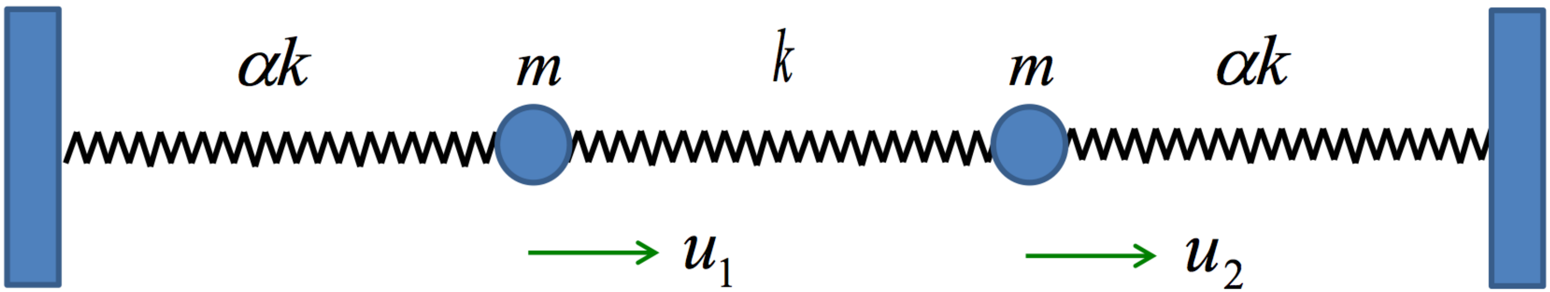
Spring-mass systems

Now consider a horizontal system in the form of masses on springs

- Again solve via decoupling and matrix methods
- Obtain the energy within the system
- Find specific solutions

Horizontal spring-mass system

Consider two masses moving without friction, between three springs, two with spring constants *αk*, one with spring constant *k*



Equations of motion:

Solutions of horizontal spring-mass system

Equations of motion:

(1)

(2)

Solve by decoupling method (add 1 and 2 and subtract 2 from 1). As before, we can write down the normal coordinates, call them q_1 and q_2

$$q_1 = \frac{1}{\sqrt{2}}(u_1 + u_2)$$

$$q_2 = \frac{1}{\sqrt{2}}(u_1 - u_2)$$

which means…

$$u_1 = \frac{1}{\sqrt{2}}(q_1 + q_2)$$

$$u_2 = \frac{1}{\sqrt{2}}(q_1 - q_2)$$

Substituting gives:

Gives normal frequencies of:

Centre of Mass

Relative

Cross-checking with matrix method

Write equations of motion as homogenous matrix equation

$$\begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{\alpha k}{m}+\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{\alpha k}{m}+\frac{k}{m} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Substitute in the below trial solution

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix}e^{i\omega t}$$

Demand that the resulting operator matrix is singular, i.e. Det{matrix}=0

$$\begin{vmatrix} -\omega^2+\frac{\alpha k}{m}+\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega^2+\frac{\alpha k}{m}+\frac{k}{m} \end{vmatrix}=0$$

Hence get eigenvalue equation

$$\left(-\omega^2+\frac{\alpha k}{m}+\frac{k}{m}\right)^2-\left(\frac{k}{m}\right)^2=0$$

We obtain same solutions as before

Energy of horizontal spring-mass system

Kinetic Energy

$$K = \frac{1}{2}m(\dot{u}_1^2 + \dot{u}_2^2)$$

$$K = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2)$$

Potential Energy

$$V = \frac{1}{2}\alpha k u_1^2 + \frac{1}{2}k(u_2 - u_1)^2 + \frac{1}{2}\alpha k u_2^2$$

$$V = \frac{1}{2}\alpha k q_1^2 + \frac{1}{2}(\alpha + 2)k q_2^2$$

$$V = \frac{1}{2}m\omega_1^2 q_1^2 + \frac{1}{2}m\omega_2^2 q_2^2$$

Total Energy:

