Waves & Normal Modes

Prof. Matt Jarvis

matt.jarvis@astro.ox.ac.uk

Hilary Term 2015

General Details

- 12 lectures
- Notes will be available to download every 1-2 weeks.
- These will NOT be complete, so pay attention in the lectures
- Unintentional mistakes may occur – please let me know in lectures or via email if you spot anything
- 3 problem sheets will be distributed. These are inherited from previous lecturers of this course – many thanks to them
- Material will be posted on https://www2.physics.ox.ac.uk/contacts/people/jarvis (under "Teaching")
- Thanks also to Guy Wilkinson for his lecture notes on this course

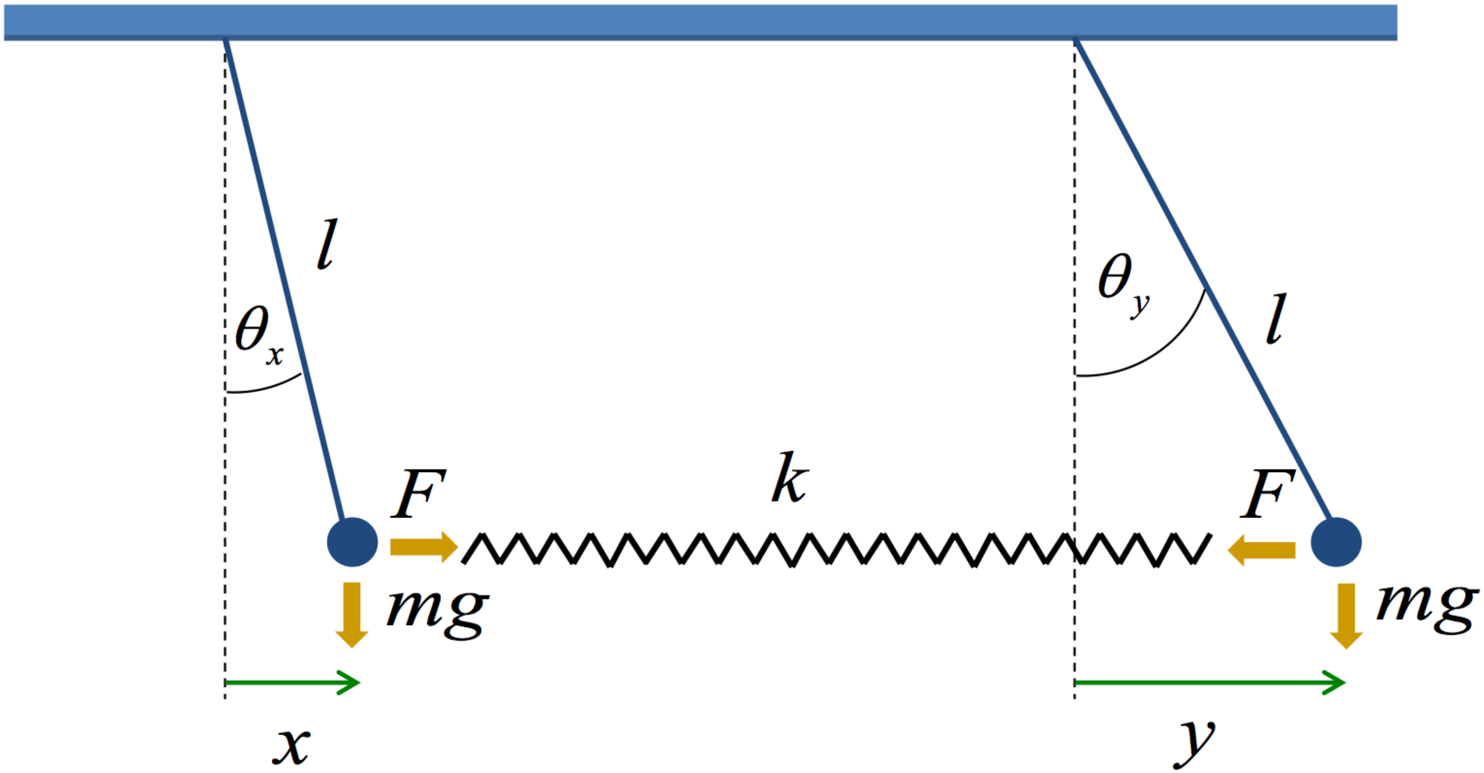
Text Books

- Vibrations & Waves, A.P. French, MIT Introductory Physics Series
- Vibrations and waves in Physics, I.G. Main, CUP
- Waves, C.A. Coulson & A. Jeffrey, Longman Mathematical Texts

