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Keble College - Trinity 2017

S3: Quantum Ideas

Tutorial 2: Schrödinger Equation, Spin, Entanglement, and Cryptography

Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.
Leave these at Keble lodge by 5pm on Monday of 4th week.
Look at the 'class problems' in preparation for the tutorial session.

Goals

- Be able to solve the Schrödinger equation for simple potentials.
- Begin to think about spin of atoms and photons and measurements of non-commuting observables.
- Begin to think about entanglement and how this can be used in cryptography.

Problems

Last week we took a look at some of the key ideas historical development of quantum mechanics. This week we will do some real quantum physics, looking at measurements of spin, entanglement and how this can be exploited to transmit a secret message, with its security guaranteed by the laws of physics! First though, we will return to our discussion of wavefunctions, and solve the Schrödinger equation to compute stationary states and time evolution of a particle in a box.

1. **1D Schrödinger equation: Particle in an infinite box potential**

   Consider a particle of mass m confined to an infinite one-dimensional well of width L. The potential is given by

$$\begin{aligned} V(x) &= V_0 \qquad |x| \leq L/2, \\ V(x) &= \infty \qquad |x| > L/2. \end{aligned}$$

   (a) Show that the allowed energies for the particle are quantized and given by

$$E_n = \frac{\hbar^2 k^2}{2m} + V_0 = \frac{\hbar^2 \pi^2 n^2}{2mL^2} + V_0, \tag{1}$$

   where n is positive integer. What are the corresponding wave functions $\psi_n(x)$? What are the dimensions of the wave functions? Describe and explain the boundary conditions at $x = \pm L/2$.

   (b) Calculate the average position $\langle x \rangle$, momentum $\langle p_x \rangle$, and associated variances $(\Delta x)^2$ and $(\Delta p_x)^2$. Show that these results are consistent with the Heisenberg uncertainty relation $\Delta x \Delta p_x \geq \hbar/2$.

   (c) For (i) the particle in the ground state and (ii) first excited state, calculate the probability of finding the particle between $-L/4$ and $L/4$.

[1] This problem set is based off the problems by Dr. Brian Smith and past exam questions.

2. Time evolution of wavefunctions

Consider the particle in an infinite potential from problem 1. Suppose we prepare the particle in a superposition state

$$\psi(x,t) = \left(\psi_1(x,t) + \psi_2(x,t)\right)/\sqrt{2}, \tag{2}$$

where the eigenstates $\psi_n(x,t) = \psi_n(x,0)\exp(-\mathrm{i}E_n t)$ now include time dependence.

(a) What are the probability P_c of finding the particle in the central half ($-L/4 < x < L/4$) of the box for the superposition state $\psi(x,t)$ as a function of time?

(b) Calculate the average position as a function of time $\langle x(t)\rangle$.

The solution to problem 1 implies that, when measured, the energy of a particle in an infinite box potential can only take one of an infinite number of discrete values (we say that the energy is "quantised"). This is a common feature of observables in quantum mechanics, including spin, which we focus on for the next couple of problems. Classically, we think of the magnetic dipole moment of an atom as a vector $\vec{\mu}$, which can point in any direction. In quantum mechanics, the components of this vector are quantised. Furthermore, there is an uncertainty relation between different components of the vector – if we measure one component of $\vec{\mu}$, then the other components become completely undetermined. The next few questions explore this idea for atomic spin and photon polarisation.

3. The Stern-Gerlach Experiment

(a) Describe the Stern-Gerlach experiment in which a beam of silver atoms is deflected by an inhomogeneous magnetic field. Compare their observations with the results anticipated from classical physics and explain their significance. (Silver atoms have a spin $s = 1/2$.)

(b) In such an experiment a collimated beam of silver atoms traveling in the y direction is deflected by a magnetic field B_z possessing a gradient in the z direction. Silver atoms of mass m, magnetic moment of magnitude μ and velocity v pass through a short magnet of length l_1 before striking an observing screen a distance l_2 from the exit of the magnet. Show that the deflection D_z of the beam at the observation screen is given to first approximation by

$$D = \frac{\mu_z}{m}\frac{\partial B_z}{\partial z}\frac{l_1 l_2}{v^2}. \tag{3}$$

(c) Consider the three Stern-Gerlach interferometers in Fig. 1, arranged to achieve a temporary separation of atoms according to their magnetic moments. Interferometers A and C separate them according to their moments in the x direction, while interferometer B separates according to their moments in the z direction.

