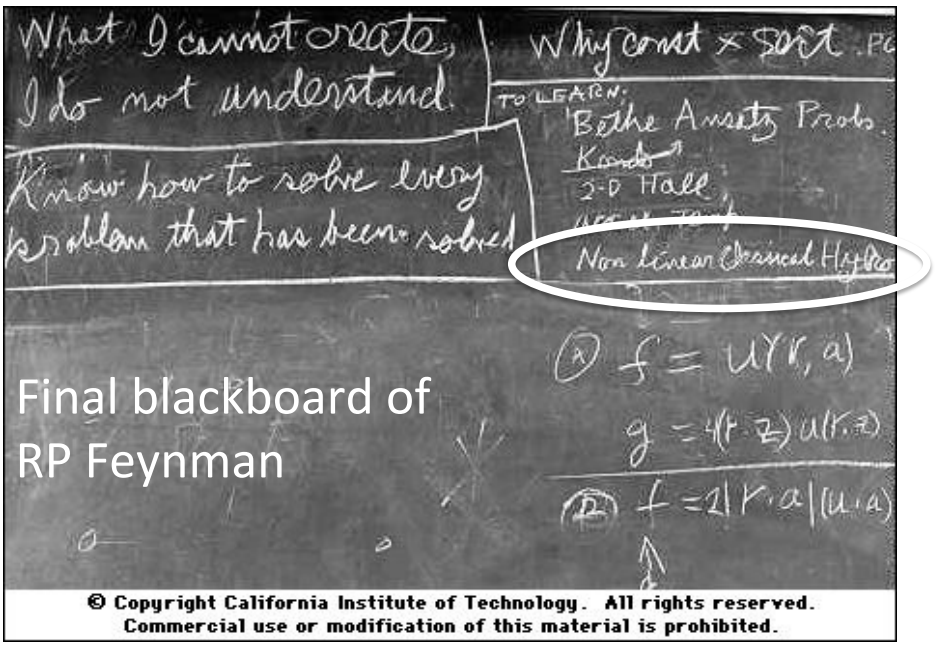

Oxford Master Course in Mathematical and Theoretical Physics

Astrophysical Fluid Dynamics

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Contents

Most of the fundamental ideas of science are essentially simple, and may, as a rule, be expressed in a language comprehensible to everyone.

— *Albert Einstein*

1 Fundamentals

1.1 Opening Comment

We arose from the gas cloud that collapsed to form our solar system, and when, in the distant future, the sun enters its red giant phase, we shall all return to gas and dust. Little wonder, then, that the topic of astrophysical gasdynamics holds our fascination.

The behaviour of a gas subject to large-scale gravitational and magnetic forces is enormously rich and full of surprises. One of my goals in giving this course is to try to give you, the student encountering the topic for the first time, a sense of both the generality and the depth of the problems we are struggling with. Truly, there isn't an area of modern astrophysics that is not touched in some way by the dynamical behaviour of gases. Astrophysical gas dynamics may be the most fundamental domain of astrophysics. It is impossible to understand star formation, stellar structure, planet formation, accretion discs, or anything in the early universe without a detailed knowledge of gas dynamics. It is an excellent way to begin a study of theoretical astrophysics.

1.2 Governing Equations

Although the ultimate fundamental objects are the atomic particles that comprise our gas, we shall work in the limit in which the matter is regarded as a *nearly* continuous fluid. The fact that this is not *exactly* a continuous fluid manifests itself in many ways, the most important of which is the equation of state of our gas, which depends upon the notion of atomic collisions. But more subtle effects are also present, like viscosity and thermal conduction, both of which are a consequence of finite mean free paths.

One of the most interesting features of astrophysical gases is that they are almost always magnetized. This allows modes of behaviour that are absent in an ordinary gas (e.g. shear waves), sometimes with profound consequences, especially in rotating systems. The dynamics of magnetized gases is known as magnetohydrodynamics, or MHD for short, and we will have much to say about this topic. The ohmic resistivity of magnetized gas is another example of a collisional process involving individual particles, in this case charged particles.

I shall assume that the reader is familiar with the basic equations of standard hydrodynamics. If not, (s)he may review a standard textbook (my favourite is *Elementary Fluid Dynamics* by D.J. Acheson), or the set of extensive notes I have prepared for my course Hydrodynamics, Instability and Turbulence. Let us begin with a very brief review.

1.2.1 Mass Conservation

The statement of mass conservation is expressed by the equation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0, \tag{1}$$

Here ρ is the mas density and $\boldsymbol{v}$ is the velocity field. The content of this equation is simply that if there is net mass influx into or mass outflux from a fixed volume, the mass within that volume must change accordingly. If the flow is divergence free, the density of an individual fluid element remains constant.

1.2.2 Newtonian Dynamics

Our second fundamental equation is a statement of Newton's second law of motion, that applied forces cause acceleration in a fluid. The acceleration refers to an individual element of fluid, hence the time derivative is expressed as a total, or *Lagrangian* derivative, following the path of the element:

$$\rho \frac{D\boldsymbol{v}}{Dt} = \rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \right] = \boldsymbol{F} \tag{2}$$

where the right side is the sum of the forces on the fluid element.

The most fundamental force acting on a fluid is the pressure. We shall almost always be working with an ideal gas in this course, and the pressure is then given by the ideal gas equation of state

$$P = \frac{\rho k T}{\mu} \tag{3}$$

where T is the temperature in Kelvins, k is the Boltzmann constant 1.38×10^{-23} J K^{-1}, and μ is the mass per particle. The quantity kT/μ has dimensions of velocity squared, and arises often enough that it is given its own name:

$$c_S^2 \equiv \frac{kT}{\mu} \tag{4}$$

where the subscript S refers to "sound" for reasons that will become clear later. c_S is the "isothermal sound speed."

The pressure arises from the kinetic energy of the gas particles themselves, which must never be confused with fluid elements. A fluid element is small enough that it has uniquely defined dynamic and thermodynamic attributes (e.g. density and pressure), but large enough to contain a vast number of atoms. A fluid element has a well-defined entropy for example, an atom does not.

There is a very simple relationship between the pressure P and internal energy density $\mathcal{E}$ of an ideal gas:

$$\mathcal{E} = \frac{P}{\gamma - 1}. \tag{5}$$

Here γ is the adiabatic index of the gas. It is equal to 5/3 for single particles, and 7/5 for diatomic molecules.

A pressure exerts a force only if it is not spatially uniform. For example, the pressure force in the x direction on a slab of thickness dx and area $dy\,dz$ is

$$[P(x - dx/2, y, z, t) - P(x + dx/2, y, z, t)]dy\,dz = -\frac{\partial P}{\partial x}dV \tag{6}$$

There is nothing special about the x direction, so the force per unit volume from a pressure is more generally $-\boldsymbol{\nabla} P\,dV$.

Other forces can be added on as needed. One force of obvious importance in astrophysics is gravity. The Newtonian gravitational acceleration $\boldsymbol{g}$ can always be derived from a potential function

$$\boldsymbol{g} = -\boldsymbol{\nabla}\Phi \tag{7}$$

If the field is from an external source, then Φ is a given function of $\boldsymbol{r}$ and t, otherwise it must be computed along with the evolution of the fluid itself. We shall discuss the problems of self-gravity later in the course.

Another force that we must consider arises from the presence of a magnetic field. Magnetic fields allow a gas to behave in ways not allowed when the field vanishes, and the additional degrees of freedom imparted to a gas mean that magnetic forces can be very important even when the field *appears* to be weak! To calculate the magnetic force per unit volume exerted by a magnetic field, start with the Maxwell equation

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \epsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t} \tag{8}$$

The effects of the displacement current are negligible for nonrelativisitic fluids, since they involve time delays associated with light propagation. Hence, the current density is determined by the magnetic field geometry:

$$\boldsymbol{J} = (1/\mu_0)\boldsymbol{\nabla}\times\boldsymbol{B} \tag{9}$$

The Lorentz force per unit volume is $\boldsymbol{J}\times\boldsymbol{B}$, assuming that the gas is everywhere locally neutral.

In the absence of dissipational processes, the equation of motion for a magnetized gas is therefore

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + (\rho\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} \tag{10}$$

1.2.3 Viscosity

The fact that there is a finite distance between collisions of the mass particles of gas creates internal so-called *viscous* stresses in the flow. As a result, a velocity gradient tends to relax and become erased with time if undisturbed. To leading order, the forces arising from the velocity gradients are found to be linear in these gradients. This is reasonable, rather like the leading term in some sort of a Taylor expansion.

To represent this, we introduce index notation i, j, k which take on vector Cartesian components x, y, z. A repeated index is summed over. Hence $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ is denoted $\partial v_i/\partial x_i$. The viscous stress tensor σ_{ij}, dimensions of momentum per unit second per unit area, takes the form first introduced by Stokes,

$$\sigma_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}\right) \tag{11}$$

where η is a scalar that may be a function of density and pressure (more below). This is the most general linear superposition of velocity gradients that vanishes for solid body rotation (no shear) and for uniform expansion (no preferred direction for momentum flow). As an *Exercise,* you should prove this.

Notice that $\sigma_{ij} = \sigma_{ji}$, i.e., the i component of the momentum being trasnported in the j direction is physically equivalent to the j component of of the momentum being transported in the i direction. The momentum being "carried" and the momentum doing the "transporting" are interchangeable.

Notice as well that the σ_{ij} tensor has a zero trace: the sum of its diagonal elements $\sigma_{ii} = 0$. This is an indication that the deformations of the flow caused by the shear that affects the viscous stress are associated with no change in volume. There can be a viscous stress associated with the velocity divergence itself under unusual circumstances, but we will not pursue this in this course (see e.g. Landau & Lifschitz, *Fluid Mechanics*).

The parameter η is known as the *dynamic viscosity* coefficient, with dimensions of mass per length per time. The viscosity is often ignorable in many applications, but is needed when small scales are important for dissipation. This may be the case for turbulent flow. In fact, astrophysicists love to represent the whole of turbulent flow phenomenologically with a fake "turbulent viscosity parameter," as a way to account for the enhanced transport turbulence often produces. Avoid joining this crowd, but if you must, at least beware of drawing detailed mathematical conclusions this way.

The equation of motion with viscosity may be written, in a mixed vector-index notation as

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + (\rho\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} + \frac{\partial\sigma_{ij}}{\partial x_j} \tag{12}$$

where i is the component of all boldface vector quantities being selected.

In the limit of no gravity, no magnetic field, constant density and constant η, we obtain the so-called Navier-Stokes equation:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}P' + \nu\nabla^2\boldsymbol{v} \tag{13}$$

where $P' = P/\rho$ and $\nu = \eta/\rho$ is the *kinematic viscosity.* As an *Exercise,* you should prove this. If you can further prove that this equation, together with $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, has no solutions developing singularities from nonsingular initial conditions (or much to everyone's surprise find such a singular solution), you can win \$1,000,000 from the Clay Mathematics Institute[1]. Without viscosity this statement is false; there are singular solutions that develop from the "inviscid" equations, as we will see later in the course.

1.2.4 Energetics

The thermal energy behaviour of the gas is described by the internal energy loss equation, which is most conveniently expressed in terms of the entropy per particle. The entropy is defined up to an (unimportant) additive constant, and is given by

$$s = \frac{S}{N} = \frac{k}{\gamma - 1}\ln P\rho^{-\gamma} \tag{14}$$

where N is the number of particles, γ is the adiabatic index (equal to $1 + 2/f$ where f is the number of degrees of freedom of a particle).

[1] http://www.claymath.org/millenium-problems/navierstokes-equation

Exercise. Derive the above expression from $dE = -PdV + TdS$, $P = \rho kT/m = (\gamma - 1)\mathcal{E}$, $E = \mathcal{E}V$.

The entropy of a fluid element is conserved unless there is a loss or gain from radiative processes, or the gas is heated by dissipation. If n is the number of particles per unit volume, then

$$nT\frac{Ds}{Dt} = \frac{P}{\gamma - 1}\frac{D\ln P\rho^{-\gamma}}{Dt} = \text{volume heating rate} \equiv \dot{Q} \tag{15}$$

If there are no radiative losses or gains and no dissipation, as is often the case when the fluid motions are too rapid for heat to escape, the fluid is said to be adiabatic and the right side of the above is zero. Note that the internal thermal energy is *not* conserved in an adiabatic fluid because of compression or expansion. As an exercise, the reader should show that c_S^2 satisfies the equation

$$\rho\frac{D}{Dt}\frac{c_S^2}{\gamma - 1} = -P\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} \tag{16}$$

for an adiabatic gas. (Use the entropy and mass conservation equations.) The temperature of a fluid element, like the density, remains fixed only if the motions are incompressible.

In the presence of "bulk" radiative losses, meaning that the photons can easily escape, a typical form for $\dot{Q}$ might be

$$\dot{Q} = n\Gamma - n^2\Lambda(T) \equiv -\rho\mathcal{L} \tag{17}$$

where Γ is an external heating rate, and the Λ term represents the effect of collisional losses (hence the n^2 dependence from binary collisions). In an early influential model of the interstellar medium, Γ was the heating rate due to cosmic rays and $n^2\Lambda$ the volumetric losses due to the excitation of CII lines. More generally, one uses the net loss function per unit mass $\mathcal{L}$ as a measure of the departure from adiabatic behaviour. Typically, $\mathcal{L}$ is written as a function of ρ and T (e.g. thermal bremsstrahlung losses are proportional to $\rho^2 T^{1/2}$), but any two thermodynamic variables will do.

To apply the energy equation to stellar interiors, the nature of radiative energy losses must be carefully assessed. If active convection is not taking place, the dominant mode of energy loss is the transport of the radiation energy density. This may seem odd at first glance, because the energy density of a stellar interior is usually completely dominated by the thermal energy of the matter particles; only in the most massive stars does the radiation energy density become comparable. The reason that energy transport from radiation is more effective is that the mean free path for photon scattering is huge compared with any collision mean free path associated with the matter. In other words, the radiation energy leaks out much more rapidly from a stellar interior, and is thence radiated at the stellar surface very much like a blackbody: the energy loss per unit area of surface is σT^4, where σ is the Stefan-Boltzmann constant and T is the surface temperature. ($\sigma = 5.67 \times 10^{-8}$ J s^{-1} m^{-2} K^{-4}.)

A stellar interior is about as close to a perfect thermodynamic equilibrium that one can imagine, but it is not *exactly* perfect. It is, after all, hotter near the core than near the surface. It is this temperature gradient that is responsible for the drift of radiation out of the interior plasma into the surrounding space. We expect that the flux of energy caused by this gradient should be of order $c\lambda\boldsymbol{\nabla}(aT^4)$, where λ is the mean free path for photon scattering. In fact, the precise radiative flux $\boldsymbol{\mathcal{F}}$ is given by

$$\boldsymbol{\mathcal{F}} = -\frac{c\lambda}{3}\boldsymbol{\nabla}(aT^4) \tag{18}$$

the factor of 3 being little more than a directional average. (See Martin Schwarzschild's classic text, *The Structure and Evolution of the Stars,* for a careful derivation.)

The mean free path is given by elementary kinetic theory as

$$\lambda = \frac{1}{n\sigma} \tag{19}$$

where n is the total of scattering particles and σ is an average cross section per particle (not to be confused with Stefan-Boltzmann constant!) It is convenient to work with the mass-related quantities. With μ being a mass per particle, we define the mass density $\rho = \mu n$ and the average cross section per particle $\kappa = \sigma/\mu$, and write

$$\boldsymbol{\mathcal{F}} = -\frac{4acT^3}{3\rho\kappa}\boldsymbol{\nabla}T \tag{20}$$

κ is called the opacity, generally a complicated function of the temperature, density and and abundances. At typical stellar temperatures, an approximation known as a Kramers law in which

$$\kappa \propto \frac{\rho}{T^{3.5}} \tag{21}$$

is often used (see e.g. *Principles of Stellar Evolution,* by D. Clayton, or good old Schwarzschild). At very high temperature however, the scattering is dominated by electron scattering, and κ is a constant. Only the most massive stars come into this regime.

Exercise. Evaluate this constant for the case (a) of a purely hydrogenic gas; (b) a gas in which helium is 10% of the total number of baryons. Answers: (a) 0.397 g cm^{-2}; (b) 0.6435 g cm^{-2}. Note the convenient cgs units! (DATA: Proton mass = 1.6726×10^{-24}g, electron cross section = 6.6526×10^{-25} cm^2, helium mass = 6.64466×10^{-24}g, electron mass = 9.109×10^{-28}g.)

Exercise. In a plasma, the ordinary nonradiative thermal heat is transported by electron Coulomb collisions at a flux (energy per area per time) of $\boldsymbol{F_C} = -\chi T^{5/2}\boldsymbol{\nabla}T$, where χ in cgs units is about 6×10^{-7}. (See Spitzer, *Physics of Fully Ionised Gases.*) Compare the radiative and Coulomb heat fluxes for a plasma at 1 gram per cm^3 and $T = 10^6$K.

In the presence of diffusive radiative losses (or any other sort of diffusive losses), the entropy equation becomes simply

$$nT\frac{Ds}{Dt} = \frac{P}{\gamma - 1}\frac{D\ln P\rho^{-\gamma}}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} \tag{22}$$

with $\boldsymbol{\mathcal{F}}$ given by (20). In radiative equilibrium, the left hand side of this equation is identically zero, so that in a spherical star, the flux magnitude $\mathcal{F}$ is proportional to $1/r^2$. In terms of the star's lumimosity L,

$$F = -\frac{4acT^3}{3\rho\kappa}\frac{dT}{dr} = \frac{L}{4\pi r^2}. \tag{23}$$

1.3 The vector "v dot grad v"

The vector $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ is more complicated than it appears. In Cartesian coordinates, matters are simple: the x component is just $(\boldsymbol{v}\cdot\boldsymbol{\nabla})v_x$, and similar for y, z. But in cylindrical

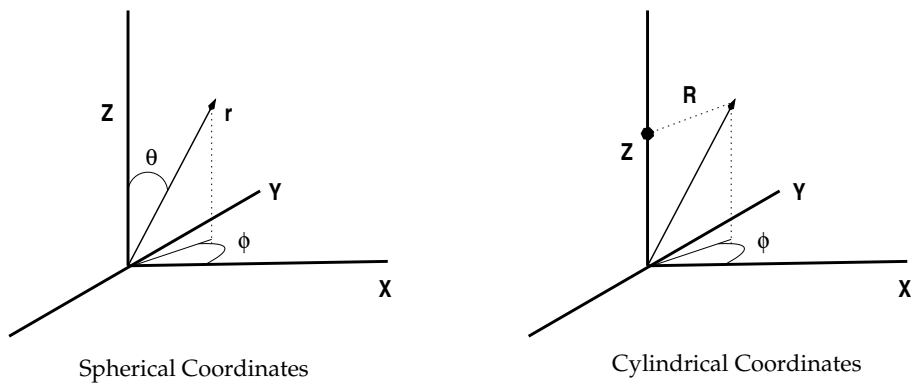

Figure 1: Spherical (r, θ, ϕ) and cylindrical coordinates (R, ϕ, z).

coordinates (R, ϕ, z), the radial R component (say) of this vector is NOT $(\boldsymbol{v}\cdot\boldsymbol{\nabla})v_R$. Rather, we must take care to write

$$(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{\nabla}(v_R\boldsymbol{e_R} + v_\phi\boldsymbol{e_\phi} + v_z\boldsymbol{e_z}) \tag{24}$$

where the $\boldsymbol{e_i}$ are unit vectors in their respective directions. In Cartesian coordinates, these unit vectors would be constant, but in any other coordinate system they change with position. You should be able to show that

$$\frac{\partial \boldsymbol{e_R}}{\partial \phi} = \boldsymbol{e_\phi}, \quad \frac{\partial \boldsymbol{e_\phi}}{\partial \phi} = -\boldsymbol{e_R}, \tag{25}$$

and that there are no other unit vector derivatives in cylindrical coordinates. Thus, the radial component of $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ is

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_R - \frac{v_\phi^2}{R}, \tag{26}$$

and the azimuthal component is

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_\phi + \frac{v_R v_\phi}{R} \tag{27}$$

The extra terms are related to centripetal and Coriolis forces, though more work is needed to extract the latter...a piece of it still remains in the gradient term!

We will use both spherical and cylindrical coordinates throughout this course, as shown in figure [1].

1.4 Rotating Frames

It is often useful to work in a frame rotating at a constant angular velocity Ω, perhaps the frame in which an orbiting planet appears at rest around its star. The same rule that applies to ordinary point mechanics applies here as well: add

$$-2\boldsymbol{\Omega}\times\boldsymbol{v} + R\Omega^2\boldsymbol{e_R} \tag{28}$$

to the applied forces operating on a fluid element (the right side of the equation). The first term is the Coriolis force, the second is the centrifugal force, $\boldsymbol{\Omega}$ is in the vertical direction, and all velocities are measured relative to the rotating frame of reference.

1.4.1 The Taylor-Proudman Theorem

Suppose that in the rotating frame the fluid velocity $\boldsymbol{v}$ is much less than $R\Omega$. Suppose further that the density is constant and the pressure term $(1/\rho)\boldsymbol{\nabla}P$ is an exact gradient. Then, the sum of the enthalpy, gravity, and centrifugal terms is expressible as a gradient, say $\boldsymbol{\nabla}\mathcal{H}$, and the steady-state fluid equation is simply

$$2\boldsymbol{\Omega}\times\boldsymbol{v}=\boldsymbol{\nabla}\mathcal{H} \tag{29}$$

Since $\boldsymbol{\Omega}$ lies along the z axis, the left hand side has no z component, so neither does the right. But that means that $\mathcal{H}$ is independent of z. Then the radial and azimuthal velocity components are independent of z as well, that is, the flow is constant on cylinders! In the case of constant density, the mass conservation equation is $\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$. But the above equations of motion imply

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0 \tag{30}$$

so that $\partial v_z/\partial z=0$, and v_z is also independent of z. If there is no v_z at the upper or lower boundaries, v_z vanishes everywhere, and the motion is two dimensional! The fact that small motions in rotating systems are often independent of height is called the Taylor-Proudman theorem, and you will often hear "Taylor Columns" referred to in fluid mechanics talks. Now you now where they come from.

It may seem that our arguments go through even if the density is not constant when $P=P(\rho)$, since it is still possible to form an H function. But now mass conservation cannot be so trivially satisfied, because of the $\boldsymbol{v}\cdot\boldsymbol{\nabla}\rho$ term. With only three quantities (v_x,v_y,ρ) we cannot satisfy four equations, so there must be a v_z present, and the flow is no longer so simple.

1.5 The Indirect Potential

A common problem of interest is a gaseous disc around a dominant central body M_* in which a secondary body M is embedded (or otherwise exerts an external gravitational force). With M_* as the origin, the acceleration on a fluid element at location $\boldsymbol{R}$ is NOT

$$\frac{d^2\boldsymbol{R}}{dt^2}=-\frac{GM_*\boldsymbol{R}}{R^3}-\frac{GM\boldsymbol{r}}{r^3}. \quad \text{(NO!)} \tag{31}$$

(See figure [2] for vector definitions.) This is the acceleration that would be measured in an *inertial frame*. The right side of this equation is actually $d^2\boldsymbol{X}/dt^2$, the acceleration at the same location from the point of view of the center-of-mass of M_* and M. Sitting on top of the central star, one is no longer in an inertial frame: the star itself is accelerating. The correct acceleration in the star's frame is given by adding a term $-GM\boldsymbol{R}_M/R_M^3$ to the right side, i.e. minus the acceleration of the star itself. The correct equation is

$$\frac{d^2\boldsymbol{R}}{dt^2}=-\frac{GM_*\boldsymbol{R}}{R^3}-\frac{GM\boldsymbol{r}}{r^3}-\frac{GM\boldsymbol{R_M}}{R_M^3} \quad \text{(YES!)} \tag{32}$$

The last term is known as the indirect term, derivable from the indirect potential $-GM/R_M$. The total perturbing potential due to M is then

$$\Phi=-\frac{GM}{r}-\frac{GM}{R_M}=-\frac{GM}{|R-R_M|}-\frac{GM}{R_M} \tag{33}$$

Note that if $R_M\gg r$, then the indirect term may be neglected; it is important when R_M is comparable to r.

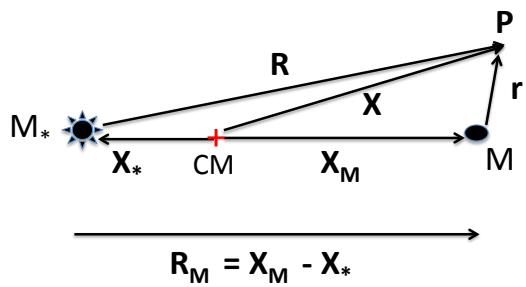

Figure 2: Central star M_* surrounded by a disc with a perturbing mass M. CM denotes the center-of-mass; all X vectors are measured relative to this point. The effect of working in the convenient but noninertial frame centered on M_* can be included by adding an "indirect" potential to the equations of motion. See text.

1.6 Local Equations in Discs and Stars

It is possible to simplify the dynamics in a disc or star by working in a small neighbourhood around a point. Often, this is all that is necessary to reveal the critical dynamics of a flow, and simplifying the problem mathematically allows for deeper physical understanding.

Consider the case of a Keplerian disc, gas flow in a central potential $-GM/r$. In a frame rotating at the angular velocity $\Omega(R_0) \equiv \Omega_0$, where R_0 is particular cylindrical location, the added rotational force terms are $-2\boldsymbol{\Omega_0} \times \boldsymbol{v}$ plus $R\Omega_0^2 \boldsymbol{e_R}$. The radial force due to gravity may be written as $-R\Omega^2(R)$, where $\Omega(R)$ is the angular velocity of the gas *as a function of* R. At $R = R_0$ there is a force balance, but at a slightly different location $R = R_0 + x$ (x small), there is an imbalance given by $-x d\Omega^2/d\ln R = 3\Omega_0^2 x$:

$$(R_0 + x)[\Omega^2(R_0) - \Omega^2(R_0 + x)] \simeq -x\frac{d\Omega^2}{d\ln R}.$$

Dropping the "0" subscript, the local equation of motion in a Keplerian disc is given by

$$\frac{D\boldsymbol{v}}{Dt} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + 2\boldsymbol{\Omega}\times\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P + 3\Omega^2 x\boldsymbol{e_R} - z\Omega^2\boldsymbol{e_z} \tag{34}$$

Exercise. Where did the $-z\Omega^2$ term come from?

In taking the Lagrangian derivative, the $1/R$ curvature terms may be ignored, since we are working in a very small neighbourhood of a patch of disc, in essence taking the limit

of $R \to \infty$ while Ω remains finite. Then we may use ordinary Cartesian coordinates with dx, dy, dz replacing $dR, Rd\phi, dz$:

$$\frac{Dv_x}{Dt} - 2\Omega v_y = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 3\Omega^2 x \tag{35}$$

$$\frac{Dv_y}{Dt} + 2\Omega v_x = -\frac{1}{\rho}\frac{\partial P}{\partial y} \tag{36}$$

$$\frac{Dv_z}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - \Omega^2 z \tag{37}$$

Local forces (perhaps magnetic) may be added to the right side at will. These equations are sometimes referred to as the *Hill system.*

Exercise. The indirect potential should be ignored if we wish to study the effects of an embedded point mass in a disc in the local approximation. Why?

A similar, but somewhat simpler reduction is used for spherical stars or planets. Here, the centrifgual terms in $R\Omega^2$ are generally negligible. Now, we may identify dx, dy, dz with $rd\theta, r\sin\theta d\phi, dr$. The "$\beta$–plane" equations for motion on a spherical surface are

$$\frac{Dv_x}{Dt} - fv_y = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{38}$$

$$\frac{Dv_y}{Dt} + fv_x = -\frac{1}{\rho}\frac{\partial P}{\partial y} \tag{39}$$

where $f = 2\Omega\cos\theta$ is the "Coriolis parameter." Note that $f = 0$ at the equator.

Exercise. Derive these equations.

An interesting two-dimensional oceanographic application is to add a tidal potential force on the right and use $P = \rho g\zeta$, where ρ is the (here constant) density of water and $\zeta = \zeta(x, y, t)$ is the (varying) height of the sea, *relative* to the equilibrium sea level.

Exercise. Show that the equation of mass conservation for this problem is

$$\frac{\partial\zeta}{\partial t} + \frac{\partial(hv_x)}{\partial x} + \frac{\partial(hv_y)}{\partial y} = 0 \tag{40}$$

where h is the *total* height of the sea. (Note that to leading order, h can be replaced by undisturbed sea depth $h_0(x, y)$. In what sense do I mean "leading order?") The set of three equations, with the additional tidal forcing terms, are known as the Laplace tidal equations.

1.7 Manipulating the Fluid Equations

For a particular problem, working in cylindrical or spherical cooridinates is often the most convenient, but for proving general theorems or identities, Cartesian coordinates are usually the simplest to use. There is a formalism that makes working with the fluid equations much easier in this case.

As in our brief introductory discussion of viscosity, we will let the index i, j, or k will represent Cartesian component x, y, or z. Hence v_i means the ith component of v, which may any of the three depending upon what value i is chosen. So v_i is a way to write $\boldsymbol{v}$. The gradient operator $\boldsymbol{\nabla}$ is written ∂_i, in a way that should be self-explanatory.

As before, if an index appears twice, it is understood that it is to be summed over all the values x, y, and z. Hence

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_i B_i = A_x B_x + A_y B_y + A_z B_z, \tag{41}$$

and

$$(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = (v_i \partial_i) v_j \tag{42}$$

In this last example i is a dummy index, and the vector component is represented by j. The dynamical equation of motion in this notation is

$$\rho[\partial_t + (v_i \partial_i)]v_j = -\partial_j P - \rho \partial_j \Phi \tag{43}$$

Sometimes the "rot" (or "curl") operator is needed. For this, we intoduce the Levi-Civita symbol ϵ^{ijk}. It is defined as follows:

- If any of the i, j, or k are equal to one another, then $\epsilon^{ijk} = 0$.
- If $ijk = 123$, 231, or 312, the so-called even permutations of 123, then $\epsilon^{ijk} = +1$.
- If $ijk = 132$, 213, or 321, the so-called odd permutations of 123, then $\epsilon^{ijk} = -1$.