Sum of energies in each normal mode

Specific Solutions for horizontal spring-mass system

General solution is the sum of the two normal modes

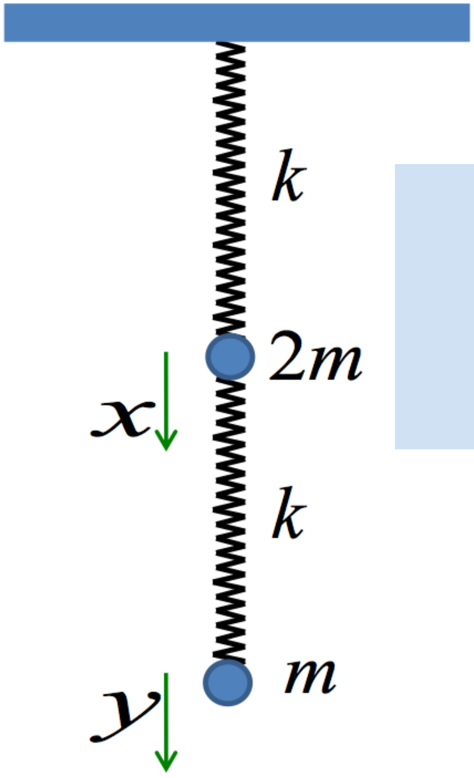
$$u_1 = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)$$
$$u_2 = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2)$$

Initial conditions: $u_1 = a \ ; \ u_2 = 0 \ ; \ \dot{u}_1 = \dot{u}_2 = 0$

These give: $A_1 = A_2 = \frac{a}{2} \ ; \ \phi_1 = \phi_2 = 0$

Both normal modes excited. The `beats' solution.
Completely analagous to the coupled pendulum

Vertical spring-mass system



x and *y* are displacements from equilibrium positions

- Find the normal frequencies of the system
- Find the ratio of the amplitudes for each normal mode

Decoupling method only works for limited cases with a sufficient amount of symmetry. You cannot solve this with decoupling, so have to go to matrix method and have a guess!

Solving with matrix method

Write equations of motion as homogenous matrix equation

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{k}{m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Substitute in the below trial solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$$

Demand that the resulting operator matrix is singular, i.e. Det{matrix}=0

$$\begin{vmatrix} -\omega^2 + \frac{k}{m} & -\frac{k}{2m} \\ -\frac{k}{m} & -\omega^2 + \frac{k}{m} \end{vmatrix} = 0$$

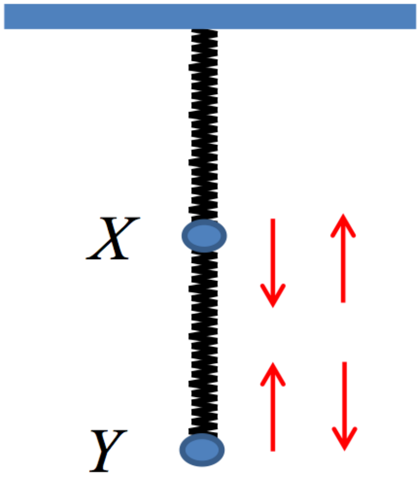
Hence get eigenvalue equation

$$\left(-\omega^2 + \frac{k}{m}\right)^2 - \frac{1}{2}\left(\frac{k}{m}\right)^2 = 0$$