Vibrations & Waves in Physics

- Although we will only use specific examples during this course, the physics of waves and vibrations underpin many different areas of physics that you will comes across over the next 3-4 years.
- For example:
    - Electromagnetism
    - Quantum Mechanics
    - Cosmology

Coupled Pendula



Equations of Motion
for each pendulum…

How to solve?

Equations of Motion:

(1)

(2)

Two equations and two unknowns, *x* and *y*

First we will use the so-called `decoupling method'

Which involves decoupling the equations from each other and solving the decoupled equations individually.

Decoupling Method

Equations of Motion:

(1)

(2)

Adding (1) and (2)

Looks very similar to the standard wave equation… define $q_1 \equiv x + y$

$$\ddot{q}_1 = -\omega_1^2 q_1 \quad \text{with} \quad \omega_1^2 = \frac{g}{l}$$

SHM!!!

A_1 & ϕ_1 are constants set by boundary conditions

Decoupling Method

Equations of Motion: (1)

(2)

Subtracting (2) from (1)

This time $q_2 \equiv x - y$

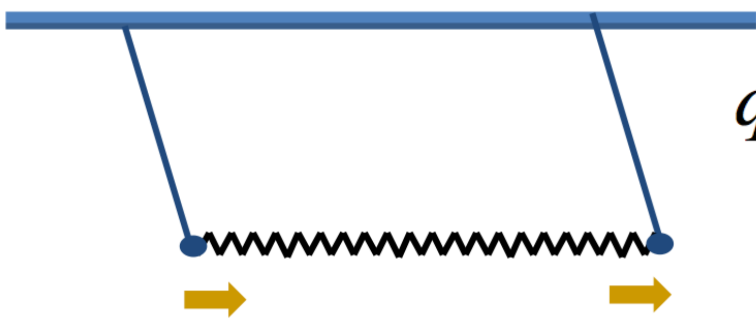
$$\ddot{q}_2 = -\omega_2^2 q_2 \quad \text{with} \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