The reader should be able to convince him(her)self that

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \epsilon^{ijk} \partial_i A_j \tag{44}$$

Here, the vector component is represented by the index k. Don't forget to sum over i and j! ϵ^{ijk} is used in the ordinary cross product as well:

$$\boldsymbol{A} \times \boldsymbol{B} = \epsilon^{ijk} A_i B_j. \tag{45}$$

Notice that

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \epsilon^{ijk} A_k B_i C_j \tag{46}$$

which proves that any even permutation of the vectors on the left side of the equation must give the same value, and an odd rearrangement gives the same value with the opposite sign.

A double cross product looks complicated:

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \epsilon^{lkm} A_l (\epsilon^{ijk} B_i C_j) = \epsilon^{mlk} \epsilon^{ijk} A_l B_i C_j. \tag{47}$$

The last equality follows because mlk is an even permutation of lkm. This looks unpleasant, but fortunately there is an identity that saves us:

$$\epsilon^{mlk} \epsilon^{ijk} = \delta_{mi} \delta_{lj} - \delta_{mj} \delta_{li} \tag{48}$$

where δ_{ij} is the Kronecker delta function (equal to zero if i and j are different, unity if they are the same). The proof of this is left as an exercise for the reader, who should be convinced after a few simple explicit examples. With this identity, our double cross product becomes

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = B_m A_j C_j - C_m A_i B_i = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}). \tag{49}$$

Our final example is to derive an expression for

$$\boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \epsilon^{ijk} A_i (\epsilon^{lmj} \partial_l B_m) = \epsilon^{kij} \epsilon^{lmj} (A_i \partial_l B_m) \tag{50}$$

Using our identity (48), this becomes

$$(\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})(A_i\partial_l B_m) = A_i\partial_k B_i - A_i\partial_i B_k = A_i\partial_k B_i - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{51}$$

One consequence of this result is a representation of $A_i\partial_k B_i$ in any coordinate system:

$$A_i\partial_k B_i = \boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{52}$$

Another particular important application of (51) is to the Lorentz force expression

$$(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} = -\frac{1}{2}\boldsymbol{\nabla}B^2 + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{53}$$

The first term on the right side has the form of a magnetic pressure gradient; the second behaves like a tension force. It depends on the derivative of $\boldsymbol{B}$ along its length, and if the magnitude of $\boldsymbol{B}$ remains fixed, the force must be perpendicular to $\boldsymbol{B}$ itself. The effect of this tension force is profound, allowing a magnetized gas to support shear waves (known as Alfvén waves) that ordinarily do not exist in a fluid.

1.8 The Conservation of Vorticity

A quantity of great interest to fluid dynamicists is the *vorticity*, $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v}$. We will see that it is intimately related to the angular-momentum-like circulation element $\boldsymbol{v}\cdot\boldsymbol{dl}$ integrated around a closed fluid loop. Under some interesting circumstances, this circulation integral is conserved.

We start with the following identity, which follows immediately from the results of the previous section:

$$\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) = \frac{1}{2}\boldsymbol{\nabla}v^2 - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \tag{54}$$

Using this result to replace $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ in the inviscid, umagnetised dynamical equation of motion results in

$$\frac{\partial\boldsymbol{v}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}v^2 - \boldsymbol{v}\times\boldsymbol{\omega} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi. \tag{55}$$

Exercise. If $P = P(\rho)$ and $d\mathcal{H} = dP/\rho$, show that

$$\frac{v^2}{2} + \mathcal{H} + \Phi$$

is constant along a velocity flow streamline. This is called the Bernoulli constant, B. B need not be the same constant on every streamline!

If we take the curl of equation (55), and remember that the curl of the gradient vanishes, we find

$$\frac{\partial\boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) = \frac{1}{\rho^2}(\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}P) \tag{56}$$

Let us once again consider the case where either ρ is constant, or when P is a function only of ρ. Then the right hand side vanishes, and:

$$\frac{\partial\boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) = 0. \tag{57}$$

To interpret this physically, consider a closed curve, a loop, frozen into the fluid. The integral $\int \boldsymbol{v} \cdot \boldsymbol{dl}$ around the loop is by Stokes' theorem $\int \boldsymbol{\omega} \cdot \boldsymbol{dA}$, taken over an area that is bounded by the loop. This is the vorticity flux. How does the vorticity flux change with time as the fluid evolves? The answer is contained within (57).

We consider more generally an equation of the form

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) + \boldsymbol{\nabla}\Phi \tag{58}$$

where Φ is a potential function. The curl of this equation leads back to equation (57) for the special case $\boldsymbol{A} = \boldsymbol{v}$, but we retain generality here. Expanding the double cross product on the right and regrouping leads to

$$\frac{DA_i}{Dt} = v_j \partial_i A_j + \partial_i \Phi \tag{59}$$

where D/Dt is the usual Lagrangian derivative and we have switched to index notation. Next, consider the change in the line integral of the vector field $\boldsymbol{A}$ over a closed loop moving with the fluid:

$$\frac{D}{Dt} \int \boldsymbol{A} \cdot \boldsymbol{dl} = \int \left[\frac{D\boldsymbol{A}}{Dt} \cdot \boldsymbol{dl} + \boldsymbol{A} \cdot \frac{D\boldsymbol{dl}}{Dt} \right] \tag{60}$$

Notice the strange derivative of the line element $\boldsymbol{dl}$! The Lagrangian derivative of the line element as it moves through the fluid is just the difference between the velocity field at the two endpoints of the segment $\boldsymbol{dl}$:

$$\frac{D\boldsymbol{dl}}{Dt} = (\boldsymbol{dl} \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{61}$$

or

$$\frac{Ddl_j}{Dt} = dl_i \partial_i v_j \tag{62}$$

From equation (59)

$$\frac{D\boldsymbol{A}}{Dt} \cdot \boldsymbol{dl} = dl_i \; v_j \partial_i A_j + dl_i \partial_i \Phi, \tag{63}$$

and we have just seen that

$$\boldsymbol{A} \cdot \frac{D\boldsymbol{dl}}{Dt} = A_j dl_i \partial_i v_j. \tag{64}$$

Adding these last two equations gives

$$\frac{D}{Dt}(\boldsymbol{A} \cdot \boldsymbol{dl}) = dl_i \partial_i (\Phi + v_j A_j) \tag{65}$$

In equation (60) we thus have a perfect gradient integrated over a closed curve, hence the integral must vanish. The line integral $\int \boldsymbol{A} \cdot \boldsymbol{dl}$ is conserved with the fluid. In particular when $\boldsymbol{A} = \boldsymbol{v}$, the velocity circulation integral along with the vorticity flux surface integral are conserved in the Lagrangian sense, moving with the fluid. We shall see very soon that the same is true for the magnetic vector potential and the magnetic flux.

The fact that the integral $\int \boldsymbol{v} \cdot \boldsymbol{dl}$ around any closed curve embedded in the fluid remains constant as the fluid flows is known as vorticity conservation. Another way to say the same thing is that the field lines of vorticity $\boldsymbol{\omega}$ are "frozen" into the fluid. Once again, this is not a general fluid result, but depends upon the fact that either ρ is constant or that P and ρ

are directly functionally related. If the pressure gradient can push with the same force along a surface of varying inertial response, vorticity will surely be generated.

With the help of our $\epsilon^{ijk}\epsilon^{lmk}$ identity and just a little work, it is straightforward to show that

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) = 0 \tag{66}$$

is the same as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = +(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{\omega}\boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{67}$$

Mass conservation may be written

$$\frac{D\ln\rho}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{v}, \tag{68}$$

so that our equation becomes

$$\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega}\frac{D\ln\rho}{Dt} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v}, \tag{69}$$

or

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{1}{\rho}(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} \tag{70}$$

In strictly two-dimensional flow, this is a very powerful constraint. Then ω has only a z component, and the right side must vanish. The density ρ may be replaced by the surface density Σ on the left side of the equation: multiply by ρ^2, expand the D/Dt derivative, integrate over height, and refold the D/Dt derivative. We find that

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\Sigma}\right) = 0 \tag{71}$$

This is known as the conservation of potential vorticity. In this case, potential vorticity (PV in the parlance) labels fluid elements. PV is extremely useful in the study of two-dimensional turbulence, and in studying long wavelength wave propagation in planetary atmospheres.

Exercise. Consider rotational flow, with the velocity $\boldsymbol{v}$ having only a ϕ component v_ϕ. In general, v_ϕ could depend upon R and z, but show that if vorticity conservation holds, under steady conditions v_ϕ cannot depend upon z. This is known as *von Zeippel's* theorem.

The nation that controls magnetism will control the universe.

— Dick Tracy, created by Chester Gould

2 Magnetohydrodynamics (MHD)

2.1 Magnetic Forces

Astrophysical gases are almost always at least partially ionized. This is not too surprising: a glass of distilled water is ionized at the level of one part in 10^7, and salty sea water is much more ionized: it is a very good conductor. A medium can be almost entirely neutral and still behave like a good conductor. All but the coolest and densest astrophysical gases (e.g., protostellar discs) are electrodynamically active.

The Lorentz force per unit volume in the gas is

$$\boldsymbol{F} = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} \tag{72}$$

where ρ_e is the charge density, $\boldsymbol{E}$ is the electric field, $\boldsymbol{J}$ is the current density, and $\boldsymbol{B}$ is the magnetic field. The gases of interest here are all electrically neutral, so that $\rho_e = 0$. This means that the only part of the Lorentz force that affects the gas is the magnetic part.

We have already encountered the Lorentz force in our discussion of the equation of motion for a magnetized gas:

$$\boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} \tag{73}$$

With our $\epsilon^{ijk}\epsilon^{lmk}$ identity, we have already seen that

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = -\frac{1}{2}\boldsymbol{\nabla} B^2 + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{B} \tag{74}$$

Thus, the dynamical equation of motion for a magnetized gas is

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) - \rho \boldsymbol{\nabla}\Phi + \left(\frac{\boldsymbol{B}}{\mu_0}\cdot\boldsymbol{\nabla}\right)\boldsymbol{B} \tag{75}$$

The first magnetic term on the right clearly behaves like a sort of pressure. Magnetic fields lines of force do not like to be squeezed any more than gas molecules do.

The $(\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{B}$ term is less obvious. It corresponds to a sort of magnetic tension. Notice that it vanishes when the magnetic field does not change along its own direction. On the other hand, when there are such changes, the resulting force acts in the direction of restoring the field line back to an unstretched position. In fact, this can be made quantitative: there is a magnetic analogue to waves propagating along an ordinary string that is under tension. In the case of "magnetic strings," these waves are called Alfvén waves.

2.2 Induction Equation

Having introduced the magnetic field, we need to know how it evolves as the flow changes. The magnetic field adds one more variable to our problem (well, three actually, since there are three components of $\boldsymbol{B}$), so we need more equations. The motion of the gas causes the charged particles to move, the ions and electrons respond differently to the applied forces, currents form, these currents in turn generate new fields that affect the currents all over that change the fields ... Help. It seems like a complicated mess!

Fortunately there is a great simplifying principle to save us: in a perfect conductor, the electric field vanishes. Actually, what we should say is that in the rest frame of the conductor, the electric field locally vanishes. In a frame in which the conductor (in our case a fluid element of conducting gas) moves, the total Lorentz force (not the electric field) must vanish. In other words,

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0. \tag{76}$$

So even though we have assumed conditions for charge neutrality, there must be an electric field: $-\boldsymbol{v}\times\boldsymbol{B}$. But if the divergence of this electric field does not vanish, then there must be a local charge density, and charge neutrality cannot hold, which looks like a contradiction. Well, maybe it just turns out that $\boldsymbol{\nabla}\cdot(\boldsymbol{v}\times\boldsymbol{B}) = 0$. Guess what? The divergence doesn't vanish. In a moment, we'll come back and explain why this is not really a contradiction after all, but for the time being let us nervously continue.

Faraday's law of induction is

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E} \tag{77}$$

and with $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$, this becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{78}$$

This is the extra set of equations we need to determine the magnetic field. By knowing how the spatial gradients of $\boldsymbol{B}$ are behaving, we may compute how the field evolves in time, thanks to the powerful constraint that the Lorentz force on the charge carriers must vanish.

2.3 Self-consistency

Why don't we have a contradiction with the fact that $\boldsymbol{\nabla} \cdot \boldsymbol{E}$ is not zero? The answer is that while not zero, it is in fact small. Small?? That answer is not good enough. How small? Well, of order v^2/c^2 (c is the speed of light), which, as we will see, *is precisely of the same order as the displacement current that was also neglected.*

To estimate $\boldsymbol{\nabla} \cdot (\boldsymbol{v} \times \boldsymbol{B})$, assume that any magnetic field gradients are as large as they can be (of order $\mu_0 J$), and that J is also as large as it can be, of order the ion charge density times v, $\rho_i v$ (it could be smaller since it is proportional to the difference between ion and electron velocities). Then

$$\boldsymbol{\nabla}\cdot(\boldsymbol{v}\times\boldsymbol{B}) \sim v\mu_0 J \sim \frac{\rho_i}{\epsilon_0}\frac{v^2}{c^2}. \tag{79}$$

That answer, that the divergence of the electric field is of order v^2/c^2 times the ion charge density, *is* good enough. Not only is it permitted to ignore the divergence of the electric field, it is required! We have already not included the displacement current, and this too is

a correction of order v^2/c^2. In this case, if L is a characteristic length and $\partial/\partial t \sim v/L$, then

$$\epsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t} \sim \epsilon_0\mu_0\frac{vE}{L} \sim \epsilon_0\mu_0\frac{v^2B}{L} \sim \epsilon_0\mu_0^2 v^2 J \tag{80}$$

which is indeed of order $(v^2/c^2)\mu_0 J$. Corrections of order v^2/c^2 are relativistic, and we *must* ignore them to be self-consistently *nonrelativistic*!

Notice something quite remarkable: the magnetic field satisfies the same equation as the vorticity. In particular, equation (78) can be recast in the form of equation (58), by "uncurling" it! That means everything we learned about vorticity, in particular that it is frozen in to the fluid, also holds for the magnetic field. Magnetic flux, $\int \boldsymbol{B}\cdot d\boldsymbol{A}$, is conserved as the area moves with the fluid. But unlike the case of vorticity conservation, which depended upon a restrictive relationship between P and ρ, magnetic flux conservation depends only upon there being no dissipation (i.e., electrical resistance) in the gas. This is generally an excellent approximation.

A Summary of the Dissipationless Equations of Motion

From now on, we shall drop the subscript "0" on μ_0, and write μ.

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0 \tag{81}$$

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v} = -\boldsymbol{\nabla}\left(P + \frac{B^2}{2\mu}\right) - \rho\boldsymbol{\nabla}\Phi + \frac{1}{\mu}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{82}$$

$$\frac{P}{\gamma-1}\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\ln P\rho^{-\gamma} = 0 \tag{83}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}) \tag{84}$$

2.4 MHD Fundamentals

Now then, Dimitri ...
— President Merkin Muffley to Premier Kissoff, Dr. Strangelove

In this section, a detailed derivation of the fundamental MHD equations is presented. The discussion will be more technical here than in most of the rest of the course, but it is very important to see how the basic governing equations of the subject arise, and much of this material is not so easy to find outside of specialized treatments. I hope the reader will have the patience to read carefully through this section, but it may be skipped the first time through without loss of continuity.

In astrophysics, we are very often interested in the MHD behaviour of a gas that is almost entirely neutral but is still a good conductor. This may seem like contradictory, since a neutral gas has no charge carriers, but the key word is "almost." Even a very small population of charge carriers will make the gas magnetized and highly conducting, as we will shortly see.

A typical environment is a gas cloud consisting of neutral particles (predominantly H_2 molecules), electrons, and ions. Each species (denoted by subscript s) is separately conserved, and obeys the mass conservation equation

$$\frac{\partial \rho_s}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho_s \boldsymbol{v}_s) = 0 \tag{85}$$

where ρ_s is the mass density for species s and $\boldsymbol{v}_s$ is the velocity. The symbols of the flow quantities (e.g. $\boldsymbol{v}$, ρ, etc.) for the dominant neutral species will henceforth be presented without subscripts.

So far, everything is simple. The dynamical equations become more coupled, however, since we need to include interactions between the different species. The dynamical equation for the neutral particles is

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla}\Phi - \boldsymbol{p_{nI}} - \boldsymbol{p_{ne}} \tag{86}$$

where P is the pressure of the neutrals, Φ the gravitational potential and $\boldsymbol{p_{nI}}$ ($\boldsymbol{p_{ne}}$) is the momentum exchange rate between the neutrals and the ions (electrons).

The ion equation is

$$\rho_I \frac{\partial \boldsymbol{v_I}}{\partial t} + \rho_I(\boldsymbol{v_I}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v_I} = eZn_I(\boldsymbol{E} + \boldsymbol{v_I} \times \boldsymbol{B}) - \boldsymbol{\nabla} P_I - \rho_I \boldsymbol{\nabla}\Phi - \boldsymbol{p_{In}} \tag{87}$$

and the electron equation is

$$\rho_e \frac{\partial \boldsymbol{v_e}}{\partial t} + \rho_e(\boldsymbol{v_e}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v_e} = -en_e(\boldsymbol{E} + \boldsymbol{v_e} \times \boldsymbol{B}) - \boldsymbol{\nabla} P_e - \rho_e \boldsymbol{\nabla}\Phi - \boldsymbol{p_{en}} \tag{88}$$

The subscript I (e) refers to the ions (electrons). When not in a subscript but used in an equation, e is the fundamental charge of a proton, i.e. *it is always positive.* The electron charge is *always* $-e$. The momentum exchange rate $\boldsymbol{p_{In}}$ is precisely $-\boldsymbol{p_{nI}}$, and the same holds for $\boldsymbol{p_{en}}$. (Why?) The quantity Z is the mean charge per ion, n is a number density, and the fluid is neutral in bulk, $eZn_I = en_e$.

The key point is that for the charge carriers, all terms proportional to the mass densities ρ_I and ρ_e are small compared with the Lorentz force and momentum exchange rates. Hence, to a very good approximation,

$$0 = eZn_I(\boldsymbol{E} + \boldsymbol{v_I} \times \boldsymbol{B}) - \boldsymbol{p_{In}} \tag{89}$$

$$0 = -en_e(\boldsymbol{E} + \boldsymbol{v_e} \times \boldsymbol{B}) - \boldsymbol{p_{en}} \tag{90}$$

Adding these two equations and using bulk neutrality leads to ,

$$0 = eZn_I(\boldsymbol{v_I} - \boldsymbol{v_e}) \times \boldsymbol{B} - \boldsymbol{p_{In}} - \boldsymbol{p_{en}} \tag{91}$$

But $eZn_I(\boldsymbol{v_I} - \boldsymbol{v_e})$ is just the current density $\boldsymbol{J}$, so that

$$\boldsymbol{p_{In}} + \boldsymbol{p_{en}} = \boldsymbol{J} \times \boldsymbol{B} \tag{92}$$

Using this in the neutral equation leads to

$$\frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla}\Phi + \boldsymbol{J} \times \boldsymbol{B} \tag{93}$$

Remarkably, the net Lorentz force appears unmodified in the equation for the *neutrals*.

For a sparsely ionized fluid, departures from ideal MHD appear mainly in the induction equation. To relate $\boldsymbol{E}$ and $\boldsymbol{B}$ it is best to use the electron force equation, since the ions may be more closely locked to the neutrals. Thus

$$\boldsymbol{E} = -\boldsymbol{v_e} \times \boldsymbol{B} - \frac{\boldsymbol{p_{en}}}{en_e} = -[\boldsymbol{v} + (\boldsymbol{v_e} - \boldsymbol{v_I}) + (\boldsymbol{v_I} - \boldsymbol{v})] \times \boldsymbol{B} - \frac{\boldsymbol{p_{en}}}{en_e} \tag{94}$$

Now matters start to get very detailed. I present these details in the following section, but for purposes of this course I view this material as entirely optional. Having made the details available to you, however, I feel free to make a quick summary of the results, leaving it for you to read the next section if you wish more explanation.

The term $\boldsymbol{v_e} - \boldsymbol{v_I}$ is $-\boldsymbol{J}/en_e$.

The term $\boldsymbol{v_I} - \boldsymbol{v}$ is related to $\boldsymbol{p_{In}}$ by an equation of the form

$$\boldsymbol{p_{In}} = \gamma\rho\rho_I(\boldsymbol{v_I} - \boldsymbol{v})$$

where γ is a coefficient that may be calculated from knowledge of the interaction cross sections. (See equation (104).) But $\boldsymbol{p_{In}}$ is simply related to $\boldsymbol{J} \times \boldsymbol{B}$ from equation (92), because $\boldsymbol{p_{In}}$ is in fact dominant over $\boldsymbol{p_{en}}$. Ultimately, the reason is that the ions are more massive than the electrons.

The final term proportional to $\boldsymbol{p_{en}}$ represents ohmic dissipation. We denote the electrical conductivity by σ_{cond}.

Putting all of this together leads to the full induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \left[\boldsymbol{v} \times \boldsymbol{B} - \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}}{\mu_0 e n_e} + \frac{[(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}] \times \boldsymbol{B}}{\mu_0 \gamma \rho \rho_I} - \frac{\boldsymbol{\nabla} \times \boldsymbol{B}}{\mu_o \sigma_{cond}}\right] \tag{95}$$

The Details....

Let us examine matters a little more closely. (Please don't worry about every last detail. My purpose here is to give you a feeling for all that goes into a calculation like this, and to be able to understand the nomenclature that you will encounter in the literature. You don't have to become an expert in the minutiae of interstellar kinetic theory for this course!) $\boldsymbol{p_{nI}}$ takes the form

$$\boldsymbol{p_{nI}} = n\mu_{nI}(\boldsymbol{v} - \boldsymbol{v_I})\nu_{nI} \tag{96}$$

where n is the number density of neutrals, and μ_{nI} is the reduced mass of an ion–neutral particle pair,

$$\mu_{nI} \equiv \frac{m_I m_n}{m_I + m_n}, \tag{97}$$

m_I and m_n being the ion and neutral mass respectively. ν_{nI} is the collision frequency of a neutral with a population of ions,

$$\nu_{nI} = n_I \langle \sigma_{nI} w_{nI} \rangle. \tag{98}$$

In equation (98), n_I is the number density of ions, σ_{nI} is the cross section for neutral-ion collisions, and w_{nI} is the relative velocity between a neutral particle and an ion. The angle brackets represent an average over all possible relative velocities in the thermal population of particles. Notice that equation (96) has the dimensions of a force per unit volume, and

that it is proportional to the velocity difference between the species: if there is no difference in their mean velocities, two population of particles cannot exchange momentum.

Why does the reduced mass μ_{nI} appear? Because the reduced mass *always* appears in any interaction between two individual particles: in the center of mass frame the equations reduce to a single particle equation with the particle mass equal to the reduced mass. In an elastic one-dimensional collision, for example, if v is initial relative velocity of the two interacting particles, then the momentum exchange is $2\mu_{12}v$, where μ_{12} is the reduced mass. (Show this.)

For neutral-ion scattering, we may approximate the cross section σ_{nI} to be geometrical, which means that the quantity in angle brackets will be proportional to $\mu_{nI}^{-1/2}$. The order of the subscripts has no particular significance in either the cross section σ_{nI}, reduced mass μ_{nI}, or relative velocity w_{nI}. But ν_{In} does differ from ν_{nI}: the former is proportional to the neutral density n, the latter to the ion density n_I.

Putting all these definitions together gives

$$\boldsymbol{p_{nI}} = nn_I\mu_{nI}\langle\sigma_{nI}w_{nI}\rangle(\boldsymbol{v} - \boldsymbol{v_I}) \tag{99}$$

In accordance with Newton's third law, this is symmetric with respect to the interchange $n \leftrightarrow I$, except for a change in sign, $\boldsymbol{p_{nI}} = -\boldsymbol{p_{In}}$. All of these considerations hold, of course, for electron-neutral scattering as well. Explicitly, we have

$$\boldsymbol{p_{ne}} = nn_e\mu_{ne}\langle\sigma_{ne}w_{ne}\rangle(\boldsymbol{v} - \boldsymbol{v_e}) \simeq nn_em_e\langle\sigma_{ne}w_{ne}\rangle(\boldsymbol{v} - \boldsymbol{v_e}). \tag{100}$$

The gas is assumed to be locally neutral, so that $n_e = Zn_i$ where Z is the number of ionizations per ion particle. In a weakly ionized gas, $Z = 1$. The reduced mass μ_{ne} is very nearly equal to the electron mass m_e. The collision rates are given by (see Draine, Roberge, & Dalgarno 1983 ApJ 264, 485 for yet more details) (note, cgs units!):

$$\langle\sigma_{nI}w_{nI}\rangle = 1.9 \times 10^{-9}\ \mathrm{cm}^3\ \mathrm{s}^{-1} \tag{101}$$

$$\langle\sigma_{ne}w_{ne}\rangle = 10^{-15}\ (128kT/9\pi m_e)^{1/2} = 8.3 \times 10^{-10}T^{1/2}\ \mathrm{cm}^3\ \mathrm{s}^{-1} \tag{102}$$

The electron-neutral collision rate is just the ion geometric cross section times an electron thermal velocity. (The peculiar factor of $(128/9\pi)^{1/2}$ is a detail of the averaging procedure.) But the ion-neutral collision rate is temperature independent, much more beholden to long range induced dipole interactions, and significantly enhanced relative to a geometrical cross section assumption. Even if the ion-neutral rate were determined only by a geometrical cross section, $|\boldsymbol{p_{nI}}|$ would exceed $|\boldsymbol{p_{ne}}|$ by a factor of order $(m_e/\mu_{nI})^{1/2}$. In fact, the dipole enhancement of the ion-neutral cross section makes this factor larger still[2].

In the astrophysical literature, it is common to write the ion-neutral momentum coupling in the form

$$\boldsymbol{p_{In}} = \rho\rho_I\gamma(\boldsymbol{v_I} - \boldsymbol{v}), \tag{103}$$

where γ is the so-called *drag coefficient*,

$$\gamma \equiv \frac{\langle\sigma_{nI}w_{nI}\rangle}{m_I + m_n} \tag{104}$$

[2] I should be a little bit more careful. The statement that $|\boldsymbol{p_{ne}}|$ is larger than $|\boldsymbol{p_{nI}}|$ by a factor of $(m_e/\mu_{nI})^{1/2}$ assumes that the velocity differences $\boldsymbol{v} - \boldsymbol{v_e}$ and $\boldsymbol{v} - \boldsymbol{v_I}$ do not introduce any mass dependencies, which is generally true.

and we will use this notation from here on. Numerically, $\gamma = 3 \times 10^{13}$ cm^3 s^{-1} g^{-1} for astrophysical mixtures (Draine, Roberge, & Dalgarno 1983).

We come next to the ions and electrons. The dynamical equations for the ions and electrons are

$$\rho_I \frac{\partial \boldsymbol{v_I}}{\partial t} + \rho_I \boldsymbol{v_I} \cdot \boldsymbol{\nabla} \boldsymbol{v_I} = -\boldsymbol{\nabla} P_I - \rho_I \boldsymbol{\nabla} \Phi + Z e n_I \left(\boldsymbol{E} + \boldsymbol{v_I} \times \boldsymbol{B} \right) - \boldsymbol{p_{In}} \tag{105}$$

and

$$\rho_e \frac{\partial \boldsymbol{v_e}}{\partial t} + \rho_e \boldsymbol{v_e} \cdot \boldsymbol{\nabla} \boldsymbol{v_e} = -\boldsymbol{\nabla} P_e - \rho_e \boldsymbol{\nabla} \Phi - e n_e \left(\boldsymbol{E} + \boldsymbol{v_e} \times \boldsymbol{B} \right) - \boldsymbol{p_{en}}, \tag{106}$$

respectively. e will *always* denote the *positive* charge of a proton, the absolute value of the electron charge, 1.602×10^{-19} Coulombs or 4.803×10^{-10} esu.[3] For a weakly ionized gas, the Lorentz force and collisional terms dominate in each of the latter two equations. Comparison of the magnetic and inertial forces, for example, shows that the latter are smaller than the former by the ratio of the proton or electron gyroperiod to a macroscopic flow crossing time. Thus, to an excellent degree of approximation,

$$Z e n_I \left(\boldsymbol{E} + \boldsymbol{v_I} \times \boldsymbol{B} \right) - \boldsymbol{p_{In}} = 0, \tag{107}$$

and

$$- e n_e \left(\boldsymbol{E} + \boldsymbol{v_e} \times \boldsymbol{B} \right) - \boldsymbol{p_{en}} = 0. \tag{108}$$

The sum of these two equations gives

$$\boldsymbol{J} \times \boldsymbol{B} = \boldsymbol{p_{In}} + \boldsymbol{p_{en}} \tag{109}$$

where charge neutrality $n_e = Z n_I$ has been used, and we have introduced the current density

$$\boldsymbol{J} \equiv e n_e (\boldsymbol{v_I} - \boldsymbol{v_e}). \tag{110}$$

The equation for the neutrals becomes

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi + \boldsymbol{J} \times \boldsymbol{B} \tag{111}$$

Due to collisional coupling, the neutrals are subject to the magnetic Lorentz force just as though they were a gas of charged particles. It is not the magnetic force *per se* that changes in a neutral gas. As well shall presently see, it is the inductive properties of the gas.