From this we obtain the normal frequencies

Normal modes of vertical spring-mass system

Normal mode 1

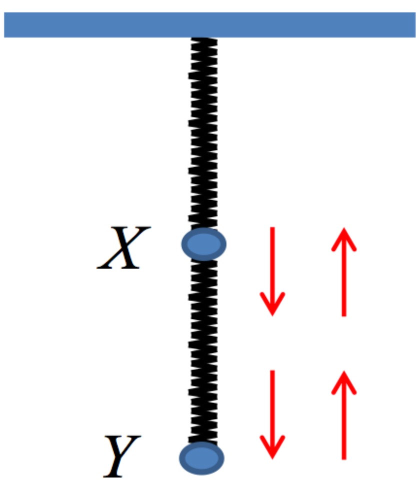


Substitute back into eigenvector equation

$$\left(-\omega_{1,2}{}^{2}+\frac{k}{m}\right)X-\left(\frac{k}{2m}\right)Y=0$$

to yield $X/Y=-1/\sqrt{2}$

Normal mode 2



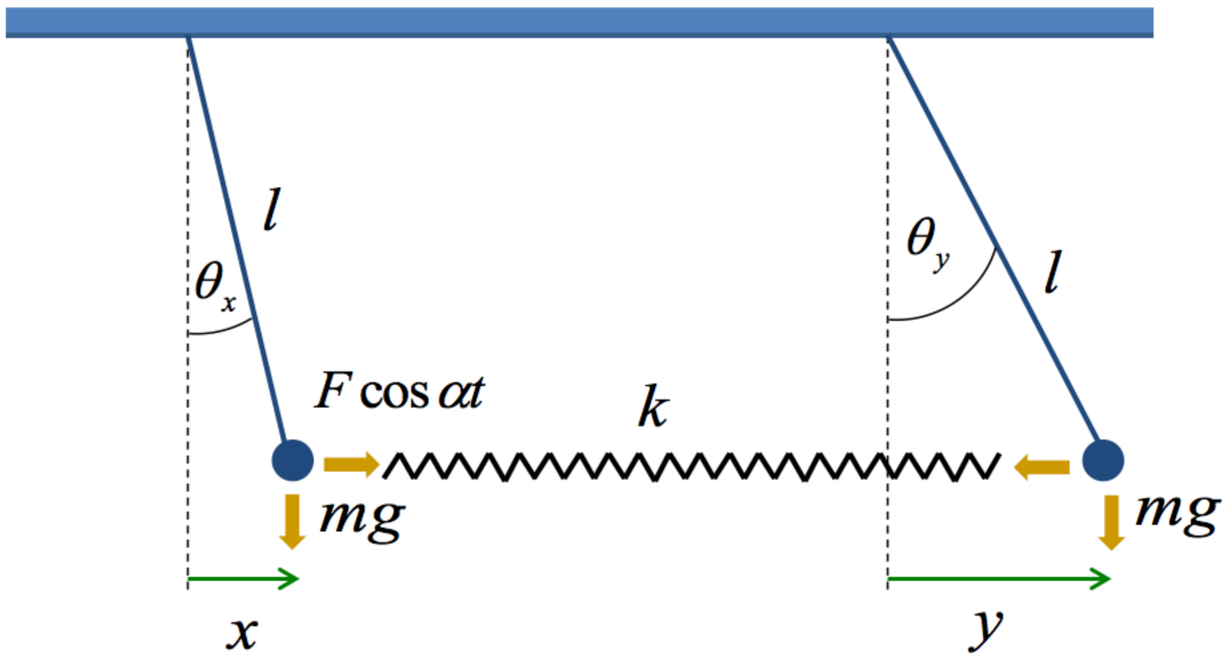
corresponds to $X/Y=1/\sqrt{2}$

Coupled Oscillators with a driving force

- So the last physical system we are going to look at in this first part of the course is the forced coupled pendula, along with a damping factor
  1. Finding the Complementary Function
  2. Finding the particular integral

- Then do the same for a horizontal spring-mass system

Damped driven coupled pendula



Both pendula experience a retarding force of γ x velocity

Equations of motion:

Damped driven coupled pendula

Let's arrange equations of motion in form $A\begin{pmatrix} x \\ y \end{pmatrix} = \text{whatever}$

We have:

and so

Contrary to before, this equation is inhomogeneous, in that RHS≠0.
To solve it we need to find both the complementary function (CF), which is solution to the homogeneous equivalent, and the particular integral (PI)

