d}^4x \, \mathcal{L}_{\text{sQED}} + \mathcal{L}_{gf} + \phi\mathfrak{J}^* + \phi^*\mathfrak{J} + \mathcal{J}^\mu A_\mu\right] \ , \tag{8}$$

satisfies the (Ward-Takahashi) identity

$$\frac{1}{\xi}\partial^2\partial^\mu\frac{\delta W}{\delta\mathcal{J}^\mu} - ie\mathfrak{J}^*\frac{\delta W}{\delta\mathfrak{J}^*} + ie\mathfrak{J}\frac{\delta W}{\delta\mathfrak{J}} + \partial_\mu\mathcal{J}^\mu = 0 \ , \tag{9}$$

where $\mathcal{J}^\mu$, $\mathfrak{J}$ are the sources for A_μ and ϕ respectively.

c) Show that the Ward-Takahashi identity implies

$$(p_\mu - p'_\mu)\,\Gamma^\mu(p,p') = i\left(\frac{1}{S(p)} - \frac{1}{S(p')}\right) , \tag{10}$$

where Γ^μ is a vertex with one photon and two scalars and S is the scalar propagator. (Hint: Apply $\frac{\delta}{\delta J^*(y)}\frac{\delta}{\delta J(z)}$ to (9) and then set the sources to zero.)

d) Check that (10) indeed holds at leading order.

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Problem 3 *Scalar QED: electron–positron annihilation*

Consider the scalar QED process: scalar-electron scalar-positron pair annihilation cross section into two photons, $\phi^+\phi^- \to \gamma\gamma$.

(a) Calculate the tree level matrix elements for $\phi^+\phi^- \to \gamma\gamma$.

(b) Calculate the differential cross section, $\mathrm{d}\sigma/\mathrm{d}\Omega$, summing over all possible outgoing photon polarizations, in the centre–of–mass frame.

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Problem 4 *Electric charge renormalisation in scalar QED*

(a) For scalar QED, draw the two Feynman diagrams at the one-loop order contributing to the photon propagator correction and evaluate them using dimensional regularisation. Show in particular that it is given by

$$\Sigma^{\mu\nu}_{1loop} = (k^2 g^{\mu\nu} - k^\mu k^\nu) \times \Pi_{1loop}(k^2) , \tag{11}$$

where in the $\overline{\mathrm{MS}}$ scheme (in $D = 4-\epsilon$ dimensions)

$$\Pi_{1loop}(k^2) = -\frac{\alpha}{12\pi}\left(\frac{2}{\epsilon} + \log\frac{\mu^2}{m^2} + I\left(\frac{k^2}{m^2}\right)\right) . \tag{12}$$

Find the undetermined function $I(k^2/m^2)$ (this is given in terms of an integral over the Feynman parameter x that it is not necessary to perform).

(b) Hence, determine the counter terms to $O(e^2)$ and write down $\Pi(k^2)$ as a function of k^2 after including these.

(c) Consider the $|k^2| \gg m^2$ limit, and show that the effective coupling $\alpha_{eff}(k^2) = \alpha/(1-\Pi(k^2))$ for scalar QED in this high momentum regime has the form

$$\frac{1}{\alpha_{eff}(k^2)} \approx \frac{1}{\alpha_0(0)} - \frac{1}{12\pi}\left(\log\left(\frac{-k^2}{m^2}\right) - \frac{8}{3}\right) . \tag{13}$$

(You may assume $Z_1 = Z_2$.)

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