Let us return to the force balance equations for the electrons:

$$- e n_e \left(\boldsymbol{E} + \boldsymbol{v_e} \times \boldsymbol{B} \right) - \boldsymbol{p_{en}} = 0. \tag{112}$$

After division by $-e n_e$, this may be expanded to

$$\boldsymbol{E} + \left[\boldsymbol{v} + (\boldsymbol{v_e} - \boldsymbol{v_I}) + (\boldsymbol{v_I} - \boldsymbol{v}) \right] \times \boldsymbol{B} + \frac{m_e \nu_{en}}{e} \left[(\boldsymbol{v_e} - \boldsymbol{v_I}) + (\boldsymbol{v_I} - \boldsymbol{v}) \right] = 0, \tag{113}$$

where we have introduced the collision frequency of an electron in a population of neutrals:

$$\nu_{en} = n \langle \sigma_{ne} w_{ne} \rangle. \tag{114}$$

[3] Beware: esu units are still commonly used in the astrophysical literature! You should become comfortable with them.

We have written the electron velocity $\boldsymbol{v_e}$ in terms of the dominant neutral velocity $\boldsymbol{v}$ and the key physical velocity differences of our problem. It has already been noted that in equation (109), $\boldsymbol{p_{en}}$ is small compared with $\boldsymbol{p_{In}}$, provided that the velocity difference $|\boldsymbol{v_e} - \boldsymbol{v}|$ is not much larger than $|\boldsymbol{v_I} - \boldsymbol{v}|$. As we argued earlier, the $\boldsymbol{p_{en}}$ term in equation (109) is small relative to $\boldsymbol{p_{In}}$:

$$\boldsymbol{J}\times\boldsymbol{B} \simeq \boldsymbol{p_{In}} = nn_I\mu nI(\boldsymbol{v_I} - \boldsymbol{v})\nu_{nI}. \tag{115}$$

It then follows that the final term in equation (113)

$$\frac{m_e\nu_{en}}{e}(\boldsymbol{v_I} - \boldsymbol{v}),$$

which is proportional to $\boldsymbol{J} \times \boldsymbol{B}$, becomes small compared with the third term

$$(\boldsymbol{v_e} - \boldsymbol{v_I})\times\boldsymbol{B},$$

which also proportional to $\boldsymbol{J}\times\boldsymbol{B}$, by a factor of order $(m_e/\mu_{In})^{1/2}$. These simplifications allow us to write the electron force balance equation as

$$\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} - \frac{\boldsymbol{J}\times\boldsymbol{B}}{en_e} - \frac{\boldsymbol{J}}{\sigma_{cond}} + \frac{(\boldsymbol{J}\times\boldsymbol{B})\times\boldsymbol{B}}{\gamma\rho\rho_I} = 0, \tag{116}$$

where the electrical conductivity has been defined as

$$\sigma_{cond} \equiv \frac{e^2 n_e}{m_e\nu_{en}} \tag{117}$$

The associated resistivity η is

$$\eta = \frac{1}{\mu_0\sigma_{cond}}, \tag{118}$$

which has units of m^2 s^{-1}. Numerically (e.g. Blaes & Balbus 1994 ApJ, 421, 163; Balbus & Terquem 2001, ApJ, 552, 235):

$$\eta = 0.0234\left(\frac{n}{n_e}\right)T^{1/2}\ \mathrm{m}^2\ \mathrm{s}^{-1} \tag{119}$$

Equation (116) is a general form of Ohm's law for a moving, multiple fluid system.

Next, we make use of two of Maxwell's equations. The first is Faraday's induction law:

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}. \tag{120}$$

We substitute $\boldsymbol{E}$ from equation (116) to obtain an equation for the self-induction of the magnetized fluid,

$$\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times\left[\boldsymbol{v}\times\boldsymbol{B} - \frac{\boldsymbol{J}\times\boldsymbol{B}}{en_e} + \frac{(\boldsymbol{J}\times\boldsymbol{B})\times\boldsymbol{B}}{\gamma\rho\rho_I} - \frac{\boldsymbol{J}}{\sigma_{cond}}\right] \tag{121}$$

It remains to relate the current density $\boldsymbol{J}$ to the magnetic field $\boldsymbol{B}$. This is accomplished by the second Maxwell equation,

$$\mu_0\boldsymbol{J} = \boldsymbol{\nabla}\times\boldsymbol{B} + \frac{\partial\boldsymbol{E}}{\partial t} \tag{122}$$

The final term in the above is the displacement current, and it may be ignored. Indeed, since we have not, and will not, use the "Gauss's Law" equation

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = (e/\epsilon_0)(Zn_I - n_e), \tag{123}$$

we *must not* include the displacement current. In Appendix B, we show that departures from charge neutrality in $\boldsymbol{\nabla}\cdot\boldsymbol{E}$ and the displacement current are both small terms that contribute at the same order: v^2/c^2. These must both be self-consistently neglected in nonrelativisitc MHD. (The final Maxwell equation $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ adds nothing new. It is automatically satisfied by equation (120), as long as the initial magnetic field satifies this divergence free condition.) These considerations imply

$$\boldsymbol{J} = \frac{1}{\mu_0}\boldsymbol{\nabla}\times\boldsymbol{B} \tag{124}$$

for use in equation (121).

To summarize, the fundamental equations of a weakly ionized fluid are mass conservation of the dominant neutrals (eq.[85])

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0, \tag{125}$$

the equation of motion (eq. [111] with [124])

$$\rho\frac{\partial\boldsymbol{v}}{\partial t} + \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}, \tag{126}$$

and the induction equation (eq. [121] with [118] and [124])

$$\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times\left[\boldsymbol{v}\times\boldsymbol{B} - \frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{\mu_0 e n_e} + \frac{[(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}]\times\boldsymbol{B}}{\mu_0\gamma\rho\rho_I} - \frac{\boldsymbol{\nabla}\times\boldsymbol{B}}{\mu_o\sigma_{cond}}\right] \tag{127}$$

It is only natural that the reader should be a little taken aback by the sight of equation (127). Be assured that it is rarely, if ever, needed in full generality: almost always one or more terms on the right side of the equation may be discarded. When only the induction term $\boldsymbol{v}\times\boldsymbol{B}$ is important, we refer to this regime as *ideal* MHD. The three remaining terms on the right are the nonideal MHD terms.

To get a better feel for the relative importance of the nonideal MHD terms in equation (121), we denote the terms on the right side of the equation, moving left to right, as I (induction), H (Hall), A (ambipolar diffusion), and O (Ohmic resistivity). We will always be in a regime in which the presence of the induction term is not in question. More interesting is the relative importance of the nonideal terms. The explicit dependence of A/H and O/H in terms of the fluid properties of a cosmic gas has been worked out by Balbus & Terquem (2001):

$$\frac{A}{H} = Z\left(\frac{9\times10^{12}\ \mathrm{cm}^{-3}}{n}\right)^{1/2}\left(\frac{T}{10^3\,\mathrm{K}}\right)^{1/2}\left(\frac{v_A}{c_S}\right) \tag{128}$$

and

$$\frac{O}{H} = \left(\frac{n}{8\times10^{17}\ \mathrm{cm}^{-3}}\right)^{1/2}\left(\frac{c_S}{v_A}\right) \tag{129}$$

Here n is the total number density of all particles, T is the kinetic temperature, v_A is the so-called *Alfvén velocity* (much more about this quantity will come later!),

$$\boldsymbol{v_A} = \frac{\boldsymbol{B}}{\sqrt{\mu_0\rho}} \tag{130}$$

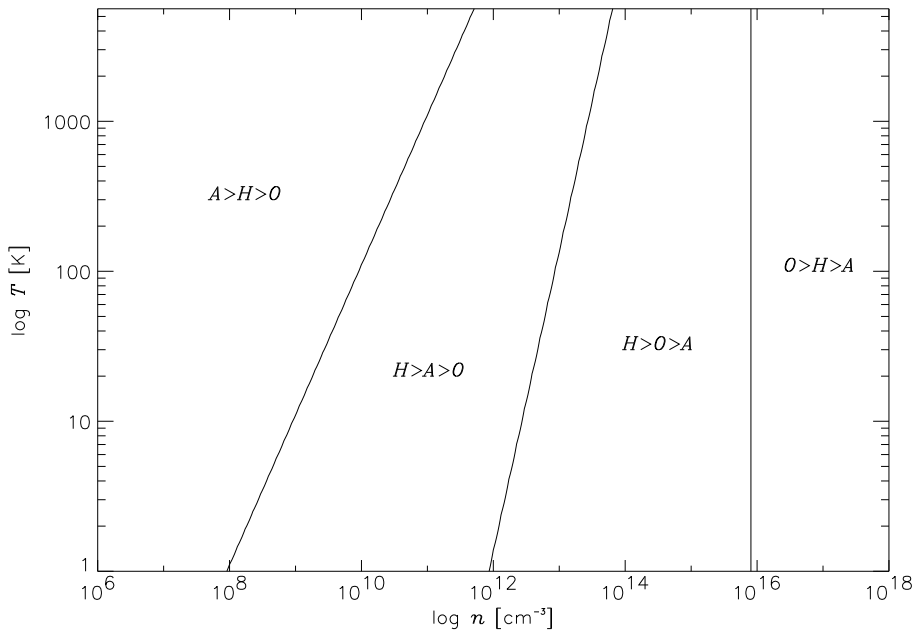

Figure 3: Parameter space for nonideal MHD. The curves correspond to the case $v_A/c_S = 0.1$. (From Kunz & Balbus 2004, MNRAS, 348, 355.)

and c_S is the isothermal speed of sound,

$$c_S^2 = 0.429\frac{kT}{m_p} \tag{131}$$

where k is the Boltzmann constant and m_p the mass of the proton. The coefficient 0.429 corresponds to a mean mass per particle of $2.33m_p$, appropriate to a molecular gas with a 10% helium admixture.

As reassurance that the fully general nonideal MHD induction equation is not needed for our purposes, note that equations (128) and (129) imply that for all three nonideal MHD terms to be comparable, $T \sim 10^8$ K! Obviously this is not a weakly ionized regime. In figure (2), we plot the domains of relative dominance of the nonideal MHD terms in the nT plane.

Our emphasis of the relative ordering of the nonideal terms in the induction equation should not obscure the fact that ideal MHD is often an excellent approximation, even when the ionization fraction is $\ll 1$. For example, the ratio of the ideal inductive term to the ohmic loss term is given by the Lundquist number

$$\boldsymbol{\ell} = \frac{v_A H}{\eta} \tag{132}$$

where H is a characteristic gradient length scale. To orient ourselves, let us consider the case of a protostellar disc and set $H = 0.1R$, where R is the radial location in the disc. (This would correspond to H being about the vertical thickness of the disc.) Then $\boldsymbol{\ell}$ is given by

$$\boldsymbol{\ell} \simeq 2.5(n_e/n)(v_A/c_S)R_{cm},$$

R_{cm} being the radius in centimeters. In other words, the critical ionization fraction at which $\boldsymbol{\ell} = 1$ is about

$$(n_e/n)_{crit} = 0.4(c_S/v_A)R_{cm}^{-1} \sim 10^{-13}(c_S/10v_A)$$

at $R = 1$ AU. The actual ionization fraction at this location may be above or below this during the course of the solar systems evolution, but the point worth noting here is that R_{cm} is a large number for a protostellar disc! Ionization fractions far, far below unity can render an astrophysical gas a near perfect electrical conductor. It therefore makes a great deal of sense to begin by examining the behaviour of an ideal MHD fluid.

Exercise. Show that the Lorentz force may be written

$$\boldsymbol{J} \times \boldsymbol{B} = \partial_i \left(\frac{B_i B_j}{\mu} - \delta_{ij} \frac{B^2}{2\mu} \right) \equiv \partial_i T^L_{ij}. \tag{133}$$

Exercise. Show that the Newtonian self-gravity force may be written

$$-\rho \boldsymbol{\nabla} \Phi = \partial_i \left(-\frac{g_i g_j}{4G\pi} + \delta_{ij} \frac{g^2}{8G\pi} \right) \equiv \partial_i T^N_{ij}. \tag{134}$$

where $g_i = -\partial_i \Phi$. (Hint: $\partial_i \partial_i \Phi = 4\pi G \rho$.)

Exercise. Show that the inertial terms in the equation of motion be written

$$\rho \partial_t v_i + \rho v_j \partial_j v_i + \partial_i P = \partial_t (\rho v_i) + \partial_i \left(\rho v_i v_j + \delta_{ij} P \right) \equiv \partial_t (\rho v_i) + \partial_i T^R_{ij}, \tag{135}$$

which defines the Reynolds stress T^R_{ij}.

Exercise. Show that the equation of motion may be written

$$\rho \partial_t v_i + \partial_i T_{ij} = 0, \tag{136}$$

where $T_{ij} = T^L_{ij} + T^N_{ij} + T^R_{ij}$ is the energy-momentum stress tensor. This form of the equation of motion is most readily generalized when relativity becomes important.

... I deduced that the forces wch keep the Planets in their Orbs must [be] reciprocally as the squares of their distances from centres about wch they revolve: & thereby compared the force requisite to keep the Moon in her Orb with the force of gravity at the surface of the earth, & found them answer pretty nearly.

— Sir Isaac Newton

We can lick gravity, but sometimes the paperwork is overwhelming.

— Dr. Werner von Braun

3 Gravity

3.1 Legendre Expansion

The gravitational field a distance $\boldsymbol{r}$ from a point mass M located at the origin is

$$\boldsymbol{g} = -\frac{GM\boldsymbol{r}}{r^3}, \tag{137}$$

or more generally,

$$\boldsymbol{g} = -\frac{GM(\boldsymbol{r}-\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r'}|^3} \tag{138}$$

if the point mass is located at $\boldsymbol{r'}$. This field is derivable from a potential function Φ,

$$\Phi = -\frac{GM}{|\boldsymbol{r}-\boldsymbol{r'}|}, \quad \boldsymbol{g} = -\boldsymbol{\nabla}\Phi \tag{139}$$

Gravity is a linear theory, and the fields (and thus the potentials) from an extended mass distribution superpose. Therefore, in general,

$$\Phi = -G\int \frac{\rho(\boldsymbol{r'})d^3r'}{|\boldsymbol{r}-\boldsymbol{r'}|}, \tag{140}$$

with

$$|\boldsymbol{r}-\boldsymbol{r'}| = (r^2 - 2rr'\cos\theta + r'^2)^{1/2} \tag{141}$$

where θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r'}$. For $r \gg r'$,

$$(r^2 - 2rr'\cos\theta + r'^2)^{-1/2} = r^{-1}\left(1 - \frac{2r'}{r}\cos\theta + \frac{r'^2}{r^2}\right)^{-1/2} \tag{142}$$

We are very often interested in the potential at great distances from the source, $r \gg r'$. The last two terms in r'/r are small, so we define

$$\delta = -\frac{2r'}{r}\cos\theta + \frac{r'^2}{r^2} \ll 1. \tag{143}$$

Then,

$$\left(1 - \frac{2r'}{r}\cos\theta + \frac{r'^2}{r^2}\right)^{-1/2} = r^{-1}(1+\delta)^{-1/2} = r^{-1}\left[1 - \frac{\delta}{2} + \frac{3\delta^2}{8} + ...\right] \tag{144}$$

Expanding δ and retaining terms through order $(r'/r)^2$, we find

$$r^{-1}\left(1 - \frac{2r'}{r}\cos\theta + \frac{r'^2}{r^2}\right)^{-1/2} = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\theta + \left(\frac{r'}{r}\right)^2 \frac{1}{2}(3\cos^2\theta - 1) + ...\right] \tag{145}$$

The expansion consists of powers of (r'/r) multiplied by a polynomial in $\cos\theta$. These latter are denoted $P_l(\cos\theta)$ and are known as *Legendre polynomials.* Their properties are discussed very clearly in Jackson's text, *Classical Electrodynamics.* The most important of these for our purposes is that the P_l are orthogonal when integrated over spherical solid angles:

$$\int P_l(\cos\theta)\, P_{l'}(\cos\theta)\, d\Omega = \frac{4\pi}{2l+1}\delta_{ll'} \tag{146}$$

Because of the symmetry between $\boldsymbol{r}$ and $\boldsymbol{r'}$, in general we must have

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r'}|} = \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\cos\theta), \tag{147}$$

where $r_>$ ($r_<$) is the greater (lesser) of r and r'.

3.2 Gauss's Law

A remarkable property of a $1/r^2$ force law is that

$$\int \boldsymbol{g}\cdot\boldsymbol{dA} = -4\pi GM \tag{148}$$

where the surface integral is over any volume containing a total mass M. To prove this, we show that it is true for a single point mass. Then by superposition, it is true for any distribution.

Note that $|\boldsymbol{g}\cdot\boldsymbol{dA}|$ is just the product of g with the area $dA*$, the projection of $\boldsymbol{dA}$ parallel to $-\boldsymbol{g}$. For a point mass, $\boldsymbol{g}$ is radial, and the projected area of $\boldsymbol{dA*}$ is precisely that of a small portion of a sphere centered on the mass point, at the same radius r as $\boldsymbol{dA}$. Since this small spherical area is $r^2 d\Omega$, where $d\Omega$ is the solid angle subtended by the area at the point mass, and $g = -GM/r^2$,

$$\boldsymbol{g}\cdot\boldsymbol{dA} = -GM\, d\Omega \tag{149}$$

This is independent of r, and the integral of the whole area just adds up the total solid angle subtended by the volume enclosing the mass: 4π. This proves Gauss's law for a point mass. By linear superposition, it is therefore true for any mass distribution interior to the surface.

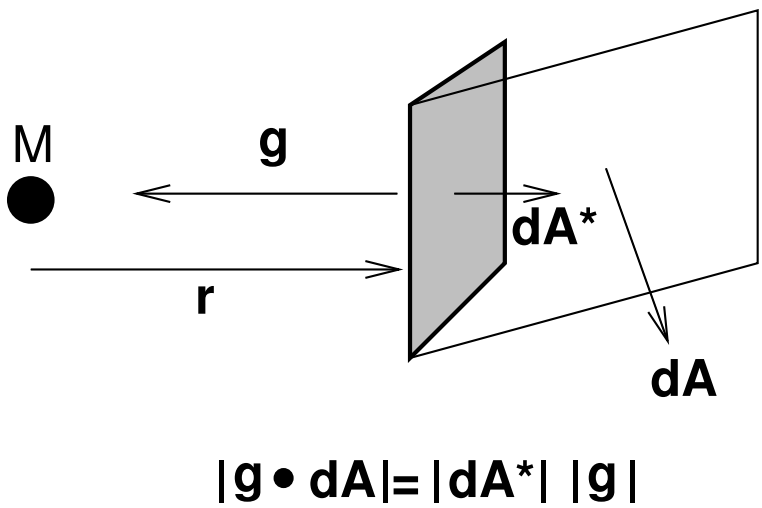

Figure 4: Gauss's law. The area element $\boldsymbol{dA}$ has projection $\boldsymbol{dA*}$ parallel to $\boldsymbol{-g}$. $dA*$ is thus a differential area element of a sphere surrounding M at distance r.

3.3 Poisson Equation

We have shown

$$\int \boldsymbol{g} \cdot \boldsymbol{dA} = -4\pi G \int \rho \, dV \tag{150}$$

for any mass distribution inside the volume V. Using the divergence theorem,

$$\int \boldsymbol{\nabla} \cdot \boldsymbol{g} \, dV = -4\pi G \int \rho \, dV \tag{151}$$

and since $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$,

$$\int \nabla^2 \Phi \, dV = 4\pi G \int \rho \, dV \tag{152}$$

The volume is arbitrary, hence we conclude

$$\nabla^2 \Phi = 4\pi G \rho, \tag{153}$$

which is known as the Poisson equation. It allows us to compute the gravitational potential, and hence the graviational forces, from a given distribution of mass. It must added to the fluid equations of motion to calculate the evolution of a self-gravitating system.

Exercise. Show that the Lorentz force may be written

$$\boldsymbol{J} \times \boldsymbol{B} = \partial_i \left(\frac{B_i B_j}{\mu} - \delta_{ij} \frac{B^2}{2\mu} \right) \tag{154}$$

and that for a self-gravitating gas the Newtonian gravitational force has a very similar form:

$$-\rho \boldsymbol{\nabla} \Phi = -\partial_i \left(\frac{g_i g_j}{4\pi G} - \delta_{ij} \frac{g^2}{8\pi G} \right) \tag{155}$$

where $g_i = -\boldsymbol{\nabla}_i \Phi$, $g^2 = g_i g_i$.

The quantities inside the ∂_i operators are known respectively as the Maxwell and gravitational stress tensors. They play a key role in momentum and energy transport in magnetic and self-gravitating systems.

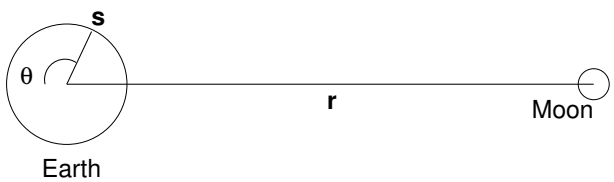

Figure 5: Geometry for tides raised on the earth by the moon.

3.4 Gravitational Tidal Forces.

As an illustration of how the expansion of the potential function can be used, let us calculate the height of the tides that are raised on the earth by the moon—though our calculation will be completely general for any two body problem apart from the numbers we use.

We define the z axis to be along the line joining the centers of the earth and the moon. The distance between the centers will be r, and a point on the earth's surface will be at a vector location $\boldsymbol{r} + \boldsymbol{s}$ relative to the center of the moon. Let $\boldsymbol{s} = (x, y, z)$ in Cartesian coordinates with origin at the center of the earth. Note that

$$\frac{1}{|\boldsymbol{r} + \boldsymbol{s}|} = (r^2 + s^2 + 2rs\cos\theta)^{-1/2} \tag{156}$$

so we need to keep track of the sign, which is different from our r, r' expansion. We regard r as fixed, and calculate forces by taking the gradient with respect to x, y, and z. We have, with $r \gg s$,

$$-\frac{GM_m}{|\boldsymbol{r} + \boldsymbol{s}|} = -\frac{GM_m}{r}\left[1 - \frac{s\cos\theta}{r} + \left(\frac{s}{r}\right)^2 P_2(\cos\theta) + ...\right] \tag{157}$$

where M_m is the mass of the moon. Differentiating with respect to $z = s\cos\theta$ gives, to first approximation

$$-\frac{\partial\Phi}{\partial z} = -\frac{GM_m}{r^2} \tag{158}$$

which looks familiar: it is the Newtonian force acting between the centers of the two bodies, directing along the line joining them. It is not the tidal force, which comes in at the next level of approximation. The tidal potential is:

$$\Phi \text{ (tidal)} = -\frac{GM_m s^2}{r^3} P_2(\cos\theta) \tag{159}$$

And the tidal force is, after carrying out the gradient operation,

$$\boldsymbol{g} \text{ (tidal)} = -\boldsymbol{\nabla}\Phi = \frac{GM_m}{r^3}(-x, -y, 2z) \tag{160}$$

Tidal forces try to squeeze matter along the directions perpendicular to the line joining the bodies, and try to stretch matter along the direction parallel to this line. Note that we speak here only of the forces; the resulting distortions can be much more complex. Not only are they sensitive to local surface features in the oceans, there are also time delays in the response of the displacement, due to the presence of dissipation.

Let us assume, however, that the new shape of the earth has adjusted so that the surface now follows an equipotential of the earth's gravitational field plus that of the moon. Let Φ_1 be the potential function of the earth's unperturbed spherical field. Let Φ_2 be the new

potential function in the presence of the moon's potential, differing slightly from Φ_1 at a given location. The tidal force causes a displacement of the original spherical equipotential surfaces by an amount $\boldsymbol{\xi}$, and this is what we wish to calculate. Let $\Phi_1(\boldsymbol{s}) = \Phi$ be constant on a sphere of radius s. The new equipotential surface with this (constant) value of Φ is $\Phi_2(\boldsymbol{r} + \boldsymbol{\xi})$, where $\boldsymbol{\xi}$ is the small displacement caused by the moon. It is this quantity that we wish to calculate. If an equipotential surface of Φ_2 has the same value as an equipotential surface of Φ_1, but only after the surface has been displaced by $\boldsymbol{\xi}$, then

$$\Phi = \Phi_2(\boldsymbol{s} + \boldsymbol{\xi}) = \Phi_2(\boldsymbol{s}) + \boldsymbol{\xi} \cdot \boldsymbol{\nabla}\Phi_2 = \Phi_1(\boldsymbol{s}) \tag{161}$$

where s is the radius of the earth. But *at the same location s:*

$$\Phi_2(\boldsymbol{s}) - \Phi_1(\boldsymbol{s}) = \Phi \text{ (tidal)}, \tag{162}$$

and to leading order we may replace Φ_2 with Φ_1 in the term proportional to $\boldsymbol{\xi}$. Then

$$-\boldsymbol{\xi} \cdot \boldsymbol{\nabla}\Phi_1 = \Phi \text{ (tidal)} \tag{163}$$

which states the physically very sensible result that the work done against the gravitational force of the earth in distorting the surface is provided by the additional tidal potential energy. Writing the potential functions explicitly:

$$\xi_s \frac{GM_e}{s^2} = \frac{GM_m s^2}{r^3} P_2(\cos\theta) \tag{164}$$

or

$$\xi_s = s\frac{M_m}{M_e}\left(\frac{s}{r}\right)^3 P_2(\cos\theta) \tag{165}$$

This works out to be

$$\xi_s = 0.32 P_2(\cos\theta) \text{ meters} \tag{166}$$

for the earth-moon system. Notice how extremely sensitive the height of the tidal displacement is to the separation distance r. When the moon was a factor of 2 closer to the earth, as it is believed to have been on a timescale of 10^9 years ago, the tidal forces were almost an order of magnitude larger.

3.5 The Virial Theorem

The Virial Theorem is one of the most useful theorems in astrophysical gasdynamics. Basically, it is an integral form of the equation of motion in full generality. When the dominant balance is between two forces, the theorem states that the associated energies must be comparable in strength. We shall use Cartesian index notation in our proof.

Begin with

$$\rho\frac{Dv_i}{Dt} = -\partial_i P - \partial_i\left(\frac{B^2}{2\mu}\right) - \rho\partial_i\Phi + \frac{B_j}{\mu}\partial_j B_i \tag{167}$$

where

$$\Phi(\boldsymbol{r}) = -G\int \frac{\rho(\boldsymbol{r'})\, d^3r'}{|\boldsymbol{r} - \boldsymbol{r'}|} \tag{168}$$

is the gravitational potential the system. Note that

$$-\partial_i\Phi = -G\int \frac{\rho(\boldsymbol{r'})\,(r_i - r_i')\, d^3r'}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \tag{169}$$

Multiply the equation of motion by r_i and sum over i,

$$\rho r_i \frac{Dv_i}{Dt} = -r_i \partial_i P - r_i \partial_i \left(\frac{B^2}{2\mu}\right) - \rho r_i \partial_i \Phi + r_i \frac{B_j}{\mu} \partial_j B_i \tag{170}$$

and then integrate over a fixed volume V. For the pressure integral,

$$\begin{aligned} -\int r_i \partial_i P \, dV &= -\int \partial_i (r_i P) \, dV + 3 \int P \, dV & (171) \\ &= -\int P \boldsymbol{r} \cdot \boldsymbol{dA} + 3 \int P \, dV \\ &= -\int P \boldsymbol{r} \cdot \boldsymbol{dA} + 2 \int U_{therm} \, dV & (172) \end{aligned}$$

where $U_{therm} = (3/2)P$ is the thermal energy density.

The integral involving the potential is

$$\int \rho r_i \frac{\partial \Phi}{\partial r_i} \, d^3 r = G \int \frac{\rho(\boldsymbol{r}) \rho(\boldsymbol{r'}) r_i (r_i - r_i')}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \, d^3 r \, d^3 r' \tag{173}$$

If we switch the labels $\boldsymbol{r}$ and $\boldsymbol{r'}$, we obtain

$$\int \rho r_i \frac{\partial \Phi}{\partial r_i} \, d^3 r = G \int \frac{\rho(\boldsymbol{r}) \rho(\boldsymbol{r'}) r_i' (r_i' - r_i)}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \, d^3 r \, d^3 r' \tag{174}$$

Adding and dividing by 2:

$$\int \rho r_i \frac{\partial \Phi}{\partial r_i} \, d^3 r = \frac{G}{2} \int \frac{\rho(\boldsymbol{r}) \rho(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|} d^3 r \, d^3 r' \equiv -V. \tag{175}$$

i.e., this is just minus the gravitational potential energy V. (The factor of 1/2 is present because each pair of interacting fluid elements occurs twice in the integration, but should only be counted once.)

On to the magnetic integrals:

$$\int r_i \partial_i (B^2/2\mu) \, d^3 r = \int (B^2/2\mu) \boldsymbol{r} \cdot \boldsymbol{dA} - 3 \int (B^2/2\mu) \, d^3 r, \tag{176}$$

where we have integrated by parts and used the divergence theorem. And

$$\begin{aligned} \int \frac{r_i}{\mu} \partial_j (B_i B_j) d^3 r &= \int (\boldsymbol{r} \cdot \boldsymbol{B}) \frac{\boldsymbol{B}}{\mu} \cdot \boldsymbol{dA} - \int \frac{\delta_{ij}}{\mu} B_i B_j \, d^3 r \\ &= \int (\boldsymbol{r} \cdot \boldsymbol{B}) \frac{\boldsymbol{B}}{\mu} \cdot \boldsymbol{dA} - \int \frac{B^2}{\mu} \, d^3 r. & (177) \end{aligned}$$

The sum of all the terms on the right side of our equation is then

$$2E_{therm} + V + M - \int \left(P + \frac{B^2}{2\mu}\right) \boldsymbol{r} \cdot \boldsymbol{dA} + \frac{1}{\mu} \int (\boldsymbol{r} \cdot \boldsymbol{B}) \boldsymbol{B} \cdot \boldsymbol{dA} \tag{178}$$

where

$$E_{therm} = \int U_{therm}\, d^3r, \quad M = \int \frac{B^2}{2\mu}\, d^3r \tag{179}$$

are the total thermal and magnetic energies.