Damped driven coupled pendula: finding CF

To find CF write down homogenous equation and solve as previously

Try $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$ and find solution when operator matrix is singular

$= 0$

Damped driven coupled pendula: finding CF

To find CF write down homogenous equation and solve as previously

$$= 0$$

$$\Rightarrow \left[-\omega^2 + \frac{i\gamma}{m}\omega + \left(\frac{g}{l} + \frac{k}{m}\right)\right]^2 - \left(\frac{k}{m}\right)^2 = 0 \qquad \Rightarrow \left[-\omega^2 + \frac{i\gamma}{m}\omega + \left(\frac{g}{l} + \frac{k}{m}\right)\right] = \pm\left(\frac{k}{m}\right)$$

gives the equations $\bar{\omega}_1^{\,2} - \frac{i\gamma}{m}\omega - \frac{g}{l} = 0$ and $\bar{\omega}_2^{\,2} - \frac{i\gamma}{m}\omega - \left(\frac{g}{l} + 2\frac{k}{m}\right) = 0$

with solutions

where $\omega_1^2 = \frac{g}{l}$, $\omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$ which are the results from the undamped scenario

There is no physical difference between +/- variants. Just use + from now on.

Damped driven coupled pendula: finding CF

Substitute eigenvalues into the below:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

to deduce one mode has X=Y, & the other X=-Y. Since $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} \exp(i\varpi_{1,2}t)$

We get the CF:

Note the exponential decay factor.

Finding CF with decoupling method

The equations of motion

can be decoupled with the normal coordinates

$$q_1 = \frac{1}{\sqrt{2}}(x+y)$$

$$q_2 = \frac{1}{\sqrt{2}}(x-y)$$

to yield the 2nd order homogeneous differential equations

$$\ddot{q}_1 + \frac{\gamma}{m}\dot{q}_1 + \omega_1{}^2 q_1 = 0 \qquad \ddot{q}_2 + \frac{\gamma}{m}\dot{q}_2 + \omega_2{}^2 q_2 = 0 \quad \text{with} \quad \omega_1^2 = \frac{g}{l}, \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

that can be solved through trial solution q=Re($e^{i\omega t}$) to give same results

Damped driven coupled pendula: finding PI

We have the CF. Now we need to find the PI, *i.e.* a solution to the full equation

$$\Bigg)\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{Re}[\exp(i\alpha t)]$$

Try this ansatz $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} P \\ Q \end{pmatrix} e^{i\alpha t}\right]$ which means solving the following

$$\Bigg)\begin{pmatrix} P \\ Q \end{pmatrix} = \frac{F}{m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Damped driven coupled pendula: finding PI

We have matrix equation of the sort: $MU=V$

$M \qquad\qquad U \qquad V$

$$\big(\;[\text{redacted}]\;\big)\begin{pmatrix}P\\Q\end{pmatrix}=\frac{F}{m}\begin{pmatrix}1\\0\end{pmatrix}$$

and so $U=M^{-1}V$, i.e. $\begin{pmatrix}P\\Q\end{pmatrix}=M^{-1}\frac{F}{m}\begin{pmatrix}1\\0\end{pmatrix}$

Damped driven coupled pendula: finding PI

Need to find the inverse of $M =$

$$M^{-1} = \frac{1}{\det M}\,\mathrm{adj}(M)$$

$$\begin{aligned}
\det M &= \left[-\alpha^2 + i\alpha\frac{\gamma}{m} + \left(\frac{g}{l} + \frac{k}{m}\right)\right]^2 - \left[\frac{k}{m}\right]^2 \\
&= \left[-\alpha^2 + i\alpha\frac{\gamma}{m} + \left(\frac{g}{l} + \frac{k}{m}\right) - \frac{k}{m}\right]\cdot\left[-\alpha^2 + i\alpha\frac{\gamma}{m} + \left(\frac{g}{l} + \frac{k}{m}\right) + \frac{k}{m}\right] \\
&= \left[-\alpha^2 + \frac{g}{l} + i\alpha\frac{\gamma}{m}\right]\cdot\left[-\alpha^2 + \left(\frac{g}{l} + \frac{2k}{m}\right) + i\alpha\frac{\gamma}{m}\right] \\
&= B_1 e^{-i\theta_1}\cdot B_2 e^{-i\theta_2}
\end{aligned}$$