The input beam is unpolarised. Deduce separately the fraction of the initial atoms which remain in the beam at point P and at point Q when the following paths are blocked:

(i) No path blocked.

(ii) Paths A2 and C1 blocked.

(iii) Paths A2, B2, and C1 blocked.

Explain how the blocking of a path can cause an *increase* in the number of atoms emerging from the system at point Q.

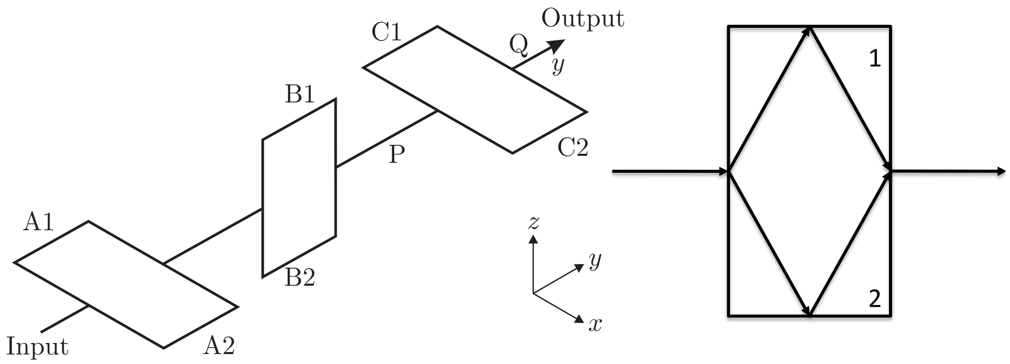

Figure 1: A sequence of three Stern-Gerlach interferometers.

4. "Stern-Gerlach with photons"

An analogous experiment can be conducted with photons (since in quantum mechanics, photon polarisation is quantised). A photon entering a polarising beamsplitter (PBS) will take one path or the other depending on its polarisation. Consider the setup shown in Fig. 2

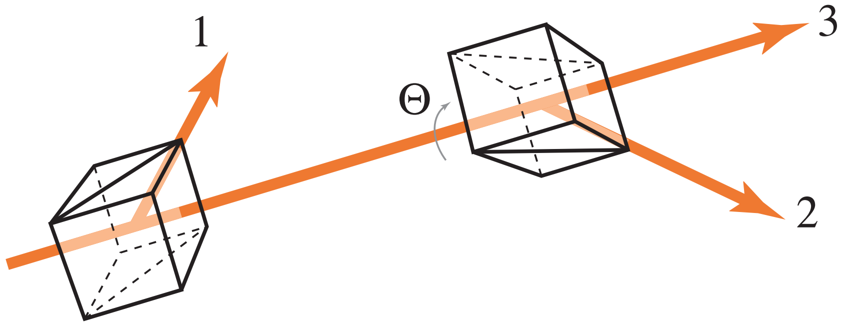

Figure 2: A pair of polarising beamsplitters orientated at an angle Θ with respect to one another.

(a) A strean of unpolarised photons passes through a pair of PBS with optical axes aligned at an angle Θ with respect to one another. Single photon counters are placed at the outputs 2 and 3 of the second PBS. Calculate the probability of recording a detection event at each detector as a function of Θ.

(b) For $\Theta = \pi/2$, one expects that no photons travel straight through the pair of PBS to output 3. How does the probability of detecting a photon in port 3 change if a third polarizer is inserted between the two existing ones, with its optical axis oriented at an angle ϕ with respect to the first polarizer? For which choice of ϕ is the intensity at port 3 a maximum?

We now come to states of multiple quantum systems and, in particular, the phenomenon of entanglement. Suppose we have two qubits, a and b. If we were to perform a measurement on both in the $\{|0\rangle, |1\rangle\}$ basis, we have, in principle, four possible distinct outcomes, and therefore four possible basis states, $|0_a, 0_b\rangle$, $|0_a, 1_b\rangle$, $|1_a, 0_b\rangle$, and $|1_a, 1_b\rangle$. In general, before measurement, we can have any superposition of these states, so we need four complex numbers to completely describe the system.[a]

Mathematically, $|0_a, 1_b\rangle$ is the tensor product of the single-qubit vectors $|0\rangle$ and $|1\rangle$, and is formally written $|0\rangle \otimes |1\rangle$, or $|0\rangle |1\rangle$ for short. You need not worry about the details of this operation for now, but it is important to note that the tensor product is distributive and associative, but not commutative. That is, $|0_a, 1_b\rangle = |0\rangle |1\rangle$ and $|1_a, 0_b\rangle = |1\rangle |0\rangle$ are physically distinct states.