A_2 & ϕ_2 are constants set by boundary conditions

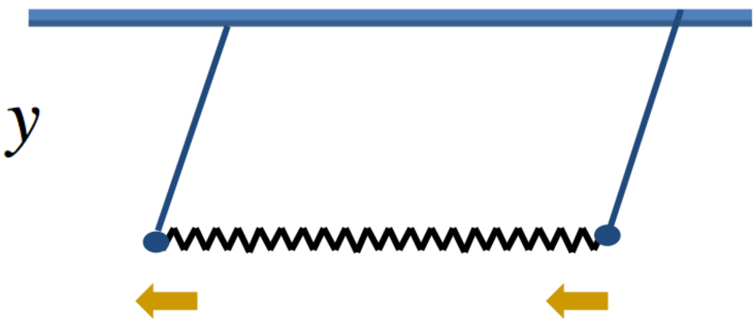
Normal Modes of Coupled Pendulum

- The first normal mode: centre-of-mass motion

$$q_1 = A_1 \cos(\omega_1 t + \phi_1) \qquad \omega_1^2 = \frac{g}{l}$$

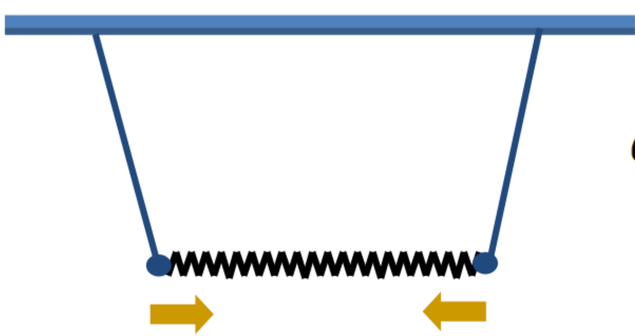


$$q_1 \equiv x + y$$

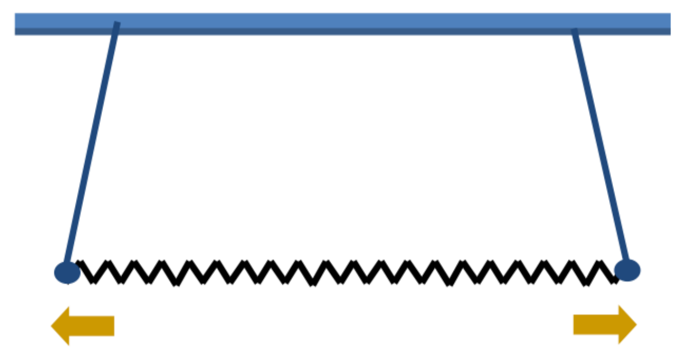


- The second normal mode: relative motion

$$q_2 = A_2 \cos(\omega_2 t + \phi_2) \qquad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$



$$q_2 \equiv x - y$$



Normal Modes of Coupled Pendulum

The variables q_1 and q_2 are called the mode, or normal, coordinates

In any normal mode only one of these coordinates is active at any one time (i.e. either q_1 is vibrating harmonically and q_2 is zero or vice versa)

It is more common to define the mode coordinates with a normalising factor in front (in this case $1/\sqrt{2}$)

$$q_1 = \frac{1}{\sqrt{2}}(x+y)$$

$$q_2 = \frac{1}{\sqrt{2}}(x-y)$$

This means that the vector defined by (q_1,q_2) has same length as that defined by (x,y), i.e. $q_1^2 + q_2^2 = x^2 + y^2$. This factor changes none of results we obtained.

