

# Problem Set 1

MMathPhys: The Standard Model

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## Quantum Chromodynamics in $e^+e^-$ collisions

Problems marked with \* are more advanced.

To be completed and handed in by Tuesday Week 5.

### 1. Two-loop renormalization group relations

- (a) In higher order of perturbation theory, the expression for the QCD  $\beta$  function is a series

$$\beta(g) = -\frac{b_0}{(4\pi)^2}g^3 - \frac{b_1}{(4\pi)^4}g^5 - \frac{b_2}{(4\pi)^6}g^7 + \dots \quad (1)$$

Integrate the renormalization group equation and show that the running coupling constant is now given by

$$\alpha_s(Q^2) = \frac{4\pi}{b_0} \left[ \frac{1}{\ln(Q^2/\Lambda^2)} - \frac{b_1}{b_0^2} \frac{\ln \ln(Q^2/\Lambda^2)}{(\ln(Q^2/\Lambda^2))^2} + \dots \right], \quad (2)$$

where the omitted terms decrease as  $(\ln(Q^2/\Lambda^2))^{-2}$ .

- (b) Combine this formula with the perturbation series for the  $e^+e^-$  annihilation cross section:

$$\sigma(e^+e^- \rightarrow \text{hadrons}) = \sigma_0 \cdot \left( 3 \sum_f Q_f^2 \right) \cdot \left[ 1 + \frac{\alpha_s}{2\pi} + a_2 \left( \frac{\alpha_s}{2\pi} \right)^2 + \mathcal{O}(\alpha_s^3) \right]. \quad (3)$$

The coefficient  $a_2$  depends on the details of the renormalization conditions defining  $\alpha_s$ . Show that the leading two terms in the asymptotic behaviour of  $\sigma(s)$  for large  $s$  depend only on  $b_0$  and  $b_1$  and are independent of  $a_2$  and  $b_2$ . Thus the first two coefficients of the QCD  $\beta$  function are independent of the renormalization prescriptions.

2\*.  $\sigma(e^+e^- \rightarrow q\bar{q}g)$

- (a) Show that the phase space for the unpolarized decay into three massless objects can be written as:

$$d^5\phi_3 = \frac{1}{(2\pi)^5} \frac{s}{32} dx_1 dx_2 d\phi_{12} d\Omega_1 \quad (4)$$

where  $s$  is the c.m.s. energy and  $x_i = 2E_i/\sqrt{s}$ .

- (b) Consider now the process  $e^+(p')e^-(p) \rightarrow q(k_1)\bar{q}(k_2)g(k_3)$ . Consider the quarks as massless. Show that the amplitude for this process is

$$\mathcal{M} = e^2 g [\bar{v}(p')\gamma^\mu u(p)] \frac{1}{q^2 + i\varepsilon} [\bar{u}(k_1)\Lambda_{\lambda\mu}v(k_2)] \varepsilon^\lambda(k_3) Q_q, \quad (5)$$

where  $Q_q$  is the fractional charge of the quarks and

$$\Lambda_{\lambda\nu} = \gamma_\lambda \frac{-1}{\not{k}_1 + \not{k}_2} \gamma_\nu + \gamma_\nu \frac{1}{\not{k}_3 + \not{k}_2} \gamma_\lambda \quad (6)$$

- (c) Using the results obtained in (a) and (b) calculate the cross-section for  $e^+e^- \rightarrow q\bar{q}g$ . First, show that the matrix element squared can be written as

$$|M|^2 = \frac{1}{s^2} L^{\mu\nu} H_{\mu\nu} \quad (7)$$

where  $L^{\mu\nu}$  and  $H^{\mu\nu}$  are the leptonic and hadronic tensors that come from the squaring of the the corresponding currents, and  $q = p + p'$ . As we are interested in azimuthal-averaged quantities, we can neglect angular correlations between the initial state plane and the final state and write

$$|M|^2 = \frac{1}{s^2} (L^{\mu\nu} g_{\mu\nu})(H^{\rho\sigma} g_{\rho\sigma}). \quad (8)$$

Using eq. (8) and the explicit expressions for the hadronic the leptonic tensor introduced in eq. (7) calculate the matrix element squared. Express your result in terms of the fractional energies for the quark and anti-quark  $x_1, x_2$ . Show that the total cross section can be written as

$$\sigma^{q\bar{q}g} = \sigma_0 C_F \frac{\alpha_s}{2\pi} \int dx_1 dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \quad (9)$$

where  $\sigma_0 = \sigma_{\text{Born}} = 4\pi\alpha^2 Q_q^2 N_c / 3q^2$ .

### 3. Jet Algorithms

The last exercise is meant to get acquainted with the event generator PYTHIA and to try different jet algorithms. You will use two different routines, FastJet and SlowJet. Their results should be equal, but their performances are very different.

- (a) In the folder `problem_set_1` you will find the file `example_jets.cc`. Run the program a first time. It will produce four plots in the folder `problem_set_1/plots`. They will contain the distributions of the number of jets  $N_{\text{jets}}$  and of the transverse momentum  $p_{T,\text{jet}}$  for SlowJet and for FastJet. Compare the plots.

- (b) Using the existing histograms as a template, create and plot an additional histogram,  $N_{\text{jets}}^{\text{Fast}} - N_{\text{jets}}^{\text{Slow}}$ . Are the two routines really identical?
- (c) Plot the histograms `tSlow` and `tFast`, which are already filled in the program. Is FastJet considerably faster than SlowJet?
- (d) Using the FastJet routine, compare the distributions of  $p_{T,\text{jet}}$  obtained using the anti- $k_T$ , Cambridge-Aachen and  $k_T$  jet algorithms.
- (e) Using the FastJet routine, plot  $N_{\text{jets}}$  and  $p_{T,\text{jet}}$  for different values of  $R$ : 0.4, 0.6, 0.8.