Now let's suppose we have a pair of qubits. The most general state of the pair is

$$|\psi_{ab}\rangle = \alpha |0\rangle |0\rangle + \beta |0\rangle |1\rangle + \gamma |1\rangle |0\rangle + \delta |1\rangle |1\rangle . \tag{4}$$

We already know how to calculate the probabilities of measurements if we measure the state of both qubits simultaneously. For example, the probability of measuring qubit a to be in state $|1\rangle$ and qubit b in state $|0\rangle$ is $|\gamma|^2$. However, commonly we will only want to measure one qubit and leave the rest of the system alone.

For example, how can we calculate the probability of finding first qubit in state 0, $P(a = 0)$? The probabilities for finding the qubits in state $|0\rangle |0\rangle$ and $|0\rangle |1\rangle$ is $|\alpha|^2$ and $|\beta|^2$ respectively. However, we have not measured the second qubit – we have gained no information about it and thus we need to add the probabilities for the two outcomes, $P(a = 0) = |\alpha|^2 + |\beta|^2$

What is the state of the pair of qubits after this measurement? Again, we know that the first qubit is in state $|0\rangle$, but we have extracted no information about the second qubit (in other words, we have not completely collapsed the wavefunction). The resulting state is therefore

$$|\psi'_{ab}\rangle = \frac{\alpha |0\rangle |0\rangle + \beta |0\rangle |1\rangle}{\sqrt{|\alpha|^2 + |\beta|^2}}, \tag{5}$$

Where we have normalised the resulting state so that $\langle \psi'_{ab} | \psi'_{ab} \rangle = 1$.

[a] If we have n qubits, we require 2^n complex numbers (take a moment to convince yourself of this). This fact makes studying quantum systems of many particles very challenging. It is also what makes quantum computers potentially much more powerful than classical computers for certain problems.

5. Entanglement and Bell States

Suppose we have a qubit in the state

$$|\psi_a\rangle \otimes |\psi_b\rangle = (r |0\rangle + s |1\rangle) \otimes (u |0\rangle + v |1\rangle). \tag{6}$$

Such a state is called a separable state.

(a) We measure *only the first qubit* in the $\{|0\rangle, |1\rangle\}$ basis. What are the possible measurement outcomes and probabilities? For each outcome, what is the state of the two qubits after the measurement?

(b) Now consider the state

$$|\psi\rangle = \alpha |0_a, 1_b\rangle + \beta |1_a, 0_b\rangle . \tag{7}$$

Again, we measure *only the first qubit* in the $\{|0\rangle, |1\rangle\}$ basis. What are the possible measurement outcomes and probabilities? For each outcome, what is the state of the two qubits after the measurement? How is this different from the situation in (a)?

(c) The four Bell states are

$$\left|\Psi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left(|0_a, 1_b\rangle + |1_a, 0_b\rangle\right), \qquad \left|\Phi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left(|0_a, 0_b\rangle + |1_a, 1_b\rangle\right),$$
$$\left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(|0_a, 1_b\rangle - |1_a, 0_b\rangle\right), \qquad \left|\Phi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(|0_a, 0_b\rangle - |1_a, 1_b\rangle\right), \tag{8}$$

These four states are maximally entangled and form a complete basis for two-qubit states. Show that none of the bell states can be written as a separable state (i.e. in the form given in Eq. (6))

The previous question was a little bit abstract and mathematical, but the existence of states of the form Eq. (8) have profound physical consequences, which have direct application in quantum information processing and cryptography, which the final question explores.

6. The EPR paradox and quantum key distribution

(a) Describe the Einstein-Podolsky-Rosen (EPR) "paradox" and explain why it worried Einstein. Discuss whether information can be transmitted faster than the speed of light in the context of the EPR thought experiment.

(b) In 1991, Artur Ekert[2] proposed a method to distribute a key between two parties using entangled photons. The secrecy of this key is guarranteed by the laws of quantum mechanics. To obtain a common key, Alice and Bob share a light source emitting polarization-entangled photon pairs in the state $|\Phi^{+}\rangle$. Alice and Bob each analyse one photon from each pair. They do this by independently and randomly selecting either the horizontal / vertical ($\oplus$) or $+45°/-45°$ ($\otimes$) basis to detect their photon.