General Solution for a Coupled Pendulum

The General Solution is a sum of the two normal modes

The constants are just set by the initial conditions

Coupled Pendulum: Different Initial Conditions

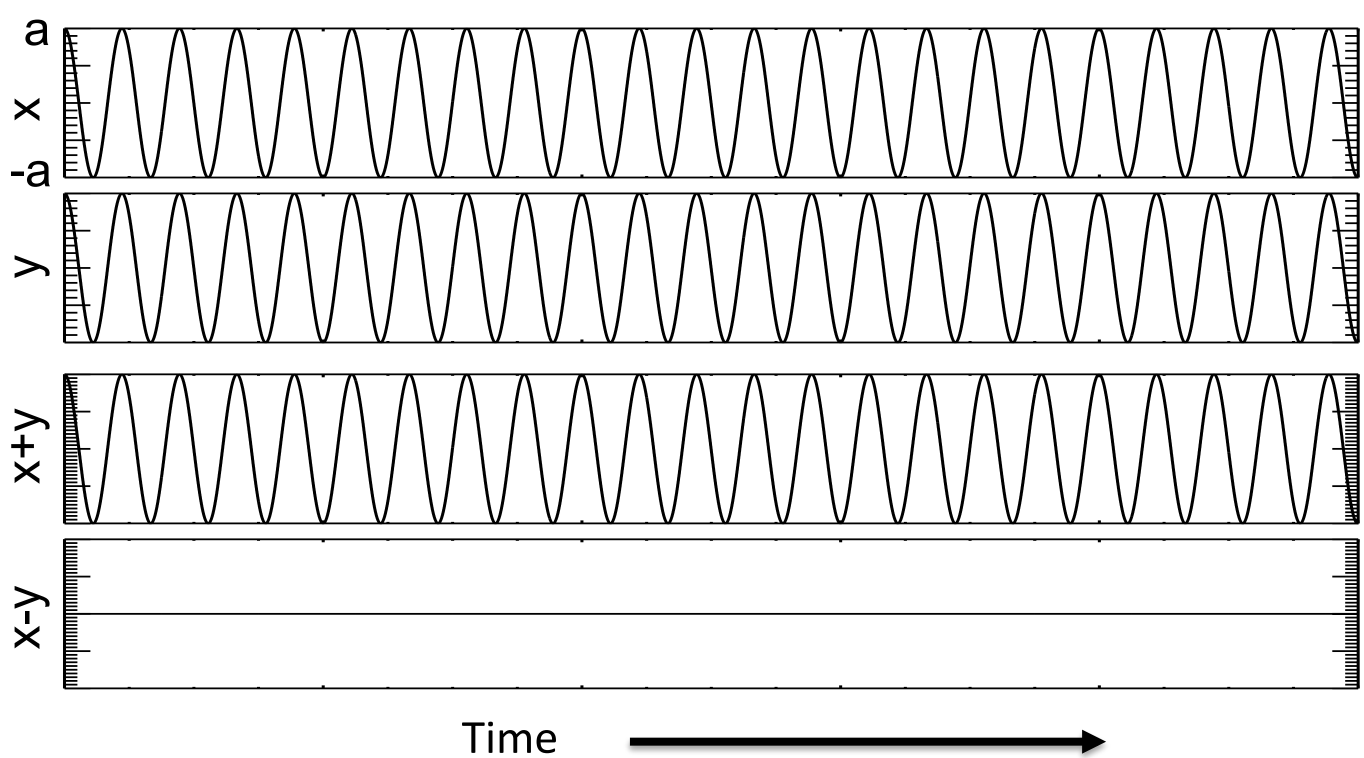
Example 1

$$x(0) = y(0) = a$$

$$\dot{x}(0) = \dot{y}(0) = 0$$

Plugging these initial conditions into the general solution gives:

$$A_1 = a \ ; \ A_2 = 0 \ ; \ \phi_1 = 0$$



Coupled Pendulum: Different Initial Conditions

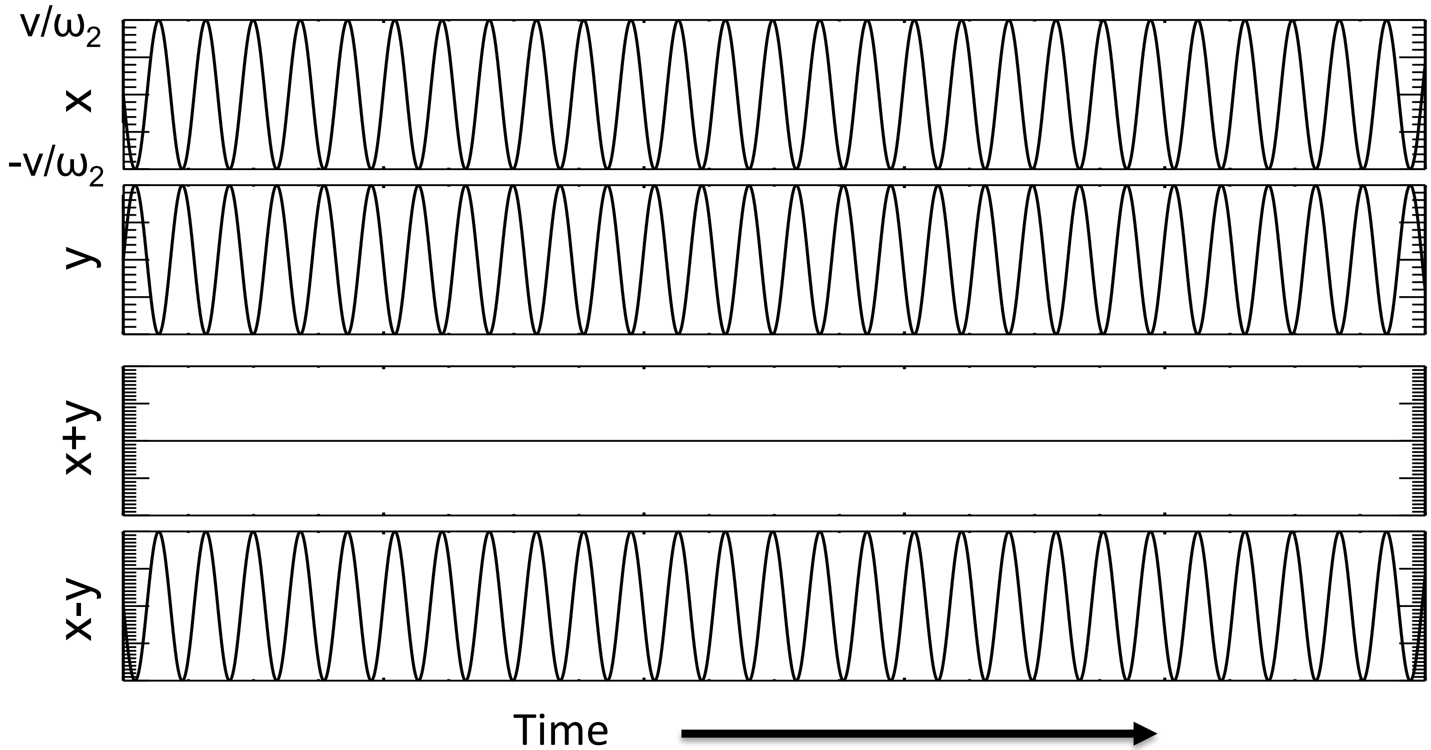
Example 2

$$x(0) = y(0) = a$$

$$\dot{x}(0) = -v \ ; \ \dot{y}(0) = v$$

Gives..