For the left side of the virial equation we start with the following identity:

$$\int \rho \frac{DQ}{Dt} d^3r = \int \rho \left(\frac{\partial Q}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} Q \right) d^3r = \int \frac{\partial(\rho Q)}{\partial t}\, d^3r + \int \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v} Q)\, d^3r \tag{180}$$

where the second equality follows from mass conservation. The last integral can be converted to a surface integral of the flux $\rho \boldsymbol{v} Q$ over a bounding area. If we choose the surface so that the velocity vanishes at this surface, then this integral vanishes, and we shall make this assumption. We then have:

$$\int \rho \frac{DQ}{Dt}\, d^3r = \int \frac{\partial(\rho Q)}{\partial t}\, d^3r = \frac{d}{dt} \int \rho Q\, d^3r \tag{181}$$

With this in hand, we perform the following manipulations:

$$\begin{aligned}
\int \rho r_i \frac{Dv_i}{Dt}\, d^3r &= \int \rho \frac{D(r_i v_i)}{Dt}\, d^3r - \int \rho v_i \frac{Dr_i}{Dt}\, d^3r && (182)\\
&= \frac{d}{dt} \int \rho r_i v_i\, d^3r - \int \rho v^2\, d^3r \\
&= \frac{d}{dt} \int \rho r_i \frac{Dr_i}{Dt}\, d^3r - 2KE \\
&= \frac{d}{dt} \int \frac{\rho}{2} \frac{Dr^2}{Dt}\, d^3r - 2KE \\
&= \frac{d^2}{dt^2} \frac{1}{2} \int \rho r^2 d^3r - 2KE \\
&= \frac{1}{2} \frac{d^2 I}{dt^2} - 2KE && (183)
\end{aligned}$$

where I is $\int \rho r^2 d^3r$ and KE denotes the total kinetic energy of the fluid. The Virial theorem is then:

$$\frac{1}{2}\ddot{I} = 2KE + 2E_{therm} + V + M - \int \left(P + \frac{B^2}{2\mu} \right) \boldsymbol{r}\cdot\boldsymbol{dA} + \frac{1}{\mu} \int (\boldsymbol{r}\cdot\boldsymbol{B})\boldsymbol{B}\cdot\boldsymbol{dA} \tag{184}$$

where the velocity is assumed to vanish over the bounding surface. The Virial theorem shows that when a dominant steady-state balance is present between two effects–pressure and gravity, say—the two associated energies are comparable. In particular, for a star in hydrostatic equilibrium,

$$E_{therm} = -\frac{1}{2}V, \quad E_{total} = E_{therm} + V = \frac{V}{2} \tag{185}$$

since the pressure vanishes at the surface, and magnetic fields are generally negligible for stellar hydrostatic equilibrium.

3.6 The Lane-Emden Equation

Consider the hydrostatic equilibrium of a star with a very simple equation of state: $P = K\rho^\gamma$. Special cases of interest include $\gamma = 1$ (isothermal spheres), $\gamma = 6/5, 2$ (elementary analytic solutions), $\gamma = 5/3, 4/3$ (nonrelativistic, relativistic white dwarfs). Our fundamental equation

$$-\boldsymbol{\nabla}\Phi = \frac{1}{\rho}\boldsymbol{\nabla}P \tag{186}$$

becomes

$$-\nabla^2\Phi = -4\pi G\rho = \boldsymbol{\nabla}\cdot\left(\frac{1}{\rho}\boldsymbol{\nabla}P\right) = \boldsymbol{\nabla}\cdot\left(\frac{1}{\rho}\boldsymbol{\nabla}K\rho^\gamma\right) \tag{187}$$

Assuming spherical geometry and carrying through the differentiation,

$$\frac{K\gamma}{\gamma-1}\frac{1}{r^2}\frac{d}{dr}r^2\left(\frac{d\rho^{\gamma-1}}{dr}\right) = -4\pi G\rho \tag{188}$$

For the case of an isothermal gas,

$$K\frac{1}{r^2}\frac{d}{dr}r^2\left(\frac{d\ln\rho}{dr}\right) = -4\pi G\rho \tag{189}$$

Next, we introduce the polytropic index,

$$n = \frac{1}{\gamma-1} \tag{190}$$

and let the density ρ be written

$$\rho = \rho_c\theta^n \tag{191}$$

where ρ_c is the central density. If we further rescale the length r as $r = a\xi$, with a a length to be determined, our equation becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\left[\frac{4\pi G\rho_c^{1-1/n}(\gamma-1)a^2}{K\gamma}\right]\theta^n \tag{192}$$

Obviously, we should choose a so that the constant in square brackets is unity, or

$$a^2 = \frac{K(n+1)\rho_c^{(1/n)-1}}{4\pi G} \tag{193}$$

The resulting equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta^n \tag{194}$$

is known as the *Lane-Emden* equation.

3.7 Solution properties of the Lane-Emden Equation

To solve the Lane-Emden equation, we need boundary conditions at the origin $\xi = 0$. Let us return to the original first order equation of hydrostatic equilibrium. Since we are in spherical symmetry, this may be written

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r} \tag{195}$$

where $M(r)$ is the mass interior to radius r. We have made use of Gauss's theorem, which assures us that the mass exterior to r contributes nothing to the force at r, and that the mass interior to r acts as though it were concentrated at $r = 0$. When r is very small, $M(r) \simeq 4\pi\rho_c r^3/3$, and thus $dP/dr \to 0$. But since $P \sim \theta^{1+n}$, it follows immediately that the ξ gradient of θ must also vanish at the origin. Our two boundary conditions on θ at $\xi = 0$ are

$$\theta(0) = 1, \quad \theta'(0) = 0. \tag{196}$$

It is now a simple matter to show at all solutions to the Lande-Emden equation must have the same asymptotic form as $\xi \to 0$:

$$\theta(\xi) \to 1 - \frac{\xi^2}{6} + ... \tag{197}$$

It is natural to ask whether there are solutions that extend to infinity, and if so, what their asymptotic form is. This can be done at once by looking for solutions of the $\theta = A\xi^p$, and solving for A and p by demanding self-consistency. We have

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = p(p+1)A\xi^{p-2} = \theta^n = -A^n\xi^{np} \tag{198}$$

This requires

$$p = \frac{2}{1-n}, \quad A^{n-1} = \frac{2(n-3)}{(n-1)^2} \tag{199}$$

In fact, this is an exact solution, not just an approximate one. But it is not the exact solution we want, because it does not satisfy the boundary conditions at the origin. Moreover, there is a subtle flaw in our analysis. The case $p = -1, n = 3$ breaks down, because what we thought was the leading order term on the right side of the Lane-Emden equation actually vanishes. We cannot exclude a solution with asymptotic behaviour $\theta \sim 1/\xi$ on the basis of our analysis; we simply have to retain the next order terms!

In fact, notice that a function of the form $(1 + \xi^2/3)^{-1/2}$ satisifes both the boundary conditions at $\xi = 0$, and has the asymptotic form $\theta \sim A/\xi$. Amazingly enough, this turns out to be an exact solution of the equation, the only known analytic solution of a nonlinear form of the Lane-Emden equation satisfying all the boundary conditions at $\xi = 0$. (The cases $n = 0$ and $n = 1$ reduce to elementary cases, which you should solve explicitly.) The solution

$$\theta = (1 + \xi^2/3)^{-1/2} \tag{200}$$

corresponds to $n = 5$ (show!). All solutions with $n \geq 5$ extend to infinity, those with $n < 5$ are finite.

Exercise. Expand equation (200) through terms of order $1/\xi^3$ to show that it satisfies the Lande-Emden equation only to *leading* order, with $n = 5$.

3.8 Polytrope Masses

The mass interior to radius r is given by

$$M(r) = -\frac{r^2}{\rho G}\frac{dP}{dr} = -\frac{Kr^2}{G}\left(1+\frac{1}{n}\right)\rho^{1/n}\frac{d\ln\rho}{dr} \tag{201}$$

Translating this to θ and ξ variables gives:

$$M(r) = \left[\frac{K^3(n+1)^3}{4\pi G^3}\right]^{1/2}\rho_c^{(3-n)/2n}\xi^2\left|\frac{d\theta}{d\xi}\right| \tag{202}$$

The total mass of a polytrope can be evaluated once the surface values of ξ and $\theta'(\xi)$ are known: no further integration is needed to determine the mass once θ is tabulated. Notice something remarkable: for $n = 3$, the mass is a universal constant, independent of whatever is chosen for the central density ρ_c!

This is a result of tremendous significance for astrophysics. For white dwarf stars, the Lande-Emden equation is not simply an approximation to help us understand more complex and realistic models. A polytropic equation of state is in fact exact for a gas of degenerate electrons in either the nonrelativistic limit, in which case $P \sim \rho^{5/3}$ and $n = 1.5$, or in the extreme relativistic limit, in which case $P \sim \rho^{4/3}$ and $n = 3$. The pressure from the nuclei in the white dwarf, is nondegenerate, and negligible compared with the degenerate electron pressure.

Imagine constructing white dwarfs by slowly increasing the value of the central density ρ_c. At first, when the the gas in nonrelativistic, the larger the value of ρ_c, the larger the value of the stellar mass M. Note however that the radius of the star, which is proportional to a, gets smaller and smaller. At some point the electrons become so confined that they become a *relativisitic* degenerate gas, and $n \to 3$. The mass of the star can increase no more! This limiting mass is called the *Chandrasekhar Mass*, and it may be evaluated by calculating the value of $\xi^2\theta = 2.01824$ at the surface of an $n = 3$ polytrope (notice that the $n = 5$ polytrope has the value 1.732 evaluated at the "surface" $\xi \to \infty$), and by using $K = 4.935 \times 10^9$ in SI units for an extreme relativistic degenerate gas. One finds

$$M_{Ch} = 1.44\, M_\odot \tag{203}$$

It is the existence of this upper mass limit, whose value depends upon the very simple solution of the $n = 3$ Lane-Emden equation, that leads to the production of neutron stars and black holes.

3.9 The Gravitational Potential Energy of a Polytrope.

It is a remarkable fact that even though we cannot express the solution to the Lane-Emden equation in a closed analytic form for arbitrary γ, there is a very simple expression for the gravitational potential energy as a function of mass M, radius R and of course γ. The calculation is very igeneous.

The Potential Energy U is given by

$$U = -\int_0^R \frac{GM_r}{r}\, 4\pi\rho r^2 dr \tag{204}$$

where M_r is shorthand for $M(r)$, the mass within radius r. We then perform a series of substitutions and integrations by parts:

$$U = -\int_0^R \frac{GM_r\rho}{r^2}\,4\pi r^3 dr = \int_0^R \frac{dP}{dr}\,4\pi r^3\,dr = -3\int_0^R \left(\frac{P}{\rho}\right) 4\pi\rho r^2\,dr \tag{205}$$

Continuing with $dM_r/dr = 4\pi\rho r^2$

$$U = -3\int_0^R \left(\frac{P}{\rho}\right)\left(\frac{dM_r}{dr}\right) dr = 3\int_0^R \frac{d}{dr}\left(\frac{P}{\rho}\right) M_r\,dr \tag{206}$$

Notice that we have yet to use anything but hydrostatic equilibrium by way of physics! Now, for the first time, we use the fact that we have a polytrope. From $P = K\rho^\gamma$ follows

$$d\left(\frac{P}{\rho}\right) = \left(\frac{\gamma-1}{\gamma}\right)\frac{dP}{\rho},$$

and we then have

$$U = 3\left(\frac{\gamma-1}{\gamma}\right)\int_0^R \frac{M_r}{\rho}\frac{dP}{dr} = -3\left(\frac{\gamma-1}{\gamma}\right)\int_0^R \frac{GM_r^2}{r^2}\,dr \tag{207}$$

A last integration by parts gives

$$U = 3\left(\frac{\gamma-1}{\gamma}\right)\int_0^R GM_r^2\frac{d}{dr}\frac{1}{r}\,dr = 3\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{GM^2}{R} + 2U\right) \tag{208}$$

where we have written U for $-\int GM_r dM_r/r$ on the right. We now have a simple equation involving only U, GM^2/R, and γ! Its solution is:

$$U = -\frac{3(\gamma-1)}{5\gamma-6}\frac{GM^2}{R} \tag{209}$$

Notice the recovery of the well-known constant density solution,

$$U = -\frac{3}{5}\frac{GM^2}{R} \qquad \text{(constant density)}$$

in the limit $\gamma \to \infty$. We can also make sense of why the case $n = 5$ in section 3.7 marks the division between finite and and infinite extent of the polytrope: it corresponds to the the gravitational potential energy becoming infinite.

3.10 Effects of Rotation

We conclude our discussion of polytropes by examining a few simple rotating systems. In the presence of rotation, the equation for hydrostatic equilibrium becomes:

$$-\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi = -R\Omega^2\boldsymbol{e_R} \tag{210}$$

If $P = P(\rho)$, the vorticity conservation equation implies that $\Omega = \Omega(R)$. (This is von Zeippel's theorem, proved in an earlier exercise.) Then

$$R\Omega^2(R) = \boldsymbol{\nabla}\left[\int^R R'\Omega^2(R')\,dR'\right] \equiv \boldsymbol{\nabla}\Phi_\Omega \tag{211}$$

which defines the velocity potential function Φ_Ω. We see that all terms are derivable from a potential function. The equation of hydrostatic equilibrium becomes

$$H + \Phi - \Phi_\Omega = C = \text{CONSTANT}, \tag{212}$$

where H is the enthalpy function

$$H \equiv \int \frac{dP}{\rho} = \frac{\gamma}{\gamma - 1}\frac{P}{\rho} \tag{213}$$

3.10.1 Example 1: Rotating Liquid

As a first example, consider the case of a constant density, uniformly rotating liquid in a constant vertical gravitational field g. (Constant density corresponds to $\gamma \to \infty$.) Then $\Phi = gz$ and $\Phi_\Omega = R^2\Omega^2/2$. If z_0 is the surface height of the liquid at the center $R = 0$, then $C = gz_0$, assuming that the pressure vanishes at the liquid surface. The solution is

$$\frac{P}{\rho} = g(z_0 - z) + \frac{1}{2}R^2\Omega^2, \tag{214}$$

and the $P = 0$ surface of the liquid is given by

$$z = z_0 + \frac{R^2\Omega^2}{2g} \tag{215}$$

The primary astronomical application of this equation is that it tells us how to make parabolic mirrors! Liquid glass is spun in huge ovens, and its surface acquires a perfectly parabolic shape. When silvered, the parabolic mirror reflects all incoming parallel rays onto a single point, the focus. In this way, an image is formed. Even a small segment of a parabolic mirror has this property. Very large telescopes may thus be constructed (and controlled) by using many small mirrors.

3.10.2 Example 2: Sub-Keplerian Disks

The Keplerian rotation law of planetary motion is

$$v_\phi^2 = \frac{GM}{R} \tag{216}$$

In a gas disc, this law is modified by gas pressure. But if we assume that the rotation law follows the same functional form (linearly proportional to central mass), the modification may be parameterized as

$$v_\phi^2 = \frac{GM\cos\beta}{R} \tag{217}$$

where β is a free parameter. Such a profile is said to be sub-Keplerian, with the additional support provided by a pressure gradient. As before, v_ϕ depends only upon cylindrical radius R for a polytropic gas. With $\Phi = -GM/r$ and $\Phi_\Omega = -GM\cos\beta/R$, the integrated equation of hydrostatic equilibrium is now:

$$\frac{\gamma}{\gamma-1}\frac{P}{\rho} - \frac{GM}{r} + \frac{GM\cos\beta}{R} = \mathcal{H}_\infty \tag{218}$$

where $\mathcal{H}_\infty$ is the (constant) enthalpy at $R = \infty$. Recall that $R = r\sin\theta$, where θ is the usual spherical angle measured from the positive z axis. We define the latitude angle $\lambda = \pi/2 - \theta$. Then, equipotential surfaces (as well as isobaric and isochoric[4] surfaces) are given by the equation

$$\left(\frac{\cos\beta}{\cos\lambda} - 1\right) = Cr \tag{219}$$

where the constant C, a combination of enthalpy terms, may be either positive or negative. When $C > 0$, then $\lambda > \beta$ everywhere, and the surface is unbounded in spherical radius, extending to the cylinder

$$R_{max} = r\cos\lambda = (\cos\beta)/C \tag{220}$$

When $C = 0$, the surface is a cone $\lambda = \beta$. Note that *all* surfaces approach this cone as $r \to 0$. Since these are surfaces of constant density, this result gives us a geometrical interpretation of what was originally a dynamical parameter: β is the opening angle of the disc wedge. Finally, when $C < 0$, then $\lambda < \beta$ everywhere, and the surface is bounded, reaching a maximum radial extent of

$$r_{max} = |C|^{-1}(1 - \cos\beta) \tag{221}$$

at $\lambda = 0$. These surfaces, which form the interior surfaces of constant density in the disc, are shaped like drops of water.

Since gas can move freely along equipotential surfaces, the unbounded surfaces suggest the possibility of subsonic winds and outflows from discs, with a small enough velocity to allow the assumption of hydrostatic equilibrium. Whether this occurs in reality depends upon how easily the escaping gas can acquire angular momentum to orbit at its asymptotic cylinder $R = R_{max}$.

The bounded surfaces correspond to the body of the disc itself. Their shape suggests that slow but significant radial mixing might occur in the disc, if gas can flow along these surfaces and exchange angular momentum as needed. The protostellar disc that formed the solar system offers some evidence that mixing has occurred because dust grains far from the sun seem to have been exposed to the higher temperatures of the inner regions. This possibility of radial mixing in discs is currently being explored by computer simulations.

Our results are summarized in figure 3.

3.11 Self-gravity and Rotation

The simplest problem with self-gravity is that of a uniformly rotating, constant density cylinder. An application of Gauss's law shows that at a distance R from the rotation axis, the gravitational field is

$$g_R = -\frac{2G\mu(R)}{R} \tag{222}$$

[4]Isochoric means constant density.

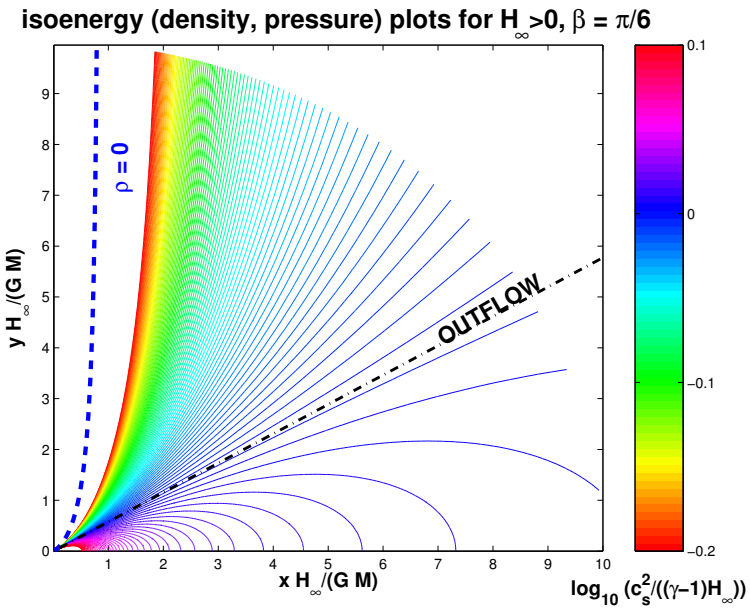

Figure 6: Equipotential disc contours for the case when Ω is 0.93 of its Keplerian value, corresponding to an opening wedge angle of $\beta = 30°$, marked by the line marked OUTFLOW. (The colour scale on the right gives contour information for $c_S^2 \equiv \gamma P/\rho$ as shown, but it may be ignored if you do not have a colour printer!) The constant H_∞ is the enthalpy at infinity, $\gamma/(\gamma-1)$ times P_∞/ρ_∞. Open contours become very closely packed near the $\rho = 0$ boundary, are not drawn.

where $\mu(R)$ is the mass per unit length interior to R,

$$\mu(R) = \int_0^R 2\pi\rho R\, dR = \pi R^2 \rho. \tag{223}$$

This gives

$$g_R = -2\pi G\rho R \tag{224}$$

and an associated potential of

$$\Phi = \pi G\rho R^2. \tag{225}$$

The potential integration constant is not important here, since it can be absorbed into the integration constant of the hydrostatic equilibrium equation, which is

$$\frac{P}{\rho} + \pi G\rho R^2 - \frac{1}{2}R^2\Omega^2 = \text{CONSTANT} = \frac{P_0}{\rho} \tag{226}$$

where P_0 is the central density. Our solution is then

$$\frac{P}{\rho} = \frac{P_0}{\rho} + \frac{1}{2}R^2\Omega^2\left(1 - \frac{2\pi G\rho}{\Omega^2}\right) \tag{227}$$

When self-gravity dominates over rotation, the pressure goes to zero at a finite radius, $R_{P=0}$:

$$[R_{P=0}]^2 = \frac{2P_0}{\rho(2\pi G\rho - \Omega^2)} \tag{228}$$

On the other hand, when rotation dominates, we find that the pressure rises sharply away from the center:

$$\frac{P}{\rho} = \frac{P_0}{\rho} + \frac{1}{2}R^2\Omega^2 \tag{229}$$

There is interesting application of this simple formula, which is to the central region of a hurricane, where the pressure can be very, very low. (At large rotation velocities the earth's Coriolis force is negligible; we also assume that the compressibility of air is unimportant.) The characteristic feature of a hurricane is a central low pressure "eye", surrounded by an "eye wall," in which the pressure rises sharply. This is reproduced in our quadratically growing solution for the pressure profile. In the exterior region of the hurricane, which is not reproduced in the above formula, significant departures from solid body rotation occur, and the pressure decreases outwards.

4 Waves and Instabilities

The mysteries come forward in waves.

— *Susan Casey, from The Wave: In Pursuit of Rogues, Freaks and Giants of the Ocean*

4.1 Small Perturbations

Disturbances are said to be linear when their amplitudes are very small compared with the corresponding equilibrium values of the background medium. When the amplitudes are comparable or in excess of the background medium, the disturbances are said to be nonlinear. For example, if at a particular point in the fluid, the equilibrium pressure is $P(\boldsymbol{r})$, and the wave disturbance at time t causes the pressure to change to $P'(\boldsymbol{r}, t)$, then in linear theory,

$$P'(\boldsymbol{r}, t) - P(\boldsymbol{r}) \equiv \delta P \ll P(\boldsymbol{r}) \tag{230}$$

For the velocity, linear theory generally requires the disturbance to be less than the speed of sound, a point we shall return to later. The background flow velocity by itself is generally irrelevant in establishing the linearity of a perturbation, since absolute motion cannot affect local physics. In linear theory, the equations for the δ amplitudes are determined by replacing all flow quantities X by $X + \delta X$, and then ignoring all terms in the resulting equation that are of order quadratic or higher in the δ amplitudes. This is what give linear theory its name.

Small disturbances can be be described in more than one way. The above equation for δP is known as an Eulerian perturbation, which is the difference between the equilibrium and altered value of a fluid quantity *at a fixed point in the background.* It is sometimes useful to work with what is known as a Lagrangian perturbation. In Lagrangian theory, we focus not upon a fixed location $\boldsymbol{r}$, but upon the displacement $\boldsymbol{\xi}$ of a particular fluid element. In above case of the pressure disturbance, for example, how does the pressure of a fluid element change when it is displaced from its equilibrium value $\boldsymbol{r}$ to $\boldsymbol{r} + \boldsymbol{\xi}$? The Lagrangian perturbation is

$$\Delta P \equiv P'(\boldsymbol{r} + \boldsymbol{\xi}, t) - P(\boldsymbol{r}) \tag{231}$$

To linear order ξ,

$$\Delta P = P'(\boldsymbol{r}, t) - P(\boldsymbol{r}) + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} P = \delta P + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} P \tag{232}$$

This, in fact, defines the Lagrangian perturbation for any flow quantity.

The Lagrangian velocity perturbation is

$$\Delta \boldsymbol{v} = \frac{D\boldsymbol{\xi}}{Dt} = \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\xi} = \delta \boldsymbol{v} + (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{233}$$

where $\boldsymbol{v}$ is any background velocity that is present. When the background velocity vanishes,

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \delta \boldsymbol{v} = \Delta \boldsymbol{v} \tag{234}$$

We may think of δ and Δ as difference operators, something like ordinary differention. For example,

$$\delta\left(\frac{1}{\rho}\right) = \frac{1}{\rho + \delta\rho} - \frac{1}{\rho} = -\frac{\delta\rho}{\rho^2}, \tag{235}$$

since we work only to linear order. But care must be taken. Note that, for example,

$$\delta\frac{\partial P}{\partial x} = \frac{\partial(\delta P)}{\partial x} \tag{236}$$

BUT(!):

$$\Delta\frac{\partial P}{\partial x} = \frac{\partial \Delta P}{\partial x} - \frac{\partial \boldsymbol{\xi}}{\partial x}\boldsymbol{\cdot}\boldsymbol{\nabla} P \neq \frac{\partial(\Delta P)}{\partial x} \tag{237}$$

In other words, δ commutes with the ordinary Eulerian partial derivatives with respect to time and space, but Δ does not. It is also possible to have precisely zero Eulerian pertubations, and yet have finite Lagrangian displacements and perturbations! (Do you see why?) In this case, there are no physical disturbances at all, instead we simply have what amounts to a relabeling of the coordinates. One must take care that true physical disturbances are in fact being calcuated.

Exercise. Prove the following commutation relations:

$$\Delta\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\Delta - \frac{\partial \boldsymbol{\xi}}{\partial t}\boldsymbol{\cdot}\boldsymbol{\nabla} \tag{238}$$

$$\Delta\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i}\Delta - \frac{\partial \boldsymbol{\xi}}{\partial x_i}\boldsymbol{\cdot}\boldsymbol{\nabla} \tag{239}$$

$$\Delta\frac{D}{Dt} = \frac{D}{Dt}\Delta \tag{240}$$

This last identity is physically obvious, stating that taking a pressure difference (say) between two fluid elements, one perturbed and one not perturbed, and then differentiating this difference with respect to time is just the same as first evaluating DP/Dt for each element, and then subtracting one from the other. Even though D/Dt is presented as an elaborate superposition of different partial derviatives, it is really just the time on a clock: measure the fluid element pressure, look at the clock, then measure it again. So this identity confirms that taking a time derivative is a linear operation when applied to the perturbed and unperturbed fluid elements. You will, however, find that it is a surprisingly lengthy exercise to prove the identity directly from the mathematical definitions with the partial derivatives, a task you should certainly do. Using this last identity and the definition of the Lagrangian perturbation, we have

$$\delta\frac{D}{Dt} = \frac{D}{Dt}\Delta - (\boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla})\frac{D}{Dt}. \tag{241}$$

4.2 Some Useful Relationships Between δ, Δ, and $\boldsymbol{\xi}$.

Using index notation, the equation of mass conservation may be written

$$\frac{D\ln\rho}{Dt} = -\partial_i v_i. \tag{242}$$

Taking the Lagrangian perturbation of both sides and using our commutation relations gives,

$$\frac{D}{Dt}\frac{\Delta\rho}{\rho} = -\partial_i(\Delta v_i) + (\partial_i \xi_j)(\partial_j v_i) \tag{243}$$

Since $\Delta v_i = D\xi_i/Dt = \partial_t \xi_i + v_j \partial_j \xi_i$, we obtain after simplification

$$\frac{D}{Dt}\frac{\Delta\rho}{\rho} = -\frac{D(\partial_i \xi_i)}{Dt} \tag{244}$$

Assuming that the fluid element's density remains unchanged when the displacement vanishes, this implies

$$\frac{\Delta\rho}{\rho} = -(\partial_i \xi_i) \tag{245}$$

This may also be written

$$\delta\rho = -\partial_i(\rho\xi_i) \tag{246}$$

We may thus relate displacements either to Eulerian or to Lagrangian density perturbations.

There is in fact a much more direct way of deriving equation (246) that works rather generally. The equation of mass conservation in Eulerian form is

$$\frac{\partial\rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) \tag{247}$$

Imagine now an equilibrium solution with the left side time derivative equal to zero. To create our Lagrangian displacements, we impart a finite velocity $\boldsymbol{u}(\boldsymbol{r})$ everywhere in the flow, but only for an infinitesimal time δt. In time δt, this causes an Eulerian change in the density (i.e. at fixed position) $\delta\rho$. But the time derivative of the above mass equation is defined at fixed position as well. Hence

$$\frac{\delta\rho}{\delta t} = -\boldsymbol{\nabla}\cdot(\rho(\boldsymbol{v}+\boldsymbol{u})) \tag{248}$$

Since $\boldsymbol{u} = 0$ corresponds to steady flow $\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0$, this equation becomes

$$\delta\rho = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi}) \tag{249}$$

where $\boldsymbol{\xi} = \boldsymbol{u}\delta t$. This is exactly equation (246).

Yet another derivation, this time of (245), follows from demanding that $\rho\mathcal{V}$ is constant for a moving fluid element, where $\mathcal{V}$ is the small elemental volume. Then

$$\frac{\Delta\rho}{\rho} = -\frac{\Delta\mathcal{V}}{\mathcal{V}} = -\frac{\int \boldsymbol{\xi}\cdot d\boldsymbol{a}}{\mathcal{V}} = -\frac{\int \boldsymbol{\nabla}\cdot\boldsymbol{\xi}\, dV}{\mathcal{V}} \to -\boldsymbol{\nabla}\cdot\boldsymbol{\xi} \tag{250}$$

as $\mathcal{V} \to 0$.

Exercise. Show that for adiabatic perturbations

$$\frac{\Delta P}{P} = \gamma\frac{\Delta\rho}{\rho} = -\gamma\boldsymbol{\nabla}\cdot\boldsymbol{\xi}, \quad \delta P = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P - \gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi}, \tag{251}$$

and that in a magnetized flow,

$$\delta\boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}). \tag{252}$$

4.3 Variational Derivation of the Equation of Motion

Using our Lagrangian formalism, the MHD equation of motion may be derived from a variational principle. This can be very useful both for practical as well as theoretical reasons.