where $B_{1,2} = \left((\omega_{1,2}{}^2 - \alpha^2)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{\frac{1}{2}}$ and $\tan\theta_{1,2} = \frac{-\alpha\gamma/m}{(\omega_{1,2}{}^2 - \alpha^2)}$

Damped driven coupled pendula: finding PI

Now $\quad \operatorname{adj} M =$

and we can write

$$-\alpha^2 + i\frac{\gamma}{m}\alpha + \frac{g}{l} + \frac{k}{m} = \frac{1}{2}\left(-\alpha^2 + i\frac{\gamma}{m}\alpha + \frac{g}{l}\right) + \frac{1}{2}\left(-\alpha^2 + i\frac{\gamma}{m}\alpha + \frac{g}{l} + \frac{2k}{m}\right)$$

$$= \frac{1}{2}\left[B_1 e^{-i\theta_1} + B_2 e^{-i\theta_2}\right]$$

$$\frac{k}{m} = \frac{1}{2}\left(-\alpha^2 + i\frac{\gamma}{m}\alpha + \frac{g}{l} + \frac{2k}{m}\right) - \frac{1}{2}\left(-\alpha^2 + i\frac{\gamma}{m}\alpha + \frac{g}{l}\right)$$

$$= \frac{1}{2}\left[B_2 e^{-i\theta_2} - B_1 e^{-i\theta_1}\right]$$

and so

$$\operatorname{adj} M = \frac{1}{2}$$

Damped driven coupled pendula: finding PI

$$M^{-1} = \frac{1}{\det M}\,\mathrm{adj}(M) =$$

$$=$$

Recall $$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{F}{m}\mathrm{Re}\left[M^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\alpha t}\right]$$

$$\begin{pmatrix} x \\ y \end{pmatrix} =$$

we have it !

with $B_{1,2} = \left((\omega_{1,2}{}^2 - \alpha^2)^2 + \left(\frac{\alpha\gamma}{m}\right)^2\right)^{\frac{1}{2}}$ and $\tan\theta_{1,2} = \dfrac{-\alpha\gamma/m}{(\omega_{1,2}{}^2 - \alpha^2)}$

Finding PI with decoupling method

Inhomogeneous equations written in terms of normal coordinates

$$\ddot{q}_1 + \frac{\gamma}{m}\dot{q}_1 + \omega_1^{\,2} q_1 = \frac{F}{m}\cos\alpha t \quad (1)$$

$$\ddot{q}_2 + \frac{\gamma}{m}\dot{q}_2 + \omega_2^{\,2} q_2 = \frac{F}{m}\cos\alpha t \quad (2)$$

$$q_1 = \frac{1}{\sqrt{2}}(x+y) \qquad q_2 = \frac{1}{\sqrt{2}}(x-y)$$

Trial ansatz for (1) $q_1 = \mathrm{Re}\left[A_1 \exp(i\alpha t)\right]$

$$\Rightarrow \left(-\alpha^2 + i\alpha\frac{\gamma}{m} + \frac{g}{l}\right)A_1 = \frac{F}{m}$$

with

$$A_1 = \frac{(F/m)\exp(i\theta_1)}{\left(\left(\omega_1^{\,2} - \alpha^2\right)^2 + (\alpha\gamma/m)^2\right)^{1/2}}$$

$$\tan\theta_1 = \frac{-(\alpha\gamma/m)}{(\omega_1^{\,2} - \alpha^2)}$$

Hence

Same procedure for (2) gives entirely analogous expression for q_2.
From these same expressions are obtained for *x* and *y* as before.

