The Standard Model and Beyond I: problem set I

Oxford MMathPhys, Trinity Term 2020

This homework is due on Friday May 22nd at 17.00. Please upload it as usual using the standard submission system. If you have questions, please contact me or the tutors (Federico Buccioni, Alexander Karlberg).

The R-ratio

Fig. 1 shows the R-ratio defined as

$$R(Q) = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)},$$

where σ is the total production cross-section and Q is the energy Q of the e^+e^- system, $Q = \sqrt{(p_{e^+} + p_{e^-})^2}$.

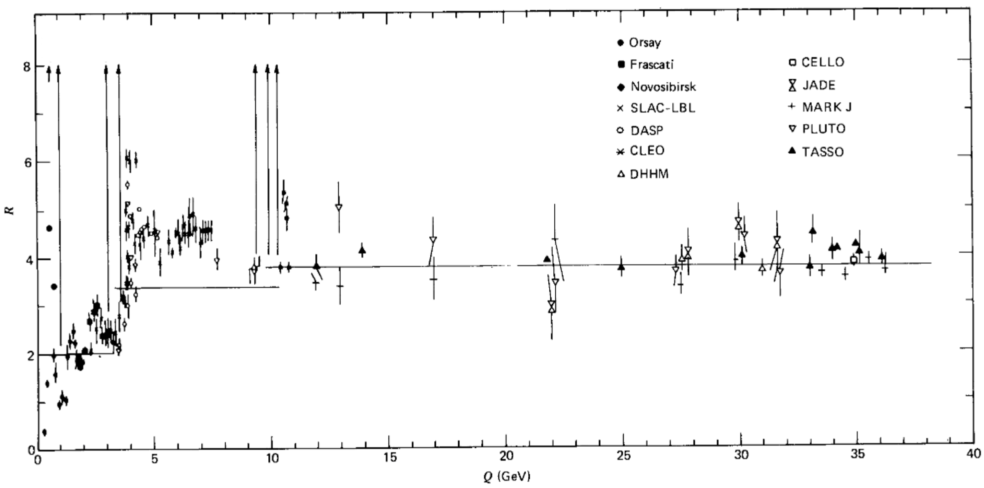

Figure 1: The R-ratio.

Explain the main features of this figure. Assuming that quarks are in the fundamental representation of a $SU(N_c)$ gauge group, what can you infer about N_c from this plot? What can you say about the charm and bottom masses? Does the Z boson play a role in your analysis? Why?

[5 marks]

Bi-unitary transformations

When we discussed the CKM matrix, we claimed that a generic matrix F can be written as a product of a hermitian matrix H and a unitary matrix U. The goal of this problem is to prove this statement.

For a generic matrix F, show that $M \equiv FF^\dagger$ is hermitian and its eigenvalues are either positive of zero. Now assume that all eigenvalues λ_i are different from zero, and define

$$U_{ik} = \frac{F_{ik}}{\sqrt{\lambda_i}}.$$

Show that U is unitary, and use this fact to prove the F can be written as the product of a unitary matrix and a hermitian matrix.

[4 marks]

EFTs, equation of motion and field redefinition

When discussing EFTs, you often hear that you can "remove redundant operators using the equations of motion". In this problem, you will see in an explicit example what this means, and in which sense it is a legitimate thing to do.

Consider the following EFT Lagrangian

$$\mathcal{L} = \mathcal{L}_4 + \sum_{i=1}^{2} \bar{c}_i \mathcal{O}_i + ...,$$

with

$$\mathcal{L}_4 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \quad \text{and} \quad \mathcal{O}_1 = \phi^3\partial^2\phi, \quad \mathcal{O}_2 = \phi^6.$$

1. What is the classical dimension of $\bar{c}_i$? Write $\bar{c}_i = c_i/\Lambda^{k_i}$, where c_i are dimensionless and Λ has mass-dimension 1.

2. Write down the equation of motion for the field ϕ obtained from the $\mathcal{L}_4$ Lagrangian.

3. Show that if you use this equation of motion in $\mathcal{L}$, you can eliminate $\mathcal{O}_1$ (at the price of re-defining the parameters in the Lagrangian. Is this an issue? Why?). In principle, can one do this manipulation in an interacting QFT? Why?

4. Now consider a new Lagrangian $\mathcal{L}' = \mathcal{L}_{\phi\to\phi'}$ where $\phi \to \phi' = \phi + \bar{c}_1\phi^3$. Do $\mathcal{L}$ and $\mathcal{L}'$ describe the same physics? Why?