Consider the quantity

$$L = \int \mathcal{L}\, d^3r\, dt \equiv \int \left(\rho\Phi + \frac{B^2}{2\mu} + \frac{P}{\gamma - 1} - \frac{\rho v^2}{2} \right) d^3r\, dt \tag{253}$$

This is a space-time integral over a Lagrangian density of the fluid, with the first three terms serving as an effective potential energy density and the last a kinetic energy density. (It is slightly more convenient here to work with "$PE - KE$" instead of $KE - PE$.) We shall assume here that Φ is a given function of position, and leave the more complicated case of a self-gravitating fluid as an exercise for the reader (see the text of Shapiro and Teukolsky for aid). Imagine now that we have a well-defined flow, and everywhere each fluid element instantaneously changes it position from $\boldsymbol{x}$ to $\boldsymbol{x} + \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ depends upon space and time. We calculate the Lagrangian changes ΔP, δPhi, etc. By demanding that the first order change in L in vanishes, we will now show that the equation of motion for an MHD fluid is recovered.

Begin by noting that since we are integrating over individual fluid elements, the volume element d^3r is active and changes as well! If we change x_i to $x_i + \xi_i$ (index notation), then since $\Delta(\rho d^3r) = 0$ by mass conservation, according to the results of the last section, d^3r changes to $(\partial_i \xi_i) d^3r$. The Lagragian change in $\rho\Phi d^3r$ is therefore

$$\Delta(\rho\Phi\, d^3r) = (\Delta\Phi)\rho d^3r = (\xi_i \partial_i \Phi)\rho d^3r \tag{254}$$

since the Eulerian change $\delta\Phi$ vanishes for any fixed function of position.

The thermal energy term is more interesting:

$$\frac{\Delta P}{\gamma - 1} = \frac{\gamma P}{\gamma - 1} \frac{\Delta\rho}{\rho} \tag{255}$$

for adiabatic perturbations. (A variational principle is possible only for this class of disturbance!) Including the volume element change,

$$\frac{\Delta(P\, d^3r)}{\gamma - 1} = \frac{\gamma P\, d^3r}{\gamma - 1} \frac{\Delta\rho}{\rho} + \frac{P}{\gamma - 1} (\partial_i \xi_i) d^3r = -P(\partial_i \xi_i) d^3r \tag{256}$$

Next we integrate by parts and assume that $P\xi_i$ vanishes when integrated over the the bounding volume. This leaves us with net change of

$$(\xi_i \partial_i) P\, d^3r. \tag{257}$$

On to the magnetic energy, the most interesting of the potential functions. The net Lagrangian change is

$$(B_i \Delta B_i/\mu)\, d^3r + (\partial_i \xi_i)(B^2/2\mu)\, d^3r. \tag{258}$$

Now,

$$\Delta \boldsymbol{B} = \boldsymbol{\delta B} + (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}) + (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{B}, \tag{259}$$

using the result of the last exercise for $\boldsymbol{\delta B}$. Hence,

$$\boldsymbol{B}\cdot\Delta\boldsymbol{B} = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B})) + \boldsymbol{B}\cdot[(\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{B}] \tag{260}$$

But

$$\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B})) = \boldsymbol{\nabla}\cdot\left[(\boldsymbol{\xi}\times\boldsymbol{B})\times\boldsymbol{B}\right] + (\boldsymbol{\xi}\times\boldsymbol{B})\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}). \tag{261}$$

The pure divergence is assumed to vanish as a surface term, and we are left with

$$(\boldsymbol{\xi}\times\boldsymbol{B})\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) = (\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{B}))\cdot\boldsymbol{\xi} = -\mu(\boldsymbol{J}\times\boldsymbol{B})\cdot\boldsymbol{\xi} \tag{262}$$

using $\boldsymbol{\nabla}\times\boldsymbol{B} = \mu\boldsymbol{J}$. The remaining magnetic terms in the Lagrangian perturbation exactly cancel because

$$\boldsymbol{B}\cdot[(\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{B}] = \xi_i\partial_i(B^2/2), \tag{263}$$

and an integration by parts (vanishing divergence once again) leaves us with a term which exactly cancels the second term of equation (258). We have thus shown that

$$\Delta\left[(B^2/2\mu)d^3r\right] = -(\boldsymbol{J}\times\boldsymbol{B})\cdot\boldsymbol{\xi}. \tag{264}$$

The net result for the first three terms is a first order change in the Lagrangian of

$$\int \boldsymbol{\xi}\cdot\left[\rho\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}P - \boldsymbol{J}\times\boldsymbol{B}\right]\, d^3r\, dt \tag{265}$$

Finally, we consider the kinetic energy term $\rho v^2/2$. Here the time integration will also play an important role, so we exhibit dt explicitly. The Lagrangian change is

$$\rho(\boldsymbol{v}\cdot\boldsymbol{\Delta v})\, d^3r\, dt = \rho\boldsymbol{v}\cdot\left(\frac{\partial\boldsymbol{\xi}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{\xi}\right)\, d^3r\, dt \tag{266}$$

For the first time derivative term, we integrate by parts and use the constraint that $\boldsymbol{\xi}$ vanish at limits of the *time* integration. (This is of course also used in standard point mechanics for Lagrangian theory.) For the second term, we perform a familiar spatial integration by parts, ignoring the exact divergence. Our expression becomes

$$\rho(\boldsymbol{v}\cdot\boldsymbol{\Delta v})\, d^3r\, dt = -\xi_i\left[\frac{\partial(\rho v_i)}{\partial t} + \partial_j(\rho v_i v_j)\right]\, d^3r\, dt \tag{267}$$

Using mass conservation, this becomes

$$-\xi_i\rho\left[\frac{\partial v_i}{\partial t} + v_j\partial_j v_i\right]\, d^3r\, dt = -\xi_i\rho\frac{Dv_i}{Dt}\, d^3r\, dt \tag{268}$$

Noting that the kinetic energy term comes in with a minus sign in our Lagrangian, the total first order change is

$$\int \boldsymbol{\xi}\cdot\left[\rho\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}P - \boldsymbol{J}\times\boldsymbol{B} + \rho\frac{D\boldsymbol{v}}{Dt}\right]\, d^3r\, dt \tag{269}$$

Since $\boldsymbol{\xi}$ is completely arbitrary over the integration domain (except at the space time boundaries), the quantity in square brackets must vanish if L is stationary. We recover the MHD equation of motion.

4.4 Sound Waves and Shock Waves in One Dimension

4.4.1 Linear Waves

The equations of an adiabatic gas in one dimension are

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \tag{270}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{271}$$

$$P = K\rho^{\gamma} \tag{272}$$

Our initial state will be the simplest possible: P and ρ both constant, $v = 0$. We introduce Eulerian linear perturbations to all flow variables. The linearized equations become

$$\frac{\partial}{\partial t}\frac{\delta\rho}{\rho} + \frac{\partial \delta v}{\partial x} = 0 \tag{273}$$

$$\frac{\partial \delta v}{\partial t} = -\frac{1}{\rho}\frac{\partial \delta P}{\partial x} \tag{274}$$

$$\frac{\delta P}{P} = \gamma\frac{\delta\rho}{\rho} \tag{275}$$

Replacing δP in favour of $\delta\rho$ in the middle equation above gives

$$\frac{\partial \delta v}{\partial t} = -a^2\frac{\partial}{\partial x}\frac{\delta\rho}{\rho} \tag{276}$$

where

$$a^2 = \gamma\frac{P}{\rho} \tag{277}$$

Differentiating (2) with respect to t and (1) with respect to x and eliminating the mixed partial derivative leads to the classical wave equation

$$\frac{\partial^2 \delta v}{\partial t^2} = a^2\frac{\partial^2 \delta v}{\partial x^2} \tag{278}$$

The most general solution to this equation is

$$\delta v = C_1 f(x + at) + C_2 g(x - at) \tag{279}$$

where C_1 and C_2 are abitrary constants, and f and g are arbitrary functions. The function f remains unchanged when $dx/dt = -a$ and represents a disturbance traveling toward negative x at the speed a, while g represents the same thing for a disturbance traveling toward positive x at velocity a. Clearly a is the characteristic disturbance velocity at which all perturbations travel, since it is easy to show that $\delta\rho$ and δP also depend only upon the arguments $x \pm a$.

4.4.2 Harmonic Solutions

For the particular example we chose, we were able to reduce the problem to a single PDE which could be solved immediately. Rarely are we this lucky! Most of the applications are far more complex. Instead, we adopt a simpler tactic. We seek plane wave solutions of the form $\exp(ikx - i\omega t)$, knowing that we may superpose them by Fourier analysis to reproduce any initial condition we choose. The wavenumber k and angular frequency ω are related to the wavelength λ and wave frequency ν by:

$$k = \frac{2\pi}{\lambda}, \quad \omega = 2\pi\nu. \tag{280}$$

Each fluid variable has the same x, t dependence in this normal mode approach, but their amplitudes will of course differ. We are free to do this procedure from the start of the analysis, once the linear equations are found, without having to reduce the problem to one variable. This is approximately true even when the background is dependent upon x (or more spatial dimensions), provided that our wavenumber is much larger than any relevant background spatial gradient. This may be formalized in a mathematical procedure known as WKB theory.

In three-dimensions the solutions have a spatial dependence $\exp(i\boldsymbol{k} \cdot \boldsymbol{r})$ where $\boldsymbol{k}$ and $\boldsymbol{r}$ are vectors. These are called plane waves because the planes of constant $\boldsymbol{k} \cdot \boldsymbol{r}$ all have the same phase.

For sound waves, if we take our wave equation (3), and look for harmonic solutions, we find that such solutions exist, provided that

$$\omega^2 = k^2 a^2 \tag{281}$$

This is known as a dispersion relation. It tells us how the frequency depends upon wavenumber, from which one may calculate the speed of the wave. In general different wavenumbers propagate at different velocities, i.e., they disperse. But in the case of sound waves, "dispersion" is a bad choice of words: all wavenumbers propagate at the same speed: a. The dispersion relation is in fact nothing more than the familiar "wave equation," $\lambda\nu = a$.

Sound waves are the way that gases, and also solid material, communicate with themselves internally. If a small disturbance occurs at one location in a gas, for example, another location a distance x away cannot be affected by this disturbance until a time x/a has passed. If a star is vibrating with a period P, the star can be no larger in size than aP, where a is an average internal sound speed. The ultimate limitation is of course the speed of light, and this argument is often made in astrophysics: if an object changes on a time scale T, the region that is affected must be smaller than cT.

What is the speed of sound in a gas? We have

$$a^2 = \gamma \frac{P}{\rho} = \gamma \frac{nkT}{mn} = \gamma \frac{kT}{m} \tag{282}$$

where n is the number density of particles, k is the Boltzmann constant, T is the temperature, and m is the average mass per particle. For a monotomic gas $\gamma = 5/3$, for a diatomic molecule like H_2, $\gamma = 7/5$. The essential point is that the speed of sound is of order the thermal speed of a typical gas atom or molecule. In an ionized gas,

$$a^2 = \gamma \frac{n_i(1+Z)kT}{m_i n_i} = \gamma \frac{(1+Z)kT}{m_i} \tag{283}$$

where m_i is the mass of an ion and Z is the average number of electrons per ion. Note that the electrons contribute to the pressure, but not to the mass density. Taking m_i to be the proton mass m_p,

$$a = 0.117(1+Z)^{1/2}T^{1/2}\text{km s}^{-1}. \tag{284}$$

For a fully ionized hydrogen gas, this is

$$0.165T^{1/2} \simeq (T^{1/2}/6)\text{km s}^{-1}.$$

Exercise. In a molecular cloud, we will have a mixture of H_2 molecules, He atoms, and trace species that we can occur for present purposes. The He atoms will be about 20% by number, thus a significant constituent. A very simple way to model the speed of sound in such a mixture is to assume (i) each consituent conserves its mass; (ii) each constituent conserves its entropy; but (iii) the constituent species exchange momentum according to the equation

$$\rho_1 \frac{D\boldsymbol{v}_1}{Dt} = -\boldsymbol{\nabla} P_1 + \alpha\rho_1\rho_2(\boldsymbol{v}_2 - \boldsymbol{v}_1)$$

with an exchange 1 and 2 for the equation for species 2. Here α is a simple constant coupling coefficient.

1.) Justify the form of the momentum exchange term, and give an order of magnitude estimate for α in terms of collision parameters. (Hint: what are the dimensions of α?)

2.) Repeat our calculation for the dispersion relation of one-dimensional sound waves with these new coupled equations, and show that the speed of sound in such a mixture, as $\alpha \to \infty$, is

$$a_{\text{mix}}^2 = \frac{\gamma_1 P_1 + \gamma_2 P_2}{\rho_1 + \rho_2}$$

where the subscripts refer to each constituent. What do you think the speed of sound would be in a gas with a mixture of of n species?

4.5 MHD Waves: Fast, Slow, Alfvén

Astrophysicists tend to think of a magnetic field as a complication added to a background flow. But the effects of magnetic field are often profound and subtle, and I believe it is best to think of the so-called background flow as the "complication" that is to be added to the physics of a magnetized medium!

In this spirit, let us begin with the behaviour of waves in a homogeneous, uniformly magnetized medium. The fundamental equations are

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}\left(P + \frac{B^2}{2\mu}\right) + \frac{1}{\mu}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{285}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{B} = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{286}$$

The last equation is simply the standard induction equation (78), with the double cross product developed. (When comparing the MHD force equation with the one generally found in the astrophysical literature in CGS units, replace $\boldsymbol{B}/\sqrt{\mu}$ with $\boldsymbol{B}/\sqrt{4\pi}$.)

We define the z axis to lie along the magnetic field, $\boldsymbol{B} = B\boldsymbol{e_z}$, the wave number and the magnetic field define the xz plane, $\boldsymbol{k} = k_x\boldsymbol{e_x} + k_z\boldsymbol{e_z}$, and $\delta\boldsymbol{v}$ has three Cartesian components.

For a gas obeying a polytropic equation of state, as we assume, $\delta P = a^2\delta\rho$. The equation of mass conservation is

$$-i\omega\frac{\delta\rho}{\rho} + ik_x\delta v_x + ik_z\delta v_z = 0 \tag{287}$$

the x and z equations of motion are

$$-i\omega\delta v_x = -i\frac{k_x}{\rho}\left(a^2\delta\rho + \frac{B_z\delta B_z}{\mu}\right) + \frac{ik_z}{\mu\rho}B_z\delta B_x, \tag{288}$$

$$-i\omega\delta v_z = -i\frac{k_z}{\rho}\left(a^2\delta\rho + \frac{B_z\delta B_z}{\mu}\right) + \frac{ik_z}{\mu\rho}B_z\delta B_z = -i\frac{k_z a^2\delta\rho}{\rho} \tag{289}$$

and the y equation of motion is

$$-i\omega\delta v_y = i(k_z B_z/\mu)\delta B_y. \tag{290}$$

The y induction equation

$$-i\omega\delta B_y = ik_z B_z\delta v_y \tag{291}$$

with its equation of motion gives us a completely decoupled mode:

$$\omega^2 = k_z^2 B_z^2/(\rho\mu) = (\boldsymbol{k}\cdot\boldsymbol{v_A})^2 \tag{292}$$

where

$$\boldsymbol{v_A} = \frac{\boldsymbol{B}}{\sqrt{\mu\rho}} \tag{293}$$

is known as the Alfvén velocity. This is a pure magnetic tension wave, and v_A is the velocity with which the disturbance propagates along the field line, like a wave on a string.

The x and z induction equations are

$$-i\omega\delta B_x = ik_z B_z\delta v_x, \tag{294}$$

$$-i\omega\delta B_z = ik_z B_z\delta v_z - B_z(ik_x\delta v_x + ik_z\delta v_z) = -iB_z ik_x\delta v_x \tag{295}$$

The x and z induction and motion equations, together with mass conservation, yield another set of modes. These satisfy the dispersion relation

$$\omega^4 - \omega^2 k^2(v_A^2 + a^2) + k^2a^2k_z^2v_A^2 = 0. \tag{296}$$

This is a quadratic equation in ω^2 and may be solved directly, but it is easiest to see the physics when v_A and a have very different magnitudes. Then, one solution is balance between the first two terms,

$$\omega_+^2 = k^2(a^2 + v_A^2), \tag{297}$$

and the other is the balance between the last two terms

$$\omega_-^2 = \frac{k_z^2a^2v_A^2}{(v_A^2 + a^2)} \tag{298}$$

Clearly $\omega_+^2 > (k_z v_A)^2 > \omega_-^2$. The ω_+ wave is known as the fast, or magnetosonic wave, and propagates more rapidly than the Alfvén wave. The ω_- wave is known as the slow wave and propagates more slowly than the Alfvén wave. Notice that as $a \to \infty$, both the Alfvén and the slow wave have the same dispersion relation. This is the incompressible limit. In a rotating disc, however, this degeneracy is broken. The slow wave can become unstable. In this form, it is known as the magnetorotational instability.

4.6 The Rankine-Hugoniot Jump Conditions for a Shock Wave

When a sound wave propagates through a medium, the denser regions of the wave propagate more rapidly than the less dense. This can be shown rigorously using a technique developed by Bernhard Riemann (see my notes for the hydrodynamics course), but it also makes intuitive sense. The quantity $a^2 = \gamma P/\rho$ is proportional to $\rho^{2/3}$ for a monotomic gas with $\gamma = 5/3$, and this shows that nonlinear theory would lead to more rapid propagation of the denser regions of a wave. This turns out to be exactly the case.

In fact, the adiabatic nonlinear equations of motion lead to an absurdity. A smooth pulse evolves in time with the high density regions moving faster than the low density regions, the pulse steepens to an infinitely large spatial gradient, and then "beyond": the density becomes double valued! Nature of course behaves in no such way. When the density gradient becomes sufficiently steep, the particulate nature of the gas and its finite mean free path cannot be ignored. Viscous diffusion extracts energy from the bulk velocity of the gas and turns it into heat. An abrupt transition occurs: the gas is compressed and slowed down. The process conserves mass, energy and momentum, but certainly not entropy! This disturbance is known as a "shock wave," a sudden irreversible heating of the gas due to the steepening of acoustic wave fronts. The speed of the gas also changes abruptly in a shock, since the mass flux must be constant in the frame of the shock. Shocks also form when an object moves faster than the speed of sound in a gaseous medium. This is because the gas must make a sudden adjustment due to the presence of a boundary—like the nose of an aircraft or the surface of a meteorite. Continuous pressure gradients can generally only make smooth changes when an object passes through gas subsonically, when the disturbances have time to propagate *before* the gas arrives at the obstacle.

In a gas without external sources, the mass flux, momentum flux and energy flux are all strictly conserved. Let us consider a shock wave, which locally can be described in one dimension. (The finite transition region is very small compared to any global length scale.) In the frame of the shock, the flow is steady. In a one-dimensional flow, the time-steady equations are particularly simple:

$$\frac{d}{dx}(\rho v) = 0 \quad \text{mass} \tag{299}$$

$$\frac{d}{dx}\left(P + \rho v^2 + f_V\right) = 0 \quad \text{momentum} \tag{300}$$

$$\frac{d}{dx}\left(\frac{1}{2}\rho v^3 + \frac{\gamma P v}{\gamma - 1} + f_E\right) = 0 \quad \text{energy} \tag{301}$$

where f_V is the viscous flux, and f_E is the dissipative energy flux. For our purposes, we don't need to know these terms in any detail, even though it is precisely these dissipative terms that mediate the shocks. The reason is that these terms are proportional to velocity and temperature gradients, and we shall integrate our perfect derivative equations from an "upstream" region (well before the shock) to a "downstream" region (well after the shock). In these two regions, the flow is steady and one-dimensional, with constant v, P, and ρ. Dissipation is therefore negligible.

The upstream and downstream flows are very different. Given the upstream quantities ρ_1, v_1, and P_1, there is a unique downstream solution, ρ_2, v_2, and P_2 that conserves mass, energy and angular momentum. Integration of equations (299)–(301) between region 1 and 2 gives

$$\rho_1 v_1 = \rho_2 v_2 \tag{302}$$

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2 \tag{303}$$

$$\frac{1}{2}\rho_1 v_1^3 + \frac{\gamma P_1 v_1}{\gamma - 1} = \frac{1}{2}\rho_2 v_2^3 + \frac{\gamma P_2 v_2}{\gamma - 1} \tag{304}$$

Eliminating $\rho_2 v_2$ and P_2 by using equations (302) and (303), equation (304) becomes a quadratic equation for $x = v_2/v_1$:

$$x^2 - \frac{2x}{\gamma + 1}\left(\gamma + \frac{1}{M^2}\right) + \frac{\gamma - 1}{\gamma + 1} + \frac{2}{M^2(\gamma + 1)} = 0 \tag{305}$$

where $M^2 = \rho_1 v_1^2/\gamma P_1$ is the square of the Mach number M, defined as the ratio of fluid velocity to adiabatic sound speed. Since one of the roots of this quadratic equation must be $x = 1$, the nontrivial solution is just the constant term (why?):

$$x \equiv \frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{M^2(\gamma + 1)} \tag{306}$$

With

$$P_2 = P_1 + \rho_1 v_1^2 (1 - x), \tag{307}$$

the pressure ratio is found to be

$$\frac{P_2}{P_1} = \frac{2\gamma M^2}{\gamma + 1} - \frac{\gamma - 1}{\gamma + 1} \tag{308}$$

and the temperature ratio is therefore

$$\frac{T_2}{T_1} = \frac{\rho_1}{\rho_2}\frac{P_2}{P_1} = \frac{(\gamma - 1)\left[\gamma(2M^2 - 1) - (2/M^2 - 1)\right] + 4\gamma}{(\gamma + 1)^2} \tag{309}$$

The equations (306), (308), and (309) are known as the *Rankine-Hugoniot* jump conditions.

When $M^2 \to 1$, all ratios approach unity, as the flow become continuous. On the other hand, for a very strong shock ($M^2 \to \infty$):

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{\gamma - 1}{\gamma + 1} \tag{310}$$

$$\frac{P_2}{P_1} = \frac{2\gamma M^2}{\gamma + 1}, \quad \frac{T_2}{T_1} = \frac{2\gamma(\gamma - 1)M^2}{(\gamma + 1)^2} \tag{311}$$

The post-shock pressure and temperature may also be written

$$P_2 = \frac{2\rho_1 v_1^2}{\gamma + 1}, \quad T_2 = \frac{2(\gamma - 1)}{(\gamma + 1)^2}\frac{m v_1^2}{k} \tag{312}$$

where m is the mass per particle and k is the Boltzmann constant. In particular, the post-shock temperature of a strong shock for a $\gamma = 5/3$ gas is

$$T_2 = \frac{3m}{16k} v_1^2. \tag{313}$$

For a supernova explosion, typically $v_1 \simeq 5000$ km s^{-1}, and $T_2 = 5.7 \times 10^8$ K. Supernova shocks are sources of energetic X-rays.

4.7 The Propagation of Strong Shocks

4.7.1 Formulating the problem

When a powerful shock wave of velocity v_S passes through a medium at rest, it imparts a velocity

$$v = v_S \left(1 - \frac{\gamma - 1}{\gamma + 1}\right) = \frac{2v_S}{\gamma + 1} \tag{314}$$

to the gas. This is simply the velocity in the frame in which the shock is at rest "boosted" to a frame in which the undisturbed fluid is at rest. The shock thus imparts a velocity to the fluid which is a fixed fraction of the the shock velocity itself. Not surprisingly, since $\gamma > 1$, the shock moves more rapidly outwards than the fluid, as it should!

A problem of considerable interest, both astrophysical and terrestrial, is to determine the precise functional form of the velocity, density, and pressure throughout the entire region interior to the shock. In astrophysics, this is a classical model for a supernova explosion; in terrestrial applications this problem is associated with nuclear weapons research.

Remarkably, a closed form solution for this problem exists. G.I. Taylor first analyzed the problem numerically after realizing that the radius of the shock must have a very simple power law dependence on time. This insight was further developed by Sedov and von Neumann (independently of one another), who produced a closed form, analytic solution.

Taylor's initial insight was the following. The post shock structure of a sufficiently powerful explosion is determined by only two physical parameters: the energy released in the shock E, and the density ρ_1 of the undisturbed medium. Nothing else matters. From E, ρ and time t, only one quantity can be constructed that has dimensions of length:

$$r_S \sim (Et^2/\rho_1)^{1/5}. \tag{315}$$

Up to a proportionality constant, this must be the radius of the shock! Although this constant cannot be determined by simple arguments, the time dependence $t^{2/5}$ is exact. In particular the shock velocity is precisely

$$v_S = \frac{2r_S}{5t}, \tag{316}$$

and the fluid velocity immediately behind the shock is

$$v(r_S) = \frac{2v_S}{\gamma + 1} = \frac{4}{5(\gamma + 1)} \frac{r_S}{t} \tag{317}$$

One can take this insight further. The density behind the shock is a function of both r and t. But there is only one parameter in the problem with units of density, and that is ρ_1. Therefore, the postshock density must be directly proportional to ρ_1 times a dimensionless function of r and t. There is no other representation that is dimensionally correct. Moreover, the dimensionless density function $\tilde{\rho}$ must itself depend upon the only dimensionless combination of r and t that can be formed for this problem:

$$\xi \equiv r(Et^2/\rho_1)^{-1/5} \tag{318}$$

Thus, the density must take the form

$$\rho = \frac{\gamma + 1}{\gamma - 1} \rho_1 \; \tilde{\rho}(\xi) \tag{319}$$

The constant factor in front is chosen so that $\tilde{\rho}$ is unity at the location of the shock, $\xi = \xi_S$. ξ_S is not known *a priori*, and must be determined as part of the problem.

The velocity field behind the shock, by similar reasoning, must take the form

$$v = \frac{4}{5(\gamma+1)}\frac{r}{t}\tilde{v}(\xi), \quad \tilde{v}(\xi_S) = 1. \tag{320}$$

The pressure immediately behind the shock is by (312),

$$P(r_S) = \frac{2\rho_1 v_S^2}{\gamma+1}, \tag{321}$$

and the pressure throughout the shock interior is therefore

$$P = \frac{8}{25(\gamma+1)}\frac{\rho_1 r^2}{t^2}\tilde{P}(\xi), \quad \tilde{P}(\xi_S) = 1 \tag{322}$$

That the detailed space-time dependencies of the fundamental flow variables are functions of very simple power law combinations of r and t is characteristic of what are known as *self-similar* flows. The name comes from the fact that the flow looks "similar" to itself at different values of r and t corresponding to a curve of constant ξ. Self-similar flows in nature are rare, since their existence generally requires that only a very small number of dimensional quantities enters into the problem, and this is usually inconsistent with boundary conditions. In particular, self-similarity does not offer a general method to solve partial differential equations, which are usually difficult to solve, even by numerical means.

4.7.2 Equations

The problem now consists of writing the adiabatic, one-dimensional fluid equations for spherical flow using our self-similar functions. The following simple identities are useful for this purpose:

$$\frac{\partial f(\xi)}{\partial r} = \frac{\xi}{r}\frac{df}{d\xi}, \quad \frac{\partial f(\xi)}{\partial t} = -\frac{2\xi}{5t}\frac{df}{d\xi} \tag{323}$$

Our final equations become coupled ordinary differential equations with ξ being the only independent variable.

The mass conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial r} + \frac{2\rho v}{r} = 0 \tag{324}$$

becomes

$$\frac{d\ln\tilde{\rho}}{d\ln\xi}\left[\tilde{v} - \left(\frac{\gamma+1}{2}\right)\right] + \frac{d\tilde{v}}{d\ln\xi} = -3\tilde{v}. \tag{325}$$

The entropy equation

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\ln P\rho^{-\gamma} = 0 \tag{326}$$

becomes

$$\frac{d\ln\tilde{P}}{d\ln\xi} - \gamma\frac{d\ln\tilde{\rho}}{d\ln\xi} = \frac{5(\gamma+1) - 4\tilde{v}}{2\tilde{v} - (\gamma+1)} \tag{327}$$

The final equation is the equation of motion, but here we will use a beautiful trick that is the key to the solution. Rather than use the equation of motion, we use *total* energy conservation. The entropy equation governs only the internal energy; to say something about the total energy, both the entropy equation and the Euler equation of motion (which is a statement about mechanical energy) must be used. The total energy equation makes explicit use of both of these equations, and so incorporates the information implicit in the equation of motion.

Consider a sphere of fixed Eulerian radius $r < r_S$. In a time dt, the fluid passing through the surface will carry away an energy

$$4\pi r^2 \left(\frac{\rho v^3}{2} + \frac{\gamma P v}{\gamma - 1} \right) dt. \tag{328}$$

This energy is lost from within the fixed volume sphere at radius r. Now use self-similarity: consider *initially* the same sphere, but in the time interval dt let your eye follow instead a shell of constant ξ, which moves out from the initial surface at a velocity of $2r/5t$. The energy inside this $\xi =$ constant moving shell must be constant! Thus the energy represented by equation (328), which has been lost from within a fixed radius r, must all be contained just beyond r in a shell of thickness $(2r/5t)dt$! This energy is

$$4\pi r^2 \left(\frac{\rho v^2}{2} + \frac{P}{\gamma - 1} \right) \left(\frac{2r}{5t} \right) dt. \tag{329}$$

Hence

$$\left(\frac{\rho v^2}{2} + \frac{\gamma P}{\gamma - 1} \right) v = \left(\frac{\rho v^2}{2} + \frac{P}{\gamma - 1} \right) \left(\frac{2r}{5t} \right). \tag{330}$$

In terms of our self-similar variables, this equation becomes

$$\frac{\tilde{P}}{\tilde{\rho}} = \frac{\tilde{v}^2 (1 + \gamma - 2\tilde{v})}{2\gamma\tilde{v} - 1 - \gamma} \tag{331}$$

Notice that this not a differential equation at all, but a sort of energy integral. Our problem thus reduces to two ordinary differential equations that are readily decoupled.