Damped driven coupled pendula: full solution

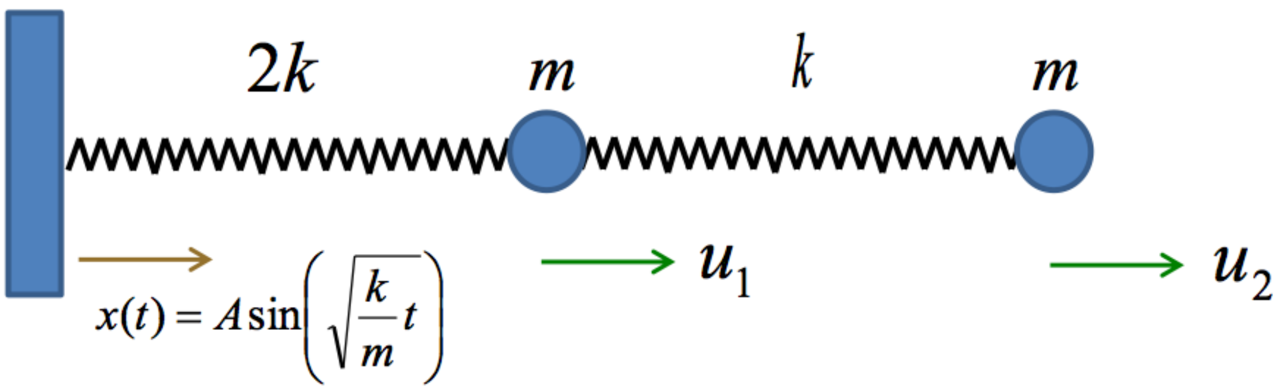
Solution = CF + PI

with $B_{1,2} = \left((\omega_{1,2}{}^2 - \alpha^2)^2 + \left(\frac{\alpha\gamma}{m} \right)^2 \right)^{\frac{1}{2}}$, $\tan\theta_{1,2} = \frac{-\alpha\gamma/m}{(\omega_{1,2}{}^2 - \alpha^2)}$, $\omega_1^2 = \frac{g}{l}$ & $\omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$

The CF part is the 'transient solution' determined by the initial conditions;
the PI part is the 'steady state solution' determined by the driving force.

Horizontal spring-mass system with driving term

Consider two masses moving without friction, with two springs of spring constants *2k* and *k* respectively, connected to wall which is driven by an external force to have time-dependent displacement $x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right)$



Equations of motion:

Horizontal spring-mass system with driving term – find the CF

Write down the homogeneous case and find CF using matrix method

Try $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathrm{Re}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} e^{i\omega t}$ gives $\left(\quad \right) = 0$

Requiring determinant = 0 yields

Substitute back in to eigenvector equ[n] $\left(\dfrac{U_2}{U_1}\right)_1 = 1-\sqrt{2}$ and $\left(\dfrac{U_2}{U_1}\right)_2 = 1+\sqrt{2}$

Horizontal spring-mass system with driving term – find the PI

We have the CF. Now we need to find the PI, *i.e.* a solution to the full equation

$$= \frac{Ak}{m}\begin{pmatrix}2\\0\end{pmatrix}\mathrm{Re}[\exp(i\sqrt{\frac{k}{m}}t)]$$

Try ansatz

which means solving

$$\begin{pmatrix}\frac{2k}{m} & -\frac{k}{m}\\ -\frac{k}{m} & 0\end{pmatrix}\begin{pmatrix}P\\Q\end{pmatrix} = \frac{kA}{m}\begin{pmatrix}2\\0\end{pmatrix}$$

Inverse of LHS 2x2 matrix is

$$\frac{1}{-(k/m)^2}\begin{pmatrix}0 & \frac{k}{m}\\ \frac{k}{m} & \frac{2k}{m}\end{pmatrix} = \frac{m}{k}\begin{pmatrix}0 & -1\\ -1 & -2\end{pmatrix}$$

and so

Horizontal spring-mass system with driving term – full solution

Solution = CF + PI