5. Expand $\mathcal{L}'$ to order $1/\Lambda^2$, and compare your result with what you obtained in (3). What can you conclude about the use of the classical equation of motion?

[5 marks]

The Noether current for chiral symmetry

Consider the linear sigma model that we discussed in class

$$\mathcal{L} = \bar{\Psi} i\hat{\partial}\Psi - g\bar{\Psi}_L \Sigma \Psi_R - g\bar{\Psi}_R \Sigma^+ \Psi_L, L(\Sigma)$$

where Ψ is the nucleon multiplet $\Psi = \begin{pmatrix} p \\ n \end{pmatrix}$, $\Sigma = \sigma + i\vec{\pi}\cdot\vec{\tau}$ with τ^i the Pauli matrices and

$$L(\Sigma) = \frac{1}{4}\mathrm{Tr}\left[\partial_\mu \Sigma \partial^\mu \Sigma^+\right] - \frac{\lambda}{4}\left(\frac{1}{2}\mathrm{Tr}[\Sigma\Sigma^+] - f_\pi^2\right)^2 .$$

Compute the Noether current $J^a_{5\mu}$ associated with the chiral symmetry, and express your result in terms of fields after symmetry breaking (i.e. shifted fields with zero vacuum expectation value) and f_π. Which transitions can this current mediate?

[4 marks]

Haag theorem and pion-nucleon interactions

Compute the tree-level scattering amplitudes for the processes

$$n^a(p_1) + n^b(p_2) \to n^c(p_3) + n^d(p_4)$$
$$\pi^i(k_1) + n^a(p_1) \to \pi^j(k_2) + n^b(p_2),$$

where π^i and n^a are pions and nucleons with generic isospin and momenta. Perform your calculation both in the linear and non-linear SU(2) σ models, and compare your results.

[14 marks]

Pion decay

Consider the decay $\pi^+ \to l^+\nu_l$ in the Fermi theory, where l^+ is a charged anti-lepton and ν_l is its relative neutrino. Compute the pion decay width as a function of f_π, m_π and m_l, where $m_\pi(m_l)$ are the pion(lepton) masses and f_π is defined through

$$\left\langle 0|J^a_{\mu,5}|\pi^b\right\rangle = i f_\pi p^\mu \delta^{ab}.$$

Compute the ratio between the decay rate to positrons and to muons using realistic input values for the masses, and comment on your result.

[14 marks]

Non-unitarity of the Fermi theory

For this problem you should consider the process

$$e^-(p_1) + \nu_\mu(p_2) \to \mu^-(k_1) + \nu_e(k_2),$$

in the Fermi theory and at lowest order in perturbation theory. You should also neglect lepton masses.

1. Argue that the scattering amplitude $\mathcal{M}$ for this process can be expanded in Legendre polynomials
$$\mathcal{M}(s,t) = 16\pi \sum_j (2j+1) a_j(s) P_j(\cos\theta),$$
where s and t are the usual Mandelstam invariants and θ is the scattering angle, $\cos\theta = 1 + 2t/s$. P_i are Legendre polynomials normalised such that
$$\int_{-1}^{1} dz P_i(z) P_j(z) = \frac{2}{2i+1}\delta_{ij},$$
and $P_i(1) = 1$.

2. Writing the scattering matrix as $S = I + iT$, and assuming that only $2 \to 2$ processes are allowed, use the optical theorem to derive the relation $|a_i(s)|^2 = \mathrm{Im}\big[a_i(s)\big]$, which implies the *unitarity bound* $|a_i(s)| \le 1$. *[Hint: consider forward scattering, and sandwich the unitarity relation for T between appropriate states.]*

3. Now explicitly compute the amplitude $\mathcal{M}$ at tree-level, and show that it violates the unitarity bound at large s. At which scale does this happen? Call this scale Λ.

4. Now repeat the calculation in the Standard Model, where this process is mediated by W exchange. Show that at low energy the result is identical to the one in the Fermi theory, but that the Standard Model amplitude never violates the unitarity bound.

5. Recalling the relation $G_F/\sqrt{2} = g^2/(8m_W^2)$ between the weak coupling constant g and the Fermi constant G_F, compute g both using a realistic value for the W mass and in a hypothetical universe where the W mass is closer to the unitarity breaking scale Λ that you derived in (3). Comment on your result.

[14 marks]