When differentiated with respect to ξ, our energy equation relates the logarithmic derivatives of $\tilde{P}$ and $\tilde{\rho}$:

$$\frac{d \ln \tilde{P}}{d \ln \xi} = \frac{d \ln \tilde{\rho}}{d \ln \xi} + \frac{d\tilde{v}}{d \ln \xi} \left(\frac{2}{\tilde{v}} - \frac{2}{1 + \gamma - 2\tilde{v}} - \frac{2\gamma}{2\gamma\tilde{v} - 1 - \gamma} \right) \tag{332}$$

Let us define the coefficient of $dv/d \ln \xi$ above as $f(\tilde{v})$. Then, if we eliminate $d \ln \tilde{P}/d \ln \xi$ in the entropy equation, we obtain

$$(1 - \gamma) \frac{d \ln \tilde{\rho}}{d \ln \xi} + \frac{d\tilde{v}}{d \ln \xi} f(\tilde{v}) = \frac{5(\gamma + 1) - 4\tilde{v}}{2\tilde{v} - \gamma - 1} \tag{333}$$

Now, recall that the mass equation is just

$$\left(\tilde{v} - \frac{(\gamma + 1)}{2} \right) \frac{d \ln \tilde{\rho}}{d \ln \xi} + \frac{d\tilde{v}}{d \ln \xi} = -3\tilde{v}. \tag{334}$$

Therefore, we may eliminate the density derivative between the last two equations, and obtain a single equation for $d\tilde{v}/d\ln\xi$:

$$\left[\frac{f(\tilde{v})(\tilde{v}-(\gamma+1)/2)-1+\gamma}{5(\gamma+1)/2+\tilde{v}(1-3\gamma)}\right]\frac{d\tilde{v}}{d\ln\xi}=1 \tag{335}$$

Substituting the explicit value of $f(\tilde{v})$ and simplifying, the equation for $\tilde{v}$ becomes

$$\frac{d\ln\xi}{d\tilde{v}}=\frac{\gamma+1}{5(\gamma+1)/2+\tilde{v}(1-3\gamma)}\left(\frac{\tilde{v}-1}{\tilde{v}}+\frac{\gamma-1}{2\gamma\tilde{v}-1-\gamma}\right) \tag{336}$$

We have reduced our problem to the solution of a separable, first order, ordinary differential equation.

The integral over $\tilde{v}$ for $\ln\xi$ may be broken up into several elementary integrals by a partial fraction expansion, and then summed. This is a completely straightforward, but lengthy, exercise. With the boundary condition $\tilde{v}(\xi_S)=1$, the result is

$$\left(\frac{\xi}{\xi_S}\right)^5=(\tilde{v})^{-2}\left(\frac{2\gamma\tilde{v}-1-\gamma}{\gamma-1}\right)^{\nu_1}\left(\frac{7-\gamma}{5(\gamma+1)-2\tilde{v}(3\gamma-1)}\right)^{\nu_2} \tag{337}$$

where

$$\nu_1=\frac{5(\gamma-1)}{1+2\gamma},\qquad \nu_2=\frac{13\gamma^2-7\gamma+12}{(3\gamma-1)(1+2\gamma)} \tag{338}$$

The self-similar density $\tilde{\rho}$ may now be determined directly from the mass conservation equation:

$$\frac{d\ln\tilde{\rho}}{d\ln\xi}=\frac{2}{\gamma+1-2\tilde{v}}\left(1+3\tilde{v}\frac{d\ln\xi}{d\tilde{v}}\right) \tag{339}$$

together with equation (336). Again, this is a straightforward partial fraction decomposition, and a series of elementary, but lengthy, integrations. The result is

$$\tilde{\rho}=\left(\frac{\gamma-1}{\gamma+1-2\tilde{v}}\right)^{\nu_3}\left(\frac{5(\gamma+1)-2\tilde{v}(3\gamma-1)}{7-\gamma}\right)^{\nu_2/(2-\gamma)}\left(\frac{2\gamma\tilde{v}-1-\gamma}{\gamma-1}\right)^{\nu_4} \tag{340}$$

where

$$\nu_3=\frac{2}{2-\gamma},\qquad \nu_4=\frac{3}{2\gamma+1} \tag{341}$$

Our final task is to determine the constant ξ_S. It is fixed by the requirement that the energy within r_S is E at any time:

$$\int_0^{r_S}\left(\frac{\rho v^2}{2}+\frac{P}{\gamma-1}\right)4\pi r^2\,dr=E \tag{342}$$

In terms of $\tilde{v}$, $\tilde{\rho}$, and $\tilde{P}$, this becomes

$$\frac{32\pi}{25(\gamma^2-1)}\xi_S^5\int_0^1 u^4(\tilde{P}+\tilde{\rho}\tilde{v}^2)\,du=1 \tag{343}$$

where $u = \xi/\xi_S$ is precisely the argument of the self-similar functions that appears in equation (337) and implicitly in equation (340). Equation (343) may be written

$$\xi_S = 0.757(\gamma^2 - 1)^{1/5} I^{-1/5} \tag{344}$$

where I is the $\int_0^1$ integral in equation (343). It is found that except near $\gamma = 1$, $I^{-1/5}$ is always very close to unity, decreasing slightly as γ decreases. Most of the dependence upon γ is the explicit factor $(\gamma^2 - 1)^{1/5}$; the $I^{-1/5}$ is found to have only a weak γ dependence. ξ_S is typically very close to unity for nonisothermal equations of state: $\xi_S = (0.995, 1.033, 1.152)$ for $\gamma = (4/3, 7/5, 5/3)$ respectively.

We close our discussion by noting that it is possible to solve the problem of a relativistic blast wave (when the shock velocity approaches the speed of light) by similar techniques. This is surprising, since another dimensional parameter enters the problem, namely the speed of light. Nevertheless, by very clever reasoning, Blandford & McKee have shown that relativistic blast waves are also self-similar in their structure. (The reference is Blandford & McKee 1976, *Physics of Fluids B,* **19,** 1130.)

4.8 The Effects of Rotation on Sound Waves

4.8.1 The Epicyclic Frequency

A rotating medium is characterized by a response frequency that is not necessarily the rotation frequency. It is known as the *epicyclic frequency,* denoted κ^2. To understand the origin of κ, consider a very simple problem: a point mass in a circular orbit in the field of a central potential Φ. The mass could be a planet, in which case our problem is Keplerian, or it might be a star in orbit in a galaxy, in which case the rotation law is non-Keplerian. The fundamental equations of motion in the orbital plane, expressed in standard polar coordinates r, ϕ, are

$$\ddot{r} - r\dot{\phi}^2 = -\frac{\partial \Phi}{\partial r} \tag{345}$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0 \tag{346}$$

where Φ is the gravitational potential.

The equilibrium orbit solution is $\dot{\phi} = \Omega$, where

$$r\Omega^2 = \frac{\partial \Phi}{\partial r} \tag{347}$$

and $r = R$, where R is the circular orbit radius. Imagine now that the mass is slightly perturbed from its circular orbit, so that

$$r = R + x, \quad \phi = \Omega t + \alpha \tag{348}$$

where x and α are very small. Then, the equations of motion are:

$$\ddot{x} - (R + x)(\Omega + \dot{\alpha})^2 = -\frac{\partial \Phi}{\partial R} - x\frac{\partial^2 \Phi}{\partial R} + \text{...higher order terms,} \tag{349}$$

$$(R + x)(\ddot{\alpha}) + 2\dot{x}(\Omega + \dot{\alpha}) = 0 \tag{350}$$

The derivatives of Φ are evaluated at $r = R$, and we are expanding Φ in a Taylor series, having retained only the first term. Indeed, we should retain only the terms that are linear in x and α everywhere. Noting that $R\Omega^2$ cancels $\partial\Phi/\partial R$, and $\partial^2\Phi/\partial R^2 = d(R\Omega^2)/dR$, this leaves the linearized equations:

$$\ddot{x} - 2\Omega R\dot{\alpha} = -x\frac{d\Omega^2}{d\ln R} \tag{351}$$

$$R\ddot{\alpha} + 2\dot{x}\Omega = 0 \tag{352}$$

The coefficients R and Ω are regarded as constants in this approximation, evaluated at the circular orbit radius R. We may therefore seek solutions of the form $\exp(i\omega t)$ for x and α, just as though we were doing a normal mode analysis. It is a simple matter to show that such solutions exist provided that

$$\omega^2 = 4\Omega^2 + \frac{d\Omega^2}{d\ln R} \equiv \kappa^2 \tag{353}$$

and

$$\alpha = \frac{2i\Omega}{\omega R}x, \tag{354}$$

where it is understood that the real part of this equation is to be taken. Equation (353) defines the *epicyclic frequency* κ. To better understand the eigenmode itself, let $x = X\exp(i\kappa t)$. Then the real solutions would be

$$x = X\cos(\kappa t), \quad \alpha = -\frac{2X\Omega}{\kappa R}\sin(\kappa t) \tag{355}$$

Since x and $R\alpha$ describe locally orthogonal cartesian axes (R radial, $R\alpha$ azimuthal), the above solution describes a retrograde elliptical path with a major to minor axis ratio of $2\Omega/\kappa$. (We will always work with decreasing outward rotation profiles, so that $\kappa < 2\Omega$.) The major axis of the ellipse lies along the azimuthal direction.

For a Keplerian disc, it so happens that $\kappa = \Omega$, but this agreement is an accident. (It allows noncircular orbits to be closed in a Newtonian force field.) The yearly radial drifting of the earth closer to and farther from the sun corresponds to epicyclic oscillation. The gravitational potential of many galaxies, on the other hand, gives rise to a rotational profile of the form $\Omega \propto 1/R$, and thus $\kappa^2 = 2\Omega^2$. Stellar orbits in such galaxies do not form simple closed curves, unless they are exactly circular.

Exercise. (Simple maths but tricky.) Prove that a $R^{-11/9}$ force law will produce closed orbits, at least to first order as per our calculation above. Can you generalise this to deduce that if the force law is proprortional to R^p, where

$$p = \frac{n^2 - 3m^2}{m^2},$$

n and m are integers, then the orbits are first-order closed? The Keplerian case is $n = m = 1$.

Exercise. Derive equation (353) directly from the local equation of motion, equation (34).

Exercise. The radial epicyclic amplitude of the earth is about 1.7% of its orbit, but the *shape* of the earth's orbit is circular to within 0.03%! Explain.

4.8.2 Density Waves

Sound waves in discs with significant rotational modification are known as density waves. To understand their dispersion relation, let us begin with the fundamental equations of motion written out in cylindrical coordinates:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0 \tag{356}$$

$$\frac{\partial v_R}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_R - \frac{v_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial P}{\partial R} - \frac{\partial \Phi}{\partial R} \tag{357}$$

$$\frac{\partial v_\phi}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_\phi + \frac{v_\phi v_R}{R} = -\frac{1}{\rho R}\frac{\partial P}{\partial \phi} - \frac{1}{R}\frac{\partial \Phi}{\partial \phi} \tag{358}$$

$$\frac{\partial v_z}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla} v_z) = -\frac{1}{\rho}\frac{\partial P}{\partial z} - \frac{\partial \Phi}{\partial z} \tag{359}$$

The first of these equations is mass conservation, followed respectively by the radial, azimuthal, and vertical equations of motion. For simplicity, we adopt a polytropic equation of state, $P = K\rho^\gamma$. If, as we shall assume, the background unperturbed disc is independent of ϕ, Eulerian linear perturbations δX can be chosen to have the form

$$\delta X(R,z)\ \exp(im\phi - i\omega t) \tag{360}$$

where m is an integer. The background angular velocity Ω must depend only upon R (why not z as well?), and the linearized equations of motion become

$$-i(\omega - m\Omega)\frac{\delta\rho}{\rho} + \frac{1}{\rho}\boldsymbol{\nabla}\cdot(\rho\delta\boldsymbol{v}) = 0, \tag{361}$$

$$-i(\omega - m\Omega)\delta v_R - 2\Omega\delta v_\phi = -\frac{\partial \delta H}{\partial R} \tag{362}$$

$$-i(\omega - m\Omega)\delta v_\phi + \frac{\kappa^2}{2\Omega}\delta v_R = -\frac{im\delta H}{R} \tag{363}$$

$$-i(\omega - m\Omega)\delta v_z = -\frac{\partial \delta H}{\partial z} \tag{364}$$

We have introduced the enthalpy function

$$H = \frac{\gamma K \rho^{\gamma-1}}{\gamma - 1} = \frac{a^2}{\gamma - 1}, \qquad \delta H = a^2\frac{\delta\rho}{\rho} \tag{365}$$

where a is the speed of sound.

We may combine these equations into a single partial differential equation for δH. Introducing the notation

$$\bar{\omega} = \omega - m\Omega, \qquad D = \kappa^2 - \bar{\omega}^2, \tag{366}$$

we find

$$\delta v_R = \frac{i}{D}\left[\bar{\omega}\frac{\partial \delta H}{\partial R} - \frac{2m\Omega}{R}\delta H\right], \tag{367}$$

$$\delta v_\phi = \frac{1}{D}\left[\frac{\kappa^2}{2\Omega}\frac{\partial \delta H}{\partial R} - \frac{m\bar{\omega}}{R}\delta H\right], \tag{368}$$

$$\delta v_z = -\frac{i}{\bar{\omega}}\frac{\partial \delta H}{\partial z} \tag{369}$$

If we combine these results with the mass conservation equation, carry through the differentiations, and then simplify, the result is a partial differential equation:

$$\left[\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{R\rho}{D}\frac{\partial}{\partial R}\right) - \frac{1}{\bar{\omega}^2}\frac{\partial}{\partial z}\left(\rho\frac{\partial}{\partial z}\right) - \frac{m^2\rho}{R^2 D} - \frac{1}{R\bar{\omega}}\frac{\partial}{\partial R}\left(\frac{2\Omega m\rho}{D}\right) - \frac{\rho}{a^2}\right]\delta H = 0 \tag{370}$$

This is a complicated but very general equation describing the behaviour of linear perturbations that depend upon all three spatial coordinates. This is more general than we shall need for our present purposes, but it is nice to have the equation without any approximations! (If you feel ambitious, and enjoy working with waves, see my review article for more: S. Balbus 2003, *Annual Reviews in Astronomy and Astrophysics,* **41,** 555.)

The simplest type of density wave is axisymmetric and independent of z. If, in addition, the radial variation is very rapid, we may approximate the spatial oscillations as a local plane wave of the form $\exp(ik_R R)$. If L_R is the local radial scale for the background disc (typically $L_R \sim R$), then $k_R L_R \gg 1$, the WKB approximation. We shall also assume that $k_R R \gg m$. Either from the linearized equations (361-365), or directly from the linear wave equation (370), the dispersion relation is found to be

$$\bar{\omega}^2 = \kappa^2 + k_R^2 a^2 \tag{371}$$

These are the simplest form of density waves, which are evidently just acoustic waves, "boosted" by the epicyclic frequency. Stated slightly differently, density waves cannot exist at frequencies below κ: they are cut-off (like, for example, electromagnetic waves in a waveguide).

Density waves are of particular importance in theories of galactic structure and protostellar discs. In galactic discs (where self-gravity may be significant), density waves are thought to initiate star formation. In protostellar discs, these waves carry away the angular momentum of planets embedded in the gas, and the planet could spiral inwards toward the inner regions of the disc—even be swallowed by the central star!

4.8.3 Inertial Waves

Let us repeat our WKB density wave calculation, but allow for a z dependence in our wave form; $\exp(ik_R R + ik_z z)$. Then, once again assuming that both components of the wavenumber are large compared to the inverse R and z length scales, we find the following dispersion relation

$$\frac{k_R^2 a^2}{\bar{\omega}^2 - \kappa^2} + \frac{k_z^2 a^2}{\bar{\omega}^2} = 1 \tag{372}$$

Let $k^2 = k_R^2 + k_z^2$. As $|k| \to \infty$, there are two very distinct solutions, or "branches" to this dispersion relation. The first is one in which $\bar{\omega}$ (and therefore ω) is also very large. These are ordinary acoustic modes, $\bar{\omega}^2 = k^2 a^2$, the large wavenumber behaviour of density waves. The second branch is new. If we assume that $\bar{\omega}$ remains of order κ (or smaller) as $|k|$ becomes large, we find that such a solution exists, provided that

$$\bar{\omega}^2 = \frac{k_z^2}{k^2}\kappa^2 \tag{373}$$

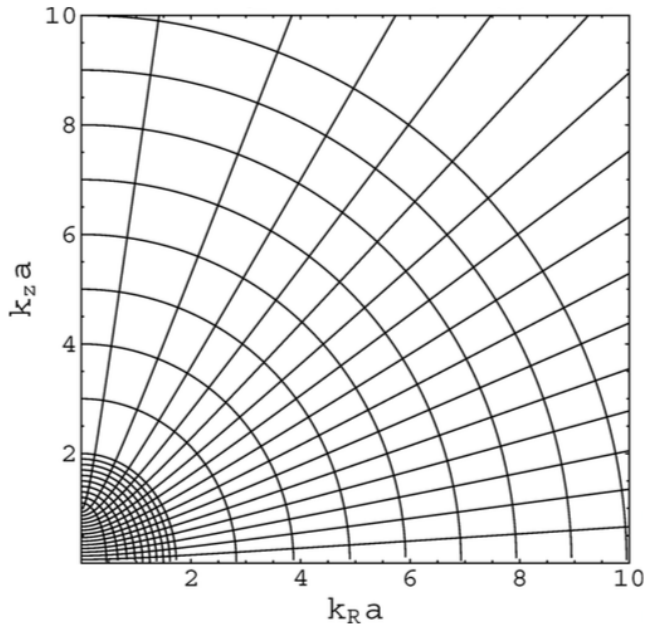

Figure 7: Contours of constant $\bar{\omega}$ for the dispersion relation (372) for a Keplerian disc. These form a set of confocal ellipses (density wave branch) and hyperbolae (inertial wave branch). The value of $\bar{\omega}$ for a particular curve is read off by where the curve intersects the $k_z a$ axis; the numerical scale is in units of $\Omega = 1$. Note that all inertial waves have $\bar{\omega} \leq 1$.

These waves are completely independent of the sound speed, and are the only response possible at frequencies less than κ^2. They are called "inertial waves" because the restoring force arises from the inertial terms in the equations of motion. (I.e., from terms in $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$.) Inertial waves are basically epicycles that have been modified by pressure forces. In general, there is a smooth continuum of waves hosted by a disc at all frequencies, but one that changes character as ω passes through κ.

Figure (7) shows the dispersion relation (372) in the form of isofrequency $\bar{\omega}$ surfaces. As an *Exercise,* the reader should show that the iso-$\bar{\omega}$ curves for inertial and density waves intersect at right angles, and that the iso-$\bar{\omega}$ curves for one family serve as streamlines for the group velocity for the other family.

4.9 Angular Momentum Transport

One of the most important properties of waves is their ability to transport angular momentum. We are are particularly interested in outward radial transport, since this may aid the process of accretion in discs. The radial flux of angular momentum is generally given by

$$F_J = \rho R v_R v_\phi. \tag{374}$$

The unperturbed disc has vanishing v_R and v_ϕ given by its Keplerian value. The passage of a small amplitude wave causes small first-order Eulerian δ-perturbations to appear, with zero time-averaged mean. Importantly, there are also *second order* changes in the velocity,

$(v_{R2}, v_{\phi 2})$, with non-vanishing mean values. Now, an adiabatic small amplitude wave does not cause a shift in the position of the fluid element it is passing through, any more than a wave moving through a spring causes the spring to permanently shift. The leading order contribution to the radial mass flux,

$$\langle \delta\rho \delta v_R \rangle + \langle \rho v_{R2} \rangle \tag{375}$$

must therefore vanish when time-averaged (the meaning of the angle brackets). Note that this is quadratic in the δ amplitudes because the terms linear in δ average (identically) to zero.

In a wave with radial velocity component δv_R and azimuthal velocity component δv_ϕ, there is only one second order term that survives,

$$F_J = \rho R \langle \delta v_R \delta v_\phi \rangle, \tag{376}$$

since all the others combine to form a vanishing mass flux!

The above expression should be averaged over one wave cycle. Of course, if we are working with complex-valued quantities, the real part must be taken before the averaging is performed. For example, let

$$\delta H = A \exp(i\psi) \tag{377}$$

where $\psi = k_R R + m\phi + k_z z - \omega t$, and A is taken to be constant to leading order. Then

$$\Re(\delta v_R) = -\frac{A k_R \bar{\omega}}{D} \cos\psi + \frac{2mA\Omega}{RD} \sin\psi \tag{378}$$

$$\Re(\delta v_\phi) = -\frac{A k_R \kappa^2}{2\Omega D} \sin\psi - \frac{mA\bar{\omega}}{RD} \cos\psi \tag{379}$$

where $\Re$ denotes the real part. On multiplying these two expressions, we note that only terms that are proportional to $\cos^2\psi$ or $\sin^2\psi$ survive the averaging. Since

$$\langle \cos^2\psi \rangle = \langle \sin^2\psi \rangle = \frac{1}{2}, \tag{380}$$

the radial angular momentum flux is

$$F_J = \frac{A^2}{2} \frac{m\rho k_R}{\bar{\omega}^2 - \kappa^2}. \tag{381}$$

At a given time in a plane of constant z, points separated by dR and $d\phi$ along a curve of constant phase ("wave crest") satisfy the constraint

$$k_R dR + m d\phi = 0. \tag{382}$$

A wave crest with the property $dR/d\phi < 0$ must therefore have $m/k_R > 0$, and is known as trailing, since following a radially outward path, it sweeps back toward negative ϕ. (The opposite of trailing is "leading.")

The presence of shear in discs kinematically distorts wavecrests, so that most disturbances evolve into trailing waves. Trailing density waves ($\bar{\omega}^2 > \kappa^2$) transport angular momentum outward. Trailing inertial waves ($\bar{\omega}^2 < \kappa^2$) transport angular momentum inward.

Now, in order for gas in the disc to accrete, angular momentum must be transported outwards while the matter moves inwards: the mass cannot take the angular momentum with

it. Trailing density waves do precisely this, but unless the disc is strongly self-gravitating, it will not generate its own density waves. These waves must be externally driven. As practical examples, this might be done by a companion star in a binary system, or by internal embedded planets in a protostellar disc. But *something* is certainly needed.

By the criterion of angular momentum transport, trailing inertial waves do more harm than good. Transporting angular momentum inward, they make it impossible for the disc to accrete. Notice that random forcing of the disc at frequencies less than the epicyclic frequency will generate a fourier superposition of inertial waves, provided that the forcing is not too strong. Such waves always become trailing and always move angular momentum the "wrong way." We will see later how all of this changes when we allow magnetic fields into the disc.

5 Classic Instabilities

Myself is thus and so, and will continue thus and so. And why fight it? My balance comes from instability.

— *Saul Bellow*

Waves are the natural response when a fluid is disturbed and restoring forces are activated. But there are circumstances in which the responding forces are not restorative: they drive the fluid yet further from its initial equilibrium. In such cases, the flow is said to be *unstable;* the process by which it evolves from the equilibrim is an *instability.* We introduce the subject with one of the most important instabilities in astrophysics: gravitational instability, otherwise known as the *Jeans instability.*

5.1 The Jeans Instability

What is the effect of self-gravity on a sound wave in an astrophysical gas? We may attack this problem in a crude but simple way by revisiting our §4.4 analysis of one-dimensional sound waves with a nearly trivial addition. (Trivial, at least, in a mathematical sense.) Consider an infinite homogeneous medium, allowing for the effects of self-gravity. (We will not concern ourselves yet with the conceptual difficulties of establishing this initial condition. They do not affect our conclusion.) Denoting the gravitational potential by Φ, the linearized equations for the Eulerian δ perturbations are mass conservation

$$\frac{\partial}{\partial t}\frac{\delta\rho}{\rho} + \frac{\partial \delta v}{\partial x} = 0 \tag{383}$$

the equation of motion,

$$\frac{\partial \delta v}{\partial t} = -\frac{1}{\rho}\frac{\partial \delta P}{\partial x} - \frac{\partial \delta \Phi}{\partial x} \tag{384}$$

the Poisson equation,

$$\frac{\partial^2 \delta\Phi}{\partial x^2} = 4\pi G \delta\rho \tag{385}$$

and the adiabatic equation of state,

$$\frac{\delta P}{P} = \gamma \frac{\delta\rho}{\rho}. \tag{386}$$

If we eliminate δP in favour of $\delta\rho$ and seek solutions of the form $\exp(ikx - i\omega t)$, the above equations above give

$$-i\omega\frac{\delta\rho}{\rho} + ik\delta v = 0 \tag{387}$$

$$-i\omega\delta v = -ika^2\frac{\delta\rho}{\rho} - ik\delta\Phi \tag{388}$$

$$-k^2\delta\Phi = 4\pi G\delta\rho \tag{389}$$

where

$$a^2 = \gamma \frac{P}{\rho} \tag{390}$$

is the sound speed. The dispersion relation emerging from equations (387)–(389) is

$$\omega^2 = k^2 a^2 - 4\pi G\rho \tag{391}$$

Comparing with the pure sound wave dispersion relation $\omega^2 = k^2 a^2$, there is a simple but striking difference: the attractive force of gravity slows down the response frequency. To first order, it speeds up the contraction and retards the dilation to equal measure, but to second order it slows the dilation phase for a longer time. The net effect is a general slowing of the oscillation. Indeed, for wavenumbers smaller than a value denoted

$$k_J = \sqrt{\frac{4\pi G\rho}{a^2}} \tag{392}$$

there is no reexpansion at all! Then, $\omega^2 < 0$ so that ω is purely imaginary. The imaginary part could be either positive or negative. Looking back at our assumed $\exp(-i\omega t)$ time dependence, we see that the positive imaginary part branch corresponds to exponential growth! The "wave" no longer propagates, it just collapses. This is the quintessential behaviour of a local instability.

For *wavelengths* in excess of the so-called *Jeans length*,

$$\lambda_J = \frac{2\pi}{k_J} = a\sqrt{\frac{\pi}{G\rho}} \tag{393}$$

the system responds with overdensities imploding and underdensities expanding. Short wavelengths are stabilised by pressure gradients, whereas long wavelengths lack this pressure gradient support. We may define the *Jeans Mass* M_J as the mass of unperturbed fluid within a cube of side $\lambda_J/2$, the factor of two restricting the sample to the overdense zone. This gives

$$M_J = \left(\frac{\pi}{4G}\right)^{3/2} \left(\frac{a^3}{\rho^{1/2}}\right) \tag{394}$$

For conditions in the dense cores of molecular clouds, in which the number densities might be of order 10^6 cm^{-3} and temperatures about 10K, M_J is typically a fraction of a solar mass. The significance of this finding is not entirely clear (star formation is very complex), but probably it is not purely coincidental.

Exercise. Verify the last comment for a molecular cloud in which He is 10% by number relative to the proton number density.

Though the mathematics is from James Jeans, Isaac Newton understood the concept over 200 years before the calculation. Arguing for the need for a truly infinite space in a letter to the cleric Richard Bentley, Newton argues that a finite self-gravitating universe would collapse toward its centre in a great lump,

"[b]ut if the matter were disposed throughout an infinite space, it could never convene into one mass, but some of it would convene into one mass and some into another, so as to make an infinite number of great masses, scattered at great distances from one another throughout all that infinite space. And thus might the sun and fixed stars be formed, supposing the matter were of a lucid nature."

The Jeans instability is an example of an important class of disturbance that acts like a wave at some wavelengths, but exhibits exponential growth at others. (Exponential growth need not be restricted to long wavelengths: anything is possible.) These distrubances that grow rapidly are sometimes called *local* instabilities. They are local because they can be understood by looking at small scale flow and background features, and uunstable because a small departure from equilibrium becomes rapidly a large one. The local approximation is a great simplification, and often allows for substantive physical insight.

But Jeans's calculation is not quite fair. We are generally interested in finite bodies, and here the fact that the density is often not even approximately constant over the wavelengths of interest is troublesome. The Jeans instability is sometimes referred to as the "Jeans swindle!" One nice example where we can see the Jean's instability at work in a rigourous calculation, and at the same time see a different mathematical venue for the instability is the calculation of gravitational instability in a uniformly expanding background: the linear stability theory of our universe.

5.2 Sound Waves and Gravity in Cosmology

5.2.1 The Bonnor-Lifschitz Equation

In the expanding universe (the "Hubble flow"), the growth of inhomogeneities involves an interplay between sound waves and self-gravity. Although the universe on large scales seems to be rather uniform, the existence of gravitationally bound matter in the form of clusters of galaxies and individual galaxies indicates that homogenity cannot hold on small scales. It is of interest to ask what the fate of a small gaseous density perturbation would be in an expanding cosmology, whether this might tell us something about the way galaxies or clusters form, and whether there might be any way to observe large scale oscillations of the universe.