Assume Alice chooses the $\oplus$ basis and detects a photon polarized at $+45°$. What are the possible outcomes and respective probabilities for Bob to detect the other photon if he chooses the $\otimes$ or $\oplus$ basis?

(c) To establish a common key Alice and Bob now analyze a sequence of 20 photon pairs. Alice measures the photon sequence

11010 00010 10101 01101

using the random sequence of detection bases

$$\otimes\oplus\oplus\otimes\otimes \quad \otimes\oplus\oplus\otimes\otimes \quad \oplus\otimes\oplus\oplus\oplus \quad \oplus\oplus\otimes\otimes\oplus$$

Bob analyzes his corresponding photons from the pair using the sequence

$$\otimes\oplus\otimes\otimes\otimes \quad \oplus\otimes\oplus\oplus\otimes \quad \otimes\otimes\oplus\otimes\oplus \quad \oplus\otimes\oplus\otimes\oplus$$

Find the sifted data set they get i.e. keeping only data in which both Alice and Bob measured in the same basis.

(d) Bob now receives the encrypted message '010.010.011.110' from Alice over a public channel. Decipher this message by performing an exclusive-or, 'XOR' with the key you found above. Assume that only a subset of the alphabet is encoded by three-bit binary numbers according to

$$000 = \mathrm{I}, 001 = \mathrm{O}, 010 = \mathrm{B}, 011 = \mathrm{G}, 100 = \mathrm{E}, 101 = \mathrm{Q}, 110 = \mathrm{D}, 111 = \mathrm{T}.$$

What is the decoded message?

(e) Assume that the shared photon source is accessible to an eavesdropper, named Eve, and that instead of letting the photon source emit the entangled photon pairs, she sends photons of known (to her) polarization to Alice and Bob. How can Alice and Bob detect her presence?

[2] who is currently a professor in the Oxford Maths department

These two problems sets should have given you a flavour of some of the concepts in quantum mechanics and (hopefully!) stoked your interest, but we have barely scratched the surface! Next year you will get a more comprehensive and rigorous introduction to quantum mechanics, and in further years you will apply the ideas you have learned to the physics of atoms, molecules and lasers, to condensed matter systems, and to subatomic particles.

Class problems

7. **Quantum Harmonic Oscillator (From Prelims 2016)**

   Write down the time-dependent Schrödinger equation for a particle of mass m free to move only along the x axis in a region with potential energy $V(x)$. Give a physical interpretation of each term and show how the time-independent Schrödinger equation (TISE) is obtained for a particle of well-defined energy. Give a physical interpretation of the particle wave function $\psi(x,t)$. What are the dimensions of the wave function $\psi(x,t)$?

   Consider a particle constrained to move along the x direction subject to a restoring force $F = -kx$.

   (a) Write down the TISE and sketch the corresponding potential for this case.

   (b) The ground and first excited state wave functions for this system are

   $$\phi_0(x) = \sqrt{\frac{1}{\pi^{1/2}\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \qquad \phi_1(x) = \sqrt{\frac{2}{\pi^{1/2}\sigma^3}} x \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

   where $\sigma = \sqrt{\hbar/(m\omega)}$, and $\omega = \sqrt{(k/m)}$. Using the TISE calculate the energies, E_0 and E_1, associated with these states expressed in terms of $\hbar\omega$.

   (c) Give expressions for the probability distributions, $p_0(x)$ and $p_1(x)$, for these states and plot them on separate graphs. What is the probability to find a particle in a small region δx centred on the origin when the particle is prepared in each state?

   (d) Calculate the average position $\langle x \rangle$ and corresponding uncertainty in position $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ for both states. Indicate these on your plot of the probability distributions.

   (e) For a particle prepared in a superposition of the ground and first-excited states, $\phi_+(x) = [\phi_0(x) + \phi_1(x)]/\sqrt{2}$, at time $t = 0$, determine the minimum time required to evolve into the orthogonal state $\phi_-(x) == [\phi_0(x) - \phi_1(x)]/\sqrt{2}$

$$\left[\text{You may use the integrals} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{\sigma^2}\right) \mathrm{d}x = \frac{\sqrt{\pi}\sigma^3}{2} \text{ and } \int_{-\infty}^{\infty} x^4 \exp\left(-\frac{x^2}{\sigma^2}\right) \mathrm{d}x = \frac{3\sqrt{\pi}\sigma^5}{4}\right]$$