The needed equations of motion are mass conservation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0, \tag{395}$$

and the force equation,

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi. \tag{396}$$

Φ is the local gravitational potential. To evaluate it, we will use a remarkable and important result which follows very generally from relativity theory and hold in a homogeneous, isotropic universe: Within a sphere of radius r, no matter how the sphere is chosen, only the matter between the (arbitrary) origin and r contributes to the gravitational field at the point r. In the unperturbed uniformly expanding flow, the density ρ is everywhere a spatial constant. The inward radial force per unit mass at r is then

$$g = -\frac{\partial \Phi}{\partial r} = -\frac{G(4\pi\rho r^3/3)}{r^2} = -\frac{4\pi}{3}G\rho r \tag{397}$$

or

$$\Phi = \frac{2\pi}{3}G\rho r^2 \tag{398}$$

In this gas, consider two points moving with the background expansion, and let their separation be l at some moment in time t_0. Then, at other times, their separation will be

$$r = R(t)l \tag{399}$$

where R is known as the scale factor ($R(t_0) = 1$). Notice that the right side of this equation is linear in l, and that l is constant over the course of the expansion. Choose one of the points as the origin, and differentiate this equation with respect to time. One obtains the vector equation

$$\frac{d\boldsymbol{r}}{dt} \equiv \boldsymbol{v} = \frac{\dot{R}}{R}\boldsymbol{r}, \tag{400}$$

since the expansion is always radially directed away from the point of origin. The scale factor $R(t)$ is different for different cosmological models, and we will leave it undetermined for the moment. Finally, the evolution of the background density ρ follows from mass conservation,

$$\rho = \rho_0/R^3 \tag{401}$$

where ρ_0 is the density at $t = t_0$. This is physically self-explanatory, but it also follows directly from the mass equation for uniformly expanding flow,

$$\frac{\partial \ln \rho}{\partial t} = -\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}. \tag{402}$$

The equation of motion governing the physical separation r of two points in an expanding universe is thus now

$$\frac{d^2 r}{dt^2} = -\frac{4\pi G\rho r}{3} \tag{403}$$

or

$$\ddot{R} = -\frac{4\pi G\rho R}{3} \tag{404}$$

With ρR^3 constant, this integrates to

$$\dot{R}^2 - \frac{8\pi G\rho R^2}{3} = \text{const.} \tag{405}$$

A universe in critical balance is one in which the constant on the right side is negligible. In fact, the early universe seems to be well described by such a model, and we shall adopt it[5]. We immediately obtain two results of interest. First, with ρR^3 constant, it follows that $R\dot{R}^2$ is constant, whence $R \propto t^{2/3}$. (The time t_0 is defined so that $R = (t/t_0)^{2/3}$.) With this scaling for R, our second important result follows by inspection:

$$\rho = \frac{1}{6\pi G t^2}, \tag{406}$$

an explicit expression for the background density of the universe as a function of time.

Next, let P, ρ, $\boldsymbol{v}$, and Φ be perturbed from their background values:

$$P \to P + \delta P \quad \rho \to \rho + \delta\rho \quad \boldsymbol{v} \to \boldsymbol{v} + \delta\boldsymbol{v} \quad \Phi \to \Phi + \delta\Phi \tag{407}$$

The linearized mass conservation becomes

$$\frac{\partial \delta\rho}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\delta\rho + \delta\rho\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} + \rho\boldsymbol{\nabla}\boldsymbol{\cdot}\delta\boldsymbol{v} = 0, \tag{408}$$

[5]If we include a contribution from the vacuum to ρ, say ρ_V, this is an excellent model for the current universe as well. The density ρ_V is independent of R.

or using equation (402),

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\right)\frac{\delta\rho}{\rho}+\boldsymbol{\nabla}\boldsymbol{\cdot}\delta\boldsymbol{v}=0. \tag{409}$$

The linearized force equation is

$$\frac{\partial\delta\boldsymbol{v}}{\partial t}+\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\delta\boldsymbol{v}+\delta\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{v}=-\frac{1}{\rho}\boldsymbol{\nabla}\delta P-\boldsymbol{\nabla}\delta\Phi \tag{410}$$

Since $(\delta\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{r}=\delta\boldsymbol{v}$, this is

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\right)\delta\boldsymbol{v}+\frac{\dot{R}}{R}\delta\boldsymbol{v}=-\frac{1}{\rho}\boldsymbol{\nabla}\delta P-\boldsymbol{\nabla}\delta\Phi \tag{411}$$

Next we change coordinates. We define $\boldsymbol{r'}$ and t' by

$$\boldsymbol{r}=R(t)\boldsymbol{r'},\quad t=t' \tag{412}$$

These primed coordinates move with the background Hubble flow; indeed r' is just our l above. Then,

$$\boldsymbol{\nabla}=\frac{1}{R}\boldsymbol{\nabla'} \tag{413}$$

and

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t'}+\frac{\partial\boldsymbol{r'}}{\partial t}\boldsymbol{\cdot}\boldsymbol{\nabla'} \tag{414}$$

But

$$\frac{\partial\boldsymbol{r'}}{\partial t}=-\frac{\boldsymbol{r}}{R^2}\dot{R}=-\frac{\boldsymbol{v}}{R} \tag{415}$$

Thus

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\right)=\frac{\partial}{\partial t'}, \tag{416}$$

a physically obvious statement: the time rate of change $\partial/\partial t$ as measured by someone moving with the Hubble flow must be the Lagrangian derivative on the left side of this equation! In primed co-moving coordinates, our equations becomes

$$\frac{\partial}{\partial t'}\frac{\delta\rho}{\rho}+\frac{1}{R}\boldsymbol{\nabla'}\boldsymbol{\cdot}\delta\boldsymbol{v}=0, \tag{417}$$

$$\left(\frac{\partial}{\partial t'}+\frac{\dot{R}}{R}\right)\delta\boldsymbol{v}=-\frac{1}{R}\left(\frac{\boldsymbol{\nabla'}\delta P}{\rho}+\rho\boldsymbol{\nabla'}\delta\Phi\right) \tag{418}$$

Notice that none of the coefficients in (417) and (418) depend upon space, only time $t'=t$. This is what was gained by our coordinate transformation.

Finally we take $\boldsymbol{\nabla'}\boldsymbol{\cdot}$ of equation (418), and use equation (417) and the perturbed Poisson equation

$$\nabla'^2\delta\Phi=4\pi G\delta\rho R^2, \tag{419}$$

to obtain

$$\left(\frac{\partial^2}{\partial t'^2}+\frac{2\dot{R}}{R}\frac{\partial}{\partial t'}\right)\frac{\delta\rho}{\rho}=\frac{1}{\rho R^2}\nabla'^2\delta P+4\pi G\delta\rho \tag{420}$$

This is the fundamental equation for the evolution of small perturbations in expanding cosmologies. It is often called the *Bonnor-Lifschitz* equation. Henceforth, we shall suppress the primes $'$, with the understanding that we are working in comoving coordinates.

5.2.2 Adiabatic Solutions of the Bonnor-Lifschitz Equation

Let us assume that the density perturbations follow a simple adiabatic law of the form $P \propto \rho^\gamma$, or

$$\frac{\delta P}{P} = \gamma \frac{\delta \rho}{\rho} \tag{421}$$

where $\gamma = 5/3$ for an ideal monotomic gas. For such a gas, the sound speed v_s is given by $v_s^2 = \gamma P/\rho$. Then, equation (420) has wave solutions proportional to $\exp(ikr)$, with the relative density amplitude satisfying

$$\left(\frac{\partial^2}{\partial t^2} + \frac{2\dot{R}}{R}\frac{\partial}{\partial t} + \frac{k^2 v_s^2}{R^2} - 4\pi G\rho \right) \frac{\delta\rho}{\rho} = 0 \tag{422}$$

Let us evaluate this equation for the case $R = (t/t_0)^{2/3}$. Then

$$v_s^2 \propto \rho^{2/3} \propto R^{-2} \propto t^{-4/3}$$

With the help of (406), our differential equation becomes

$$\left(\frac{\partial^2}{\partial t^2} + \frac{4}{3t}\frac{\partial}{\partial t} + \frac{\alpha^2}{t^{8/3}} - \frac{2}{3t^2} \right) \frac{\delta\rho}{\rho} = 0 \tag{423}$$

where

$$\alpha^2 = k^2 a_0^2 \, t_0^{8/3}.$$

The interplay between pressure and gravity is found in the relative importance of the last two terms of the equation. At early times, the wavenumber was very large (it is in a sense frozen into the expansion of the universe) and pressure effects dominated. Disturbances oscillated as sound waves, ultimately losing energy to the background expansion. At later times, the α^2 term is negligible, since pressure forces are less important over longer wavelengths. Then, self-gravity prevails. Our equation becomes

$$\left(\frac{\partial^2}{\partial t^2} + \frac{4}{3t}\frac{\partial}{\partial t} - \frac{2}{3t^2} \right) \frac{\delta\rho}{\rho} = 0 \tag{424}$$

has the solution

$$\frac{\delta\rho}{\rho} = At^{-1} + Bt^{2/3}. \tag{425}$$

The growing solution $t^{2/3}$ corresponds to gravitational instability, but it is very mild. In particular, small perturbations do *not* run away exponentially, as they would in a static background. The fact that small density perturbations do not grow spontaneously into nonlinear structures is a central difficulty for theories for the origin of galaxies, and it indicates that deeper physics—in particular dissipation and radiative processes—are likely to play central roles in galaxy formation models. Gravity and pressure are not enough!

Finally, we invite the reader to show that the general solution of equation (423) is

$$\frac{\delta\rho}{\rho} = t^{-1/6} \left(AJ_{5/2}(3\alpha t^{-1/3}) + BJ_{-5/2}(3\alpha t^{-1/3}) \right) \tag{426}$$

where $J_{\pm 5/2}(x)$ are Bessel functions, and that the solutions $1/t$ and $t^{2/3}$ are recovered at late times. What is the asymptotic behaviour of our general solution as $t \to 0$? (For the reader who is not familiar with Bessel functions, a very clear summary of their properties can be found in the text by Jackson, *Classical Electrodynamics.*)

5.3 Rayleigh-Taylor Instability

We have just seen an instability at work in a perfectly homogeneous background. Another important class of instability is triggered by the presence of a sharp feature in the flow. Two classic examples are the Rayleigh-Taylor instability and the Kelvin-Helmholtz instability, which is the topic of the next section.

We begin with Rayleigh-Taylor. Consider a fluid of density ρ_2 directly on top of a fluid of density ρ_1; there is a density discontinuity present. There is a gravitational field g present pointing downward from region 2 to region 1. The effect of a gravitational field could be mimicked by an acceleration at the interface. For example, a density jump at a shock front that is decelerated by an external, low density medium is a prime candidate for Rayleigh-Taylor instability. Supernova blast waves are destabilised by this mechanism.

A more mundane but perhaps more intuitive venue is an ordinary glass of water. Turn it over. The water falls out. (You may have noticed this before, on your own.) But why? There is more than enough air pressure to keep it inside an upside-down glass! Indeed, fill a glass to the brim, place a piece of cardboard flush on the rim, turn the glass over slowly holding the cardboard in place, and remove your hand without sliding the card. You should find that the water *does* stay in place.

What is going on? When the carboard is in place, there is a firm surface on which the air pressure may act, and it does indeed prevent the water from flowing out. Without the cardboard, the energetically unfavourable arrangement of a "heavier" fluid on top of a "lighter" fluid has a chance to right itself and follow a path to the lower energy state with the water on the floor. The path that is followed is that tiny ripples at the surface form, drip down, and fingers of air move up into the glass. The air and water fluids can thus interchange their positions, *unless* the ripple formation is suppressed — by the placement of a piece of cardboard, say — at the interface.

Back to our example. At the interface $z = 0$ between the two regions, the 2–1 interface is rippled with a vertical displacement ξ satisfying

$$\xi \propto \exp(ikx) \tag{427}$$

where k is a real constant. What is the subsequent development?

We assume a time dependence $\exp(-i\omega t)$ in all variables. The x dependence follows from that of the interface, $\exp(ikx)$. The equations of motion in each of the fluids are

$$-i\omega\delta v_x = -ik\frac{\delta P}{\rho} \tag{428}$$

$$-i\omega\delta v_z = -\frac{1}{\rho}\frac{\partial(\delta P)}{\partial z} \tag{429}$$

Evidently, the velocity perturbations are derivable from a gradient in each of the media 1 and 2 (the peturbed pressure gradient). With

$$\boldsymbol{\nabla}\cdot\boldsymbol{\delta v} = 0 \tag{430}$$

from mass conservation, the z dependence emerges. The boundary conditions at large $|z|$ of non-infinite perturbations, together with δP satisfying the Laplace equation (show!), imply that the fluid variables must have a dependence of $\exp(|k|z)$ for the bottom fluid in medium 1, and a dependence of $\exp(-|k|z)$ for the top fluid, medium 2.

At the interface we have two boundary conditions to enforce. First, the fluid displacements (and the velocities) must be continuous at $z = 0$. Second, the force exerted on fluid 1 by fluid 2 must be equal to the force exerted by fluid 1 on fluid 2. In other words, the pressures of the displaced fluid elements must be the same. The force equation at the interface is then

$$\Delta P_1 = \Delta P_2 \tag{431}$$

where the Lagrangian perturbation ΔP is used. This is defined by

$$\Delta P = \delta P + \xi \frac{\partial P}{\partial z} \tag{432}$$

(Why do we need to use the Lagrangian perturbation and not the Eulerian?) Since the perturbed vertical velocity $\delta v_z = -i\omega\xi$ is continuous at $z = 0$, the z equations of motion for regions 1 and 2 state that at the interface

$$-i\omega\delta v_z = -\frac{|k|}{\rho_1}\delta P_1 \tag{433}$$

$$-i\omega\delta v_z = \frac{|k|}{\rho_2}\delta P_2 \tag{434}$$

The equilibrium pressure gradient is

$$\partial P_i/\partial z = -\rho_i g,$$

where i refers to either 1 or 2. (N.B.: The equilibrium pressure at the interface is continuous at the interface, even though its gradient is not.) Putting everything together, equation (431) becomes

$$\frac{i\omega}{|k|}\,\rho_1\,\delta v_z - i\frac{\delta v_z}{\omega}\rho_1 g = -\frac{i\omega}{|k|}\,\rho_2\,\delta v_z - i\frac{\delta v_z}{\omega}\rho_2 g \tag{435}$$

which leads to

$$\omega^2(\rho_1 + \rho_2) = g|k|(\rho_1 - \rho_2) \tag{436}$$

This is our dispersion relation. When $\rho_1 > \rho_2$, these disturbances propagate as waves. (A neat example: these low frequency waves can be excited by slow-moving boats coming into a port, when a nearby river or mountain run-off produces fresh water lying on top of denser salt water. The generation of these waves cause a significant drain of energy from the boat.) But if $\rho_1 < \rho_2$, we produce a negative ω^2. This means exponential behaviour in time, and small disturbances grow explosively. Well, or the water just falls out of the glass. Once again we see the classic behaviour of a fluid instability: small departures from equilibrium growing exponentially with time.

The underlying cause of the Rayleigh-Instability is that a heavy fluid lying on top of a less dense fluid is energetically unfavourable. The same system with the heavy fluid on the bottom is a state of lower (potential) energy. If a path to the lower state is opened, the system will not hesitate to exploit it. In our example the ripples between the two fluids create such a path, and they grow into long fingers of upwelling low density fluid and downwelling high density fluid, allowing the system to reach an equilibrium of lower energy.

5.4 The Kelvin-Helmholtz Instability

The Rayleigh-Taylor instability emerges when there is a density discontinuity in the presence of gravity, real or effective, when the top fluid is denser than the bottom. A process known as the Kelvin-Helmhlotz instability emerges when there is a velocity discontinuity, even in a homogeneous medium.

Consider a fluid which is at rest in the half space $z < 0$, and moves with a velocity U in the $+x$ direction for $z > 0$. This is certainly a possible equilibrium state, but it is, in fact, always unstable.

Denote the fluid at rest by subscript 1, and the fluid in motion by subscript 2. The density ρ is constant. Once again, the interface between fluids 1 and 2 is rippled with an assumed space-time dependence of $\exp(ikx - i\omega t)$. In fluid 1, the equation of motion is

$$-i\omega\boldsymbol{\delta v_1} = -\frac{1}{\rho}\boldsymbol{\nabla}\delta P_1 \tag{437}$$

while in fluid 2 it is

$$-i(\omega - kU)\boldsymbol{\delta v_2} = -\frac{1}{\rho}\boldsymbol{\nabla}\delta P_2 \tag{438}$$

With $\boldsymbol{\nabla}\cdot\boldsymbol{\delta v} = 0$, we see that δP in both fluids satisfies $\nabla^2\delta P = 0$, so that the z dependence of all variables is $\exp(-|kz|)$, as before. In particular, the z equations of motion are

$$-i\omega\delta v_{z1} = -\frac{|k|}{\rho}\delta P_1 \tag{439}$$

$$-i(\omega - kU)\delta v_{z2} = \frac{|k|}{\rho}\delta P_2 \tag{440}$$

The *displacement* of the interface ξ must obviously be the same viewed from region 1 or 2, but not so the perturbed velocities δv_z. Indeed, in region 1,

$$\delta v_{z1} = \Delta v_{z1} = \frac{D\xi}{Dt} = -i\omega\xi \tag{441}$$

where the equality between Lagrangian and Eulerian perturbations follows because there are no velocity gradients in the background flow. In region 2, on the other hand,

$$\delta v_{z2} = \Delta v_{z2} = \frac{D\xi}{Dt} = -i(\omega - kU)\xi, \tag{442}$$

that is $\delta v_{z2} = \delta v_{z1} + ikU\xi$.

The condition of pressure balance in this case is entirely Eulerian (more precisely there is no distinction between Eulerian and Lagrangian)

$$\delta P_1 = \delta P_2, \tag{443}$$

since there are no pressure gradients in the unperturbed equilibrium. Combining the last four equations gives a very unusual dispersion relation:

$$\omega^2 + (\omega - kU)^2 = 0. \tag{444}$$

Figure 8: Kelvin-Helmholtz instability in clouds in an atmosphere shear layer.

Obviously this cannot be satisfied by any real value of ω! The solution is

$$2\omega = kU(1 \pm i). \tag{445}$$

There is always an exponentially growing branch for any finite U.

This is the *Kelvin-Helmholtz* instability in its simplest form: two fluids in relative shear motion tend to be unstable. The source of free energy is obviously the shear itself. But what is the actual mechanism, why should shear be unstable? The answer can be found by a Bernoulli argument. An upward directed distortion of the interface into the upper region 2 causes a slight constriction for the x directed velocity. The fluid moves a little faster round the bump, to conserve mass. When it moves a little faster, the pressure drops, in accordance with Bernoulli's law. In region 1, just underneath, the distortion causes a dilation in any flow that is present, and the flow slows down. (We can always analyze the flow in a frame where there is flow on both sides.) Thus the pressure rises from below, and drops from above. The upward displacement is driven yet farther upward, and an instability ensues.

Kelvin-Helmholtz instabilities are common in planetary atmospheres and can be easily seen, for example, in the flow around Jupiter's great red spot. Figure [10] shows a beautiful example closer to home. They can also form as secondary instabilities in the nonlinear development of the Rayleigh-Taylor instability. (See the Wikipedia article on Rayleigh-Taylor instabilities for striking photos.)

5.5 Stability of Continuous Shear Flow

The discussion of the Kelvin-Hemlholtz instability focused on a flow with a discontinuity in the velocity shear profile. The question naturally arises of whether instability occurs when the velocity changes continuously. You wouldn't expect Rayleigh's name to be out of this discussion for very long, and sure enough, the good Baron showed that for an inviscid flow, there is a very simple necessary, though not sufficient, instability condition: the velocity profile must contain an point of inflection at which its second derivative vanishes. This is known as the *Rayleigh inflection point criterion.* Unlike the name, the argument is fiendishly clever.

5.5.1 Analysis of the inflection point criterion

Consider a constant density velocity flow in the xy plane,

$$v_x = V(y)\boldsymbol{e_x} \tag{446}$$

We consider the behaviour of small perturbations to this flow which depend upon x and t as $\exp[i(kx-\omega t)]$, with an amplitude that depends upon y. The linearized equations of motion are ($' \equiv d/dy$):

$$ik\delta v_x + \delta v_y' = 0 \tag{447}$$

$$-i(\omega - kV)\delta v_x + \delta v_y V' = -ik\delta P/\rho \tag{448}$$

$$-i(\omega - kV)\delta v_y = -\delta P'/\rho \tag{449}$$

These three equations can be reduced to a single equation for δv_y:

$$\delta v_y'' + \delta v_y \left(-k^2 + \frac{kV''}{\omega - kV}\right) = 0 \tag{450}$$

Rayleigh's argument now proceeds as follows. Multiply the above differential equation by the complex conjugate δv_y^*, and integrate between upper and lower boundaries $\pm L$. Note that

$$\int_{-L}^{L} \delta v_y^* \, \delta v_y'' \, dy = \left[\delta v_y^* \, \delta v_y'\right]_{-L}^{L} - \int_{-L}^{L} |\delta v_y'|^2 \, dy \tag{451}$$

If either δv_y or its derivative vanishes at the boundaries, or if the boundary conditions are periodic, the integrated part vanishes. Under any of these conditions, the result of transforming the differential equation is

$$-\int_{-L}^{L} |\delta v_y'|^2 \, dy + \int_{-L}^{L} \left(-k^2 + \frac{kV''}{\omega - kV}\right) |\delta v_y^2| \, dy = 0. \tag{452}$$

If there is an instability present, then ω must have an imaginary part, ω_I. Writing $\omega = \omega_R + i\omega_I$, the imaginary part of this equation is

$$\omega_I k \int_{-L}^{L} \left(\frac{|\delta v_y^2| V''}{|\omega - kV|^2}\right) dy = 0. \tag{453}$$

A necessary condition for this equation to be satisfied is that V'' must be positive over part of the range of integration, and negative over other parts. In other words, it must pass through *zero*. The flow must have a point of inflection at which $V'' = 0$. Simple shear flows, $V'' = 0$, are stable, at least in the linear regime (They can, however, be very sensitive to nonlinear perturbations.) Flows that asymptotically approach different constant velocities at different values of y do satisfy the necessary requirement for linear instability, but there is no proof of instability. In principle, flows can be destabilised by the addition of an inflection point, which may have only a very modest effect on the behaviour of the fluid in bulk.

5.6 Rotational Instability

5.6.1 The Rayleigh Criterion

Consider a rotating fluid about a central object, a common environment in astrophysical systems. It is well known that shear flow is subject to nonlinear instability, as we have just seen, and turbulence in pipe flows is seen all the time. Are gas discs subject to, say, a $1/r^2$ force law unstable?

To address this question in a simple setting, write out equation (34) in component form:

$$\frac{\partial v_x}{\partial t} - 2\Omega v_y = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 3\Omega^2 x \tag{454}$$

$$\frac{\partial v_y}{\partial t} + 2\Omega v_x = -\frac{1}{\rho}\frac{\partial P}{\partial y} \tag{455}$$

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - \Omega^2 z \tag{456}$$

Note that we have dropped the terms in $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$, as we shall be looking at infinitesimally small perturbations about circular orbits. The simplest case to begin with is one in which the motions are in the xy plane of the disc, and all variables have the plane wave form $\exp(ikz - i\omega t)$. Then we may ignore the z equation as well as the x and y pressure gradients. With $-i\omega x = v_x$, the equations become

$$-i\omega v_x - 2\Omega v_y = 3\Omega^2 \frac{i v_x}{\omega} \tag{457}$$

$$-i\omega v_y + 2\Omega v_x = 0 \tag{458}$$

from which we quickly learn that $\omega^2 = \Omega^2$, and that is that. No instability. More generally, the discussion above equation (34) indicates that $3\Omega^2$ should be replaced by $-d\Omega^2/d\ln R$ for a non-Keplerian force law. In that case, we find that

$$\omega^2 = \kappa^2 \equiv 4\Omega^2 + \frac{d\Omega^2}{d\ln R} = \frac{1}{R^3}\frac{d(\Omega^2 R^4)}{dR} \tag{459}$$

where κ^2 is the the epicyclic frequency. This calculation is already contained in our study of inertial waves in section 4.9.3, but it is helpful same thing more simply here, with our emphasis on stability. What is now apparent is that one *can* have instability, if $R^2\Omega$, the specific angular momentum, *decreases* outward! Then $\omega^2 < 0$ and exponential behaviour ensues. The criterion that the angular momentum increasing outward is necessary for stability is known as the Rayleigh criterion. (Yes, same guy.)

Exercise. Consider a simple disc system with one ring of fluid elements at R_1 and another ring of equal total mass at $R_2 > R_1$. Consider another system with the same rings interchanging their locations, 1 going to 2 and 2 going to 1, while conserving their orginal angular momenta during the exchange. Show that the energy of the system *decreases* after the exchange if and only if the Rayleigh criterion is violated in the original system. Explain the significance of this finding.

5.7 The Magnetorotational Instability

We consider the simplest possible case: a disc threaded by a uniform magnetic field in the z direction, and a waveumber also in this direction: $\boldsymbol{k} = k\boldsymbol{e_z}$. The magnetic field can exert no force along itself (why not?), so we consider only motion in the disc plane. Recall the results of equation (252), given as an exercise,

$$\delta\boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}) = i(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{\xi} \tag{460}$$

But the magnetic force per unit mass is then just

$$\frac{1}{\mu\rho}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\delta\boldsymbol{B} = \frac{i(\boldsymbol{k}\cdot\boldsymbol{B})}{\mu\rho}\delta\boldsymbol{B} = -(\boldsymbol{k}\cdot\boldsymbol{v_A})^2\boldsymbol{\xi} \tag{461}$$

In other words, the magnetic tension acts as though it were a harmonic spring, with a restoring force opposite to the fluid displacement. (In our case this force is just proportional to k_z^2, but in general it is only the wavenumber component along the magnetic field that enters, so we will leave it in this dot product form.)

The diplacement $\boldsymbol{\xi}$ is precisely the displacement we studied when we encountered the epicyclic frequency for the first time: a deviation of a particle from a circular orbit. Pressure forces are unimportant, since there are no pressure gradient forces from the perturbations in the disc plane. Therefore, to include the effect of a magnetic field, add $-(\boldsymbol{k}\cdot\boldsymbol{v_A})^2\boldsymbol{\xi}$ to the right side of equations (351) and (352). The Cartesian components of the displacement are $\boldsymbol{\xi} = (x, y)$, and the equations become

$$\ddot{x} - 2\Omega\dot{y} = -x\left(\frac{d\Omega^2}{d\ln R} + (\boldsymbol{k}\cdot\boldsymbol{v_A})^2\right)x \tag{462}$$

$$\ddot{y} - 2\Omega\dot{x} = -(\boldsymbol{k}\cdot\boldsymbol{v_A})^2 y \tag{463}$$

If now we look for solutions of the form $\exp(i\omega t)$, we obtain the dispersion relation

$$\omega^4 - \omega^2\left(\kappa^2 + 2(\boldsymbol{k}\cdot\boldsymbol{v_A})^2\right) + (\boldsymbol{k}\cdot\boldsymbol{v_A})^2\left((\boldsymbol{k}\cdot\boldsymbol{v_A})^2 + \frac{d\Omega^2}{d\ln R}\right) = 0. \tag{464}$$

When the magnetic field vanishes, we recover $\omega^2 = \kappa^2$. When the rotation Ω vanishes, we recover the double root $\omega^2 = (\boldsymbol{k}\cdot\boldsymbol{v_A})^2$, the degenerate Alfvén and slow modes. But the truly remarkable feature of this dispersion relation is that for small enough wavenumbers, the final constant term in the quartic equation will always be negative, if the angular velocity decreases outwards. This means that one of the modes, and it can be shown to correspond to the slow wave, is unstable. This is the *magnetorotational instability,* or MRI.

At small k the instability growth rate grows linearly with k, but when

$$k^2 v_A^2 > \left|\frac{d\Omega^2}{d\ln R}\right| \tag{465}$$

the growth rate goes to to zero and the instability is no longer present. Therefore, there is some value of k for which the growth rate is a maximum. For a Keplerian disc, this maximum growth rate is 0.75Ω at $kv_A = 0.97\Omega$. *The maximum growth rate is independent of the magnetic field strength!* The magnetic field simply sets the scale of the wavenumber at which the growth rate is maximized, but does not affect the magnitude of the growth rate

itself. According to our results, *any* magnetic field at all will destabilize the disc, with the most unstable mode amplitudes growing by a factor in excess of 100 each orbit.

Can this *really* be true? If the field were 10^{-100}T would the disc really become tubulent in a few orbits? Something seems to have been left out.

What has been left out is finite resistivity. The presence of electrical resistance produces a term that is proportional to k^2 in the induction equation, with a very small proportionality constant. But for very weak fields, the wavenumber of maximum growth is at such small scales, that this term becomes important, and it damps modes with wavenumbers $k \simeq \Omega/v_A$. Small wavenumbers (long wavelengths) remain unaffected, however, and still unstable. But their time $\sim 1/(kv_A)$, becomes very long—infinite, in fact, in the limit of vanishing field. So there is some limit at which a very weak field is incapable of disrupting the disc. As a practical astrophysical matter, however, the resistivity is so small in an ionized disc (and it is often very small even in a disc with a low fraction of ions) that any weak magnetic field of reasonable astrophysical strength will render the disc unstable.

The dispersion relation can be solved exactly for $\omega(k)$, and analyzed in detail. The fastest growing mode of the dispersion relation occurs when

$$(kv_A)^2 = -\frac{1}{4}\left(1 + \frac{\kappa^2}{4\Omega}\right)\left(\frac{d\Omega^2}{d\ln R}\right) \tag{466}$$

and corresponds to

$$|\omega_{max}| = \frac{1}{2}\left|\frac{d\Omega}{d\ln R}\right| \tag{467}$$

We have already noted that for a Keplerian disc, this is $kv_A = 0.97\Omega$ and $|\omega_{max}| = 0.75\Omega$. The exponential amplification factor is impressive. In terms of the orbital period T,

$$\exp(0.75 \times 2\pi t/T) = \exp(4.712t/T)$$

or a factor of 111 each orbit. Remember, this is all independent of the strength of the magnetic field!

Why does a simple spring-like force, whose strength is linear proportional to the distance, result in such a powerful instability? Since the magnetic force is exactly analogous to a spring, let us consider the behaviour of two masses connected by a spring. In figure (4), we see two masses bound by a spring, m_i and m_o. m_i starts very slightly closer to the central mass M_c. Thus, m_i orbits slightly faster than m_o. The spring stretches, and exerts a torque on both masses, pulling m_i backwards and m_o forwards. This is a transfer of angular momentum from m_i to m_o. The loss of angular momentum forces m_i to move to an orbit closer to M_c, whereas the gain of angular momentum moves m_o outward. In a more tightly bound orbit, m_i moves even more rapidly, whereas m_o, in a more distant orbit, moves more slowly. The spring therfore continues to stretch, and the angular momentum transfer is a process that "runs away." A magnetic field binding two fluid elements does exactly the same thing that our spring does. This is underlying cause of accretion disc turbulence.

5.8 Thermal Instability

A gaseous medium subject to bulk heating and cooling is vulnerable to another type of instability, stemming entirely from thermodynamics as opposed to ordinary gas dyanmics. This "thermal instability" arises as follows. From (15) and (17), the governing entropy equation is

$$\frac{Ds}{dt} = -(\gamma - 1)\frac{\rho\mathcal{L}}{P} \tag{468}$$

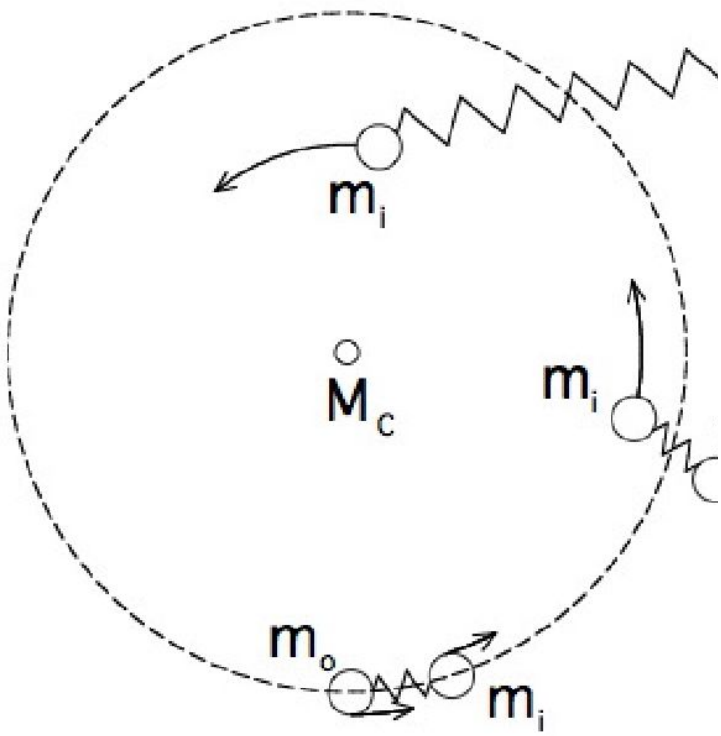

Figure 9: The magnetorotational instability. Magnetic fields in a disc bind fluid elements precisely as though they were masses in orbit connected by a spring. The inner element m_i orbits faster than the outer element m_o, and the spring causes a net transfer of angular momentum from m_i to m_o. This transfer is unstable, as described in the text. The inner mass continues to sink, whereas the outer mass rises farther outward.

where $s = \ln P\rho^{-\gamma}$, and we recall that $\mathcal{L}$ is the energy loss rate per unit mass. In the case of a homogeneous medium, initially in radiative equilibrium $\mathcal{L} = 0$, Eulerian perturbations δX, with assumed time dependence $\exp(\sigma t)$, satisfy the equation

$$\sigma\left(\frac{\delta P}{P} - \gamma\frac{\delta\rho}{\rho}\right) = -(\gamma-1)\frac{\rho}{P}\left[\left(\frac{\partial\mathcal{L}}{\partial P}\right)_T \delta P + \left(\frac{\partial\mathcal{L}}{\partial T}\right)_P \delta T\right] \tag{469}$$

where the subscripts on the partial derivatives indicate the quantity held constant. Notice that we have chosen to regard $\mathcal{L}$ as a function of P and T. The reason for this is that in the usual case of interest, the cooling time scale is long compared with the dynamical sound crossing time over the wavelength size of the disturbance, so it is a good approximation to treat δP as small: pressure equilibrium is maintained with the surroundings. With

$$\frac{\delta T}{T} = -\frac{\delta\rho}{\rho} \tag{470}$$

which follows from $\delta P/P = 0$, and ignoring the δP term in the left, our equation becomes

$$\gamma\sigma\frac{\delta\rho}{\rho} = -(\gamma-1)\frac{\rho T}{P}\left(\frac{\partial\mathcal{L}}{\partial T}\right)_P \frac{\delta\rho}{\rho} \tag{471}$$

In other words if

$$\left(\frac{\partial\mathcal{L}}{\partial T}\right)_P < 0, \tag{472}$$

the radiative losses *decrease* with increasing temperature at constant pressure, the growth rate σ is positive and the system is thermally unstable. This is known as the *Field crterion* for thermal instability, who made an extensive study of this phenomenon in 1965. Dynamics enters into the argument only via the constant pressure assumption, which can be relaxed at the price of a more complicated calculation with no difference in conclusion.

Physically, the picture is simple. Imagine a slightly cooler, overdense region that maintains pressure equilibrium with its surroundings. As the region cools, if the Field criterion is satisfied, the cooling increases more ever rapidly as the temperature decreases and the region rapdily condenses into a sort of cool cloud where, one presumes, the nature of the cooling law changes and $\mathcal{L}$ stabilises by decreasing with decreasing temperature.

In 1969, Field, Goldsmith and Habing used this insight to fashion a model of the interstellar medium (ISM) meant to explain why two stable phases of gas exist: cool clouds and a warm, confining, dilute medium. Start with the purely mathematical identity

$$\left(\frac{\partial\mathcal{L}}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_{\mathcal{L}} \left(\frac{\partial P}{\partial\mathcal{L}}\right)_T = -1 \tag{473}$$

The derivative $(\partial T/\partial P)_{\mathcal{L}}$ is taken along the PT curve under the constraint of thermal equilibrium. The derivative $(\partial P/\partial\mathcal{L})_T$ is always positive, since at constant T we may replace P in the numerator by ρ, and the astrophysical cooling function $\mathcal{L}$ is invariably an increasing function of ρ at constant T. (Denser regions always cool more rapidly.) Thus, if the Field criterion instability criterion (472) is satisfied, then

$$(\partial T/\partial P)_{\mathcal{L}} > 0 \tag{474}$$

for thermal instabilty.

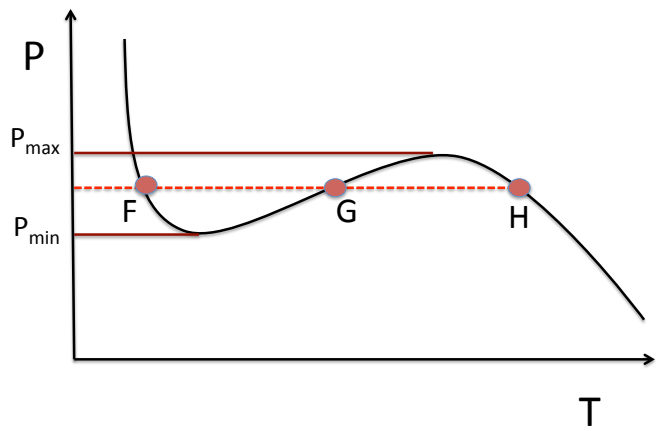

Figure 10: The FGH model of the interstellar medium. Points F and H represent stable gas phases. See text for further details.

Field, Goldsmith, and Habing studied the thermal equlibrium PT curve that would be generated by models of the ISM in which cosmic ray heating was balanced by radiative cooling from atomic line radiation, and found curves of the form shown in figure (x). Above this curve at higher P, heating exceeds cooling. Below the curve, cooling exceeds heating. For pressures above P_{max}, there is only one phase possible, a cool dense gas phase, and it is thermally stable since $dP/dT < 0$. Similarly, for pressures below P_{min}, only a dilute hot phase is present, and it is likewise thermally stable. But for $P_{min} < P < P_{max}$, *three* gas phases are possible, labelled F,G,H. Phases F and H are thermally unstable, but G is thermally unstable, falling on the part of the curve with $dP/dT > 0$. Matter in this part of the curve would spontaneously heat, move to the right, and continue toward H, or cool leftwards, and in the same runaway manner, fall to F.

This elegant model not only explained why there were clouds and an intercloud medium, in predicted that the range of possible ISM pressures would have to be very narrow. Alas, nature turned out to be messy with most of the ISM gas in self-gravitating molecular clouds (not the neutral atomic hydrogen clouds of FGH) and an appreciable fraction in a hot, X-ray emitting phase. Moreover, to make the FGH model work at all required a larger cosmic ray heating rate than that inferred from observations. But the dynamics and thermodynamics is sound, and it teaches us something important about S-shaped equilibrium curves that resurfaces in many guises with very different underlying physics. Equilibrium curves with S-shaped geometries often show multi-phase behaviour.

6 Classical Flow Solutions

You cannot step twice into the same river.

— Heraclitus

6.1 Accretion

This year, our members have put more things on top of other things than ever before. But, I should warn you: this is no time for complacency!

— The President of the Royal Society for Putting Things on Top of Other Things, M. Python

6.2 Spherical Accretion: The Bondi Problem

Consider an extended gaseous medium with a central gravitating point mass M. The pressure and density in the medium are P_∞ and ρ_∞ respectively. The gravitational field of the point mass draws on the surroundings, and gas either accumulates onto the surface of the star, or is lost through the horizon of a black hole. This process is referred to as *accretion.* Accretion heats the gas by compression, and if there is turbulence, by dissipation as well. Gravitational accretion is thought to be the power source responsible for active galactic nuclei, the most luminous objects known.

6.3 Formulation

In its simplest form, accretion flow is spherical and time-steady. Under these conditions, the mass accretion rate

$$\dot{m} = -4\pi\rho r^2 v \tag{475}$$

is constant. Here, v is the radial velocity, and the minus sign is inserted so that $\dot{m} > 0$. The gas is assumed to follow a simple polytropic law

$$P = K\rho^\gamma \tag{476}$$

where the constant K may be defined by the gas properties at $r = \infty$:

$$K = P_\infty \rho_\infty^{-\gamma} \tag{477}$$

The speed of sound, an important parameter in this problem, is

$$a^2 = \frac{dP}{d\rho} = \gamma K \rho^{\gamma-1} \tag{478}$$

Thus, another useful way to write the constant K is

$$\gamma K = a_\infty^2 \rho_\infty^{1-\gamma}, \tag{479}$$

or more simply,

$$a^2 = a_\infty^2 \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} \tag{480}$$

The equation of motion is

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM}{r^2} \tag{481}$$

But,

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{d}{dr}\left(\frac{\gamma K \rho^{\gamma-1}}{\gamma-1}\right) = \frac{d}{dr}\left(\frac{a^2}{\gamma-1}\right), \tag{482}$$

which allows the equation of motion to be integrated immediately:

$$\frac{v^2}{2} + \frac{a^2}{\gamma-1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma-1} \tag{483}$$

where the integration constant is chosen by evaluating the left side at $r = \infty$. This is simply the Bernoulli constant.

The Bernoulli equation can be used to relate v and r. Expressing a in terms of v, we find

$$\frac{v^2}{2} + \frac{a_\infty^2}{\gamma-1}\left(\frac{\dot{m}}{4\pi r^2 |v| \rho_\infty}\right)^{\gamma-1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma-1} \tag{484}$$

It is very instructive to look at the case $\gamma = 1.5$, which has a simple, explicit solution:

$$r = \frac{2\left[a_\infty^2 (\dot{m}/\pi\rho_\infty|v|)^{1/2} - GM\right]}{4a_\infty^2 - v^2} \tag{485}$$

This expression for r has a singular denominator, and as a consequence has two types of solution. The first has $|v| < 2a_\infty$ everywhere. The velocity goes to zero at large distances, increases as r decreases, and reaches a constant value $|v_0|$ satisfying

$$\dot{m} = \pi\rho_\infty (GM)^2 \frac{|v_0|}{a_\infty^4} \tag{486}$$

as $r \to 0$. Since $|v_0|$ approaches a constant, the density increases sharply as $1/r^2$ at small values of r. These solutions, which are characterized by a limited range of velocity and a large density are known as *settling solutions*.

There are many settling solutions,since $\dot{m}$ is not uniquely determined. If we now regard v_0 as a free parameter, as $|v_0|$ increases from below, approaching $2a_\infty$, the associated mass accretion rate $\dot{m}$ also increases. In the limiting case $|v_0| = 2a_\infty$, $\dot{m}$ reaches its maximum value,

$$\dot{m}_{max} = 2\pi\rho_\infty (GM)^2 a_\infty^{-3} \tag{487}$$

and the singular behaviour simply vanishes from equation (485)! We find

$$r = \frac{2GM}{(|v| + 2a_\infty)(|v| + \sqrt{2|v|a_\infty})} \tag{488}$$

This is well-behaved for $|v| = 2a_\infty$. In fact, there is no reason to stop there. The solution extends to infinite $|v|$ as $r \to 0$ and the flow approaches a pure free-fall,

$$v^2 \simeq \frac{2GM}{r} \tag{489}$$

What is the significance of $|v| = 2a_\infty$? When $|v| = 2a_\infty$, equations (488) an (483) imply $a = 2a_\infty$ at the same point. In other words, the point $|v| = 2a_\infty$ is the sonic point $M^2 = 1$. We have shown that for $\dot{m} < \dot{m}_{max}$, the flow remains subsonic everywhere, but becomes transonic at the maximum accretion rate possible. The flow remains transonic from inside the $M^2 = 1$ sonic radius down to $r = 0$, and can be brought to subsonic levels only through the mediation of a shock wave (for example, the surface of the star).

The $\gamma = 1.5$ example is special only in the sense that it is relatively easy to solve. Settling and transonic solutions are found for all values of γ between 1 and 5/3. In fact, while it is not possible to solve the Bondi accretion problem in any simple way for general γ, we can always obtain the value of $\dot{m}_{max}$. For a polytropic equation of state,

$$dP = \frac{\gamma P}{\rho} d\rho = a^2 d\rho \tag{490}$$

Hence,

$$v\frac{dv}{dr} = -a^2 \frac{d\ln\rho}{dr} - \frac{GM}{r^2} = a^2 \frac{d\ln(r^2 v)}{dr} - \frac{GM}{r^2} \tag{491}$$

where mass conservation has been used in the second equality. This leads to the equation

$$v\frac{dv}{dr} = \frac{(v^2/r)(2a^2 - GM/r)}{(v^2 - a^2)} \tag{492}$$

The sonic point S with $v^2 = a^2$ can be crossed only if

$$a_S^2 = \frac{GM}{2r_S} \tag{493}$$

which will ensure that the numerator and denominator vanish together. Combining this with the Bernoulli relation (483) yields

$$a_S^2 = \frac{2a_\infty^2}{5 - 3\gamma} \tag{494}$$

The relation $P = K\rho^\gamma$ implies that

$$\rho_S = \rho_\infty \left(\frac{a_S}{a_\infty}\right)^{2/(\gamma-1)} \tag{495}$$

so that the accretion rate

$$\dot{m} = 4\pi\rho_S r_S^2 a_S = \pi\rho_S (GM)^2 a_S^{-3} \tag{496}$$

can be expressed entirely in terms of a_∞, ρ_∞, and GM, the given parameters of the problem. Upon substitution and simplification,

$$\dot{m} = \alpha^{\alpha/(1-\gamma)} \pi\rho_\infty (GM)^2 a_\infty^{-3}, \qquad \alpha = (5 - 3\gamma)/2. \tag{497}$$

This is the maximum spherical accretion rate possible for any value of γ. The coefficient in front of π varies from $\exp(1.5) = 4.48$ for $\gamma = 1$ to 1 for $\gamma = 5/3$.

Exercise. The center of our galaxy has a black hole of mass 2.6×10^6 solar masses. There is an ambient gas with a number density of about 100 cm^{-3} and a temperature of 10^7. Estimate the Bondi accretion rate onto the Galactic Center black hole by assuming $\gamma = 1.5$. Typically, a black hole might convert 5% of the incoming rest mass energy into radiation. Using this, calculate the expected luminosity of the black hole. The actual luminosity is $\sim 2 \times 10^{33}$ ergs s^{-1}, which should be much less than your result! The Galactic Center is accreting well below its Bondi value.

6.4 The Parker Wind Problem

There is an interesting counterpart to Bondi accretion that involves an outflow from the surface of a star (or perhaps something larger like a cluster or galaxy). This idea was first developed by E. Parker in 1956 and predicted something truly remarkable: the outer layers of the Sun extend throughout and beyond the solar system! The idea is that solar corona can become sufficiently hot near the Sun's surface that the gas can escape to infinity as a cold but rapidly moving (hypersonic) fluid.

The mathematics is very similar to the Bondi problem, but with different boundary conditions. We consider a spherical flow around a central mass M with a sound speed of a_0 at the solar surface r_0. Let the flow velocity at infinity be v_∞ (assuming the gas can in fact escape!). Our Benoulli equation may either be written

$$\frac{v^2}{2} + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = \frac{a_0^2}{\gamma - 1} - \frac{GM}{r_0} \tag{498}$$

or

$$\frac{v^2}{2} + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = \frac{v_\infty^2}{2} \tag{499}$$

which means that

$$v_\infty^2 = 2\left(\frac{a_0^2}{\gamma - 1} - \frac{GM}{r_0}\right) \tag{500}$$

A nearly isothermal gas can have a temperature corresponding to just a small fraction of the formal escape velocity and still become unbound.

At the sonic point of the outflow, $a_S^2 = GM/2r_S$ still holds as before, and Bernoulli's equation gives

$$a_S^2 = \frac{\gamma - 1}{5 - 3\gamma} v_\infty^2 \tag{501}$$

But since the velocity must rise monotonically, $a_S < v_\infty$, and this means that $\gamma < 3/2$ for a wind. For accretion, recall that $\gamma \leq 5/3$.

The outflow rate from the surface of the star can be determined in just the same way that we determined the mass accretion rate for the Bondi problem: go to the sonic point and evaluate

$$\dot{m} = 4\pi r_S^2 \rho_S a_S \tag{502}$$

with

$$\rho_S = \left(\frac{a_S}{a_0}\right)^{\frac{2}{\gamma - 1}} \tag{503}$$

This gives

$$\dot{m} = \pi \alpha^{\alpha/2} G^2 M^2 \rho_0 a_0^{-2/\gamma-1} v_\infty^{5-3\gamma/\gamma-1} \tag{504}$$

with $\alpha = (\gamma - 1)/(5 - 3\gamma)$.

The effects of a magnetic field in a stellar wind have been studied by Weber and Davis (1967 ApJ, 148, 217). The problem is complex because there is not one, but *three* critical points! This is because there are three types of propagating disturbances: fast waves, slow waves, and Alfvén waves. Each has its own associated critical point.

We will not present the details here, but refer to the reader to this clearly written and comprehensive paper. The astrophysical significance of this solution is that it shows that magnetic torques can, over the lifetime of the Sun exert a significant angular momentum loss.

6.5 Disk Accretion: Hydrodynamical Model

6.5.1 Turbulent Model

The presence of angular momentum dramatically changes the nature of accretion. Because fluid elements tend to retain their angular momentum but radiate their internal energy, a thin disc forms in the vicinity of the central gravitating object. The disc is almost entirely supported by rotation, with the azimuthal velocity v_ϕ much in excess of the sound speed a. Under these conditions, how does accretion proceed?

Unlike the spherical case, there is a perfectly acceptable hydrostatic equilibrium for the disc: a Keplerian rotation profile. We will see that this rotation law is stable, and would persist almost indefinitely if there were no other forces acting upon the flow.

What might these other processes be? Any real gas has a finite ultimate viscosity, which will both dissipate energy and transport momentum—in this case *angular* momentum. But ordinary particle viscosity is much too small to cause accretion on the time scale that is needed. It is for this reason that early investigators assumed that Keplerian discs are, for some unknown reason, turbulent. Turbulence has some important properties in common with viscosity. It is certainly highly dissipative, and it can be associated with greatly enhanced transport rates.

Disk theorists now believe that at least the *onset* of turbulence in discs is understood: it is due to the presence of magnetic fields in the gas. The key point, which is most surprising, is that even a very weak magnetic field can be highly disruptive to Keplerian orbits. A more detailed explanation is provided later in these notes, but for now, let us work with a hypothetical disc that is purely hydrodynamical but nevertheless turbulent. The essence of the problem can be retained that way.

What do we mean when we say that the disc is turbulent? We shall take this to mean that each component of the velocity field consists of a systematic flow plus a fluctuation with zero mean value. For mathematical simplicity, we will suppose that the flow is locally incompressible, so that density fluctuations may be ignored. (They can be included without any significant change in the physical content of the model, but at a price of greater mathematical complexity.) The most important assumption of all is that *the radial velocity fluctuation u_R is correlated in a time-averaged sense with the azimuthal velocity fluctuation u_ϕ*. This is indeed what happens as a consequence of the magnetic instability, but for now we will accept this as an article of faith. The time-averaged correlation of u_R and u_ϕ is written $\langle u_R u_\phi \rangle$. It is the central parameter of our model.

6.5.2 Equations

The azimuthal velocity v_ϕ in our problem is

$$v_\phi = R\Omega + u_\phi \tag{505}$$

where Ω is the Keplerian angular velocity

$$\Omega^2 = \frac{GM}{R^3} \tag{506}$$

and u_ϕ is a fluctuation with vanishing mean value. The disc is dominated by the Keplerian rotation in the sense that

$$R\Omega \gg \langle u_\phi^2 \rangle^{1/2} \tag{507}$$

The radial velocity, on the other hand is

$$v_R = u_R + v_2 \tag{508}$$

where u_R is the fluctuating component and v_2 is the second-order mean inward radial drift velocity, which is generally very *small* compared with the fluctuation amplitude:

$$\langle u_R^2 \rangle^{1/2} \gg v_2 \tag{509}$$

"Second order" means that v_2 is proportional to square of the fluctuation amplitudes, as we shall see.

In steady flow, the equation of mass conservation is

$$\boldsymbol{\nabla}\boldsymbol{\cdot}(\rho \boldsymbol{v}) = 0 \tag{510}$$

If integrate this equation over the vertical extent of the disc and average over the angle ϕ, this reduces to

$$\frac{d(R\Sigma v_2)}{dR} = 0 \tag{511}$$

where Σ is the total surface density of the disc: the z integral of ρ. The total mass accretion rate $\dot{M}$ integrated over 2π azimuth is

$$\dot{M} = -2\pi R\Sigma v_2 \tag{512}$$

with $v_2 < 0$.

The azimuthal equation of motion in steady flow is

$$\rho \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} v_\phi + \rho \frac{v_R v_\phi}{R} = -\frac{1}{R}\frac{\partial P}{\partial \phi} \tag{513}$$

Since

$$R\rho \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} v_\phi + \rho v_R v_\phi = \boldsymbol{\nabla}\boldsymbol{\cdot}(R\rho v_\phi \boldsymbol{v}) \tag{514}$$

(Prove!), the equation of motion takes the form

$$\boldsymbol{\nabla}\boldsymbol{\cdot}[R(\rho v_\phi \boldsymbol{v} + P\boldsymbol{e_\phi})] = 0 \tag{515}$$

which is a statement of angular momentum conservation. Vertical integration and azimuthal averaging gives

$$\frac{d}{dR}\left(R^2\Sigma\langle v_\phi v_R\rangle\right) = 0 \tag{516}$$

Now,

$$\langle v_\phi v_R\rangle = R\Omega v_2 + \langle u_\phi u_R\rangle \tag{517}$$

Integrating the differential equation for the angular momentum flux,

$$(\Sigma R^2)(R\Omega v_2 + \langle u_\phi u_R\rangle) = C \tag{518}$$

where C is a constant. Extending this equation to very small R, assuming that the velocity correlation is not singular, and using $\Sigma v_2 R$ is constant, we see that C must be very small, and we shall take $C = 0$ for simplicity. Our equation may be written

$$R\Omega v_2 = -\langle u_\phi u_R\rangle \tag{519}$$

We now see why we are justified in regarding v_2 as being small compared with the u fluctuations: it is of order $u^2/v_K^2 \times v_K$, where v_K is the Keplerian velocity. By contrast, the u fluctuations themselves are of order $u/v_K \times v_K$.

Finally, we turn to the energy equation. Our disc is completely dominated by rotational energy. The flux is therefore

$$\left(\frac{\rho v_\phi^2}{2} - \frac{GM\rho}{R}\right)\boldsymbol{v} \tag{520}$$

Through order u_ϕ, the leading terms of v_ϕ^2 are

$$v_\phi^2 = R^2\Omega^2 + 2R\Omega u_\phi \tag{521}$$

The energy density terms (inside parentheses in equation [520]) are:

$$\frac{\rho R^2\Omega^2}{2} - \frac{GM\rho}{R} + R\rho\Omega u_\phi = -\frac{GM\rho}{2R} + R\rho\Omega u_\phi. \tag{522}$$

The vertically integrated and azimuthally averaged radial component of the energy flux is therefore (remember that the u fluctuations vanish on average, but that $\langle u_R u_\phi\rangle$ does not!):

$$-\frac{GM\Sigma v_2}{2R} + R\Sigma\Omega\langle u_\phi u_R\rangle = -\frac{3GM\Sigma v_2}{2R} = \frac{3GM\dot{M}}{4\pi R^2} \tag{523}$$

where equation (519) has been used in the first equality. The divergence of this energy flux is *not* zero:

$$\frac{1}{R}\frac{d}{dR}\left(R \times \frac{3GM\dot{M}}{4\pi R^2}\right) = -\frac{3GM\dot{M}}{4\pi R^3} \tag{524}$$

Indeed, this quantity is the mechanical energy lost per unit time per unit area in the disc due to turbulent dissipation. Notice that it is independent of the correlation $\langle u_R u_\phi\rangle$, about which we knew very little. Thanks to angular momentum conservation, we now know that this correlation is simply related to the inward drift velocity, and therefore to the accretion rate $\dot{M}$.

If each side of the disc radiates like a local black body at a surface temperature T_s, then the total energy flux radiated is $2\sigma T_s^4$, where σ is the Stefan-Boltzmann radiation constant. Equating this to the mechanical energy loss, we obtain

$$T_s^4 = \frac{3GM\dot{M}}{8\pi\sigma R^3} \tag{525}$$

which is a classic result of accretion disc theory: the surface temperature falls off like $R^{-3/4}$.

It is possible to reconstruct the temperature profiles in accretion discs in some eclipsing binary systems, when different parts of the disc are hidden at different times. The $R^{-3/4}$ law works reasonably well. In other types of discs, it works much less well, but perhaps this is not surprising. The surface of the disc may be irradiated, and the surface temperature fixed by processes other than turbulent dissipation.

Clearly, understanding accretion in disc systems is much more complicated than spherical inflow. Let us begin a more systematic study of accretion discs, so that we may better understand why our assumptions about turbulence in such systems are well-founded.

6.6 MHD Theory of Accretion Disk Turbulence

In the linear phase of the MRI, δv_R and δv_ϕ are strongly correlated, growing in phase with one another, whereas in simple epicyclic motion, they are uncorrelated (out of phase by $\pi/2$ radians). The radial and magnetic field components are also highly correlated. This is the reason that there is a flux of angular momentum outwards in the linear regime, and this outward flux persists in the nonlinear regime. Thus, the nonlinear turbulent state of the disc is characterized by well-correlated velocity and magnetic field radial and azimuthal correlations. These MHD turbulent correlations may be treated in a manner very similar to the hydrodynamic case.

For example, the exact, time independent azimuthal equation of motion is

$$\boldsymbol{\nabla}\cdot[R(\rho v_\phi \boldsymbol{v} - \rho v_{A\phi}\boldsymbol{v_A} + (P + B^2/2\mu)\boldsymbol{e_\phi})] = 0 \tag{526}$$

which includes the terms from the magnetic tension $(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}$ arising from derivatives of unit vectors. Notice that the Alfvén velocities enter into angular momentum flux with the kinetic velocities, but with the opposite sign.

The steps that we took in section 8.3.2 can be repeated with the inclusion of the magnetic field. The magnetic field, however, is *not* treated as a mean field plus a fluctuation. Instead, the Alfvén velocities are assumed to be comparable in magnitude to the u fluctuations, much larger than v_2, and much smaller than $R\Omega$. No component of the magnetic field is assumed to vanish when averages are taken.

Integration of the angular momentum equation now gives (cf. [518])

$$(\Sigma R^2)(R\Omega v_2 + \langle u_\phi u_R - v_{AR}v_{A\phi}\rangle) = C \tag{527}$$

Let us introduce the notation

$$W_{R\phi} = \langle u_\phi u_R - v_{AR}v_{A\phi}\rangle \tag{528}$$

Arguing as before that the constant C must be very small, we conclude

$$R\Omega v_2 = -W_{R\phi} \tag{529}$$

This is the same result that we found in the hydro treatment, with $\langle u_R u_\phi \rangle$ replaced by $W_{R\phi}$.

The energy flux with a magnetic field contains an additional electromagnetic component, the Poynting flux:

$$F_E(\text{Poynting}) = \frac{1}{\mu}(\boldsymbol{E} \times \boldsymbol{B}) \tag{530}$$

With $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$, this becomes

$$F_E(\text{Poynting}) = \frac{1}{\mu}[\boldsymbol{B} \times (\boldsymbol{v} \times \boldsymbol{B})] = \frac{1}{\mu}\left[B^2\boldsymbol{v} - (\boldsymbol{B}\cdot\boldsymbol{v})\boldsymbol{B}\right] \simeq -R\rho\Omega v_{A\phi}\boldsymbol{v_A} \tag{531}$$

where only the largest term has been retained. (Do you understand why this is the largest term?) But the form of this term is exactly what is needed if we wish to replace $\langle u_R u_\phi \rangle$ by $W_{R\phi}$ in the energy equation as well as the angular momentum equation! For example, equation (523) for the radial energy flux becomes with a magnetic field:

$$-\frac{GM\Sigma v_2}{2R} + R\Sigma\Omega W_{R\phi} = -\frac{3GM\Sigma v_2}{2R} = \frac{3GM\dot{M}}{4\pi R^2}, \tag{532}$$

which is exactly what we found before. In particular, the classical result of accretion disc theory, equation (525), is a consequence of MHD turbulence. What is critical contribution of the magnetic field? Through the development of the MRI, it provides the fundamental physical basis for the correlations that enter into $W_{R\phi}$ to exist.