$$A_1 = 0 \ ; \ A_2 = -\frac{v}{\omega_2} \ ; \ \phi_2 = \pi/2$$



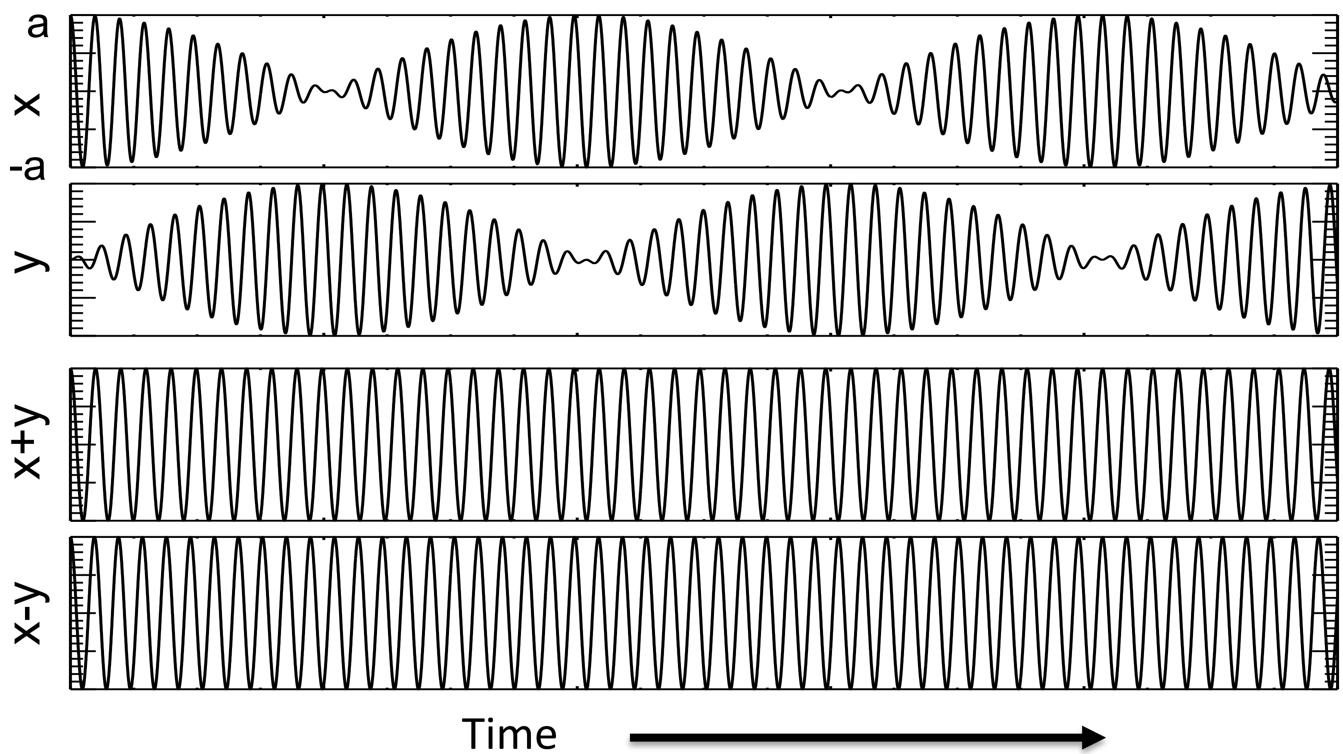
Coupled Pendulum: Different Initial Conditions

Example 3

$$x(0) = a \; ; \; y(0) = 0$$

$$\dot{x}(0) = \dot{y}(0) = 0$$

Gives… $A_1 = A_2 = \dfrac{a}{2} \; ; \; \phi_1 = \phi_2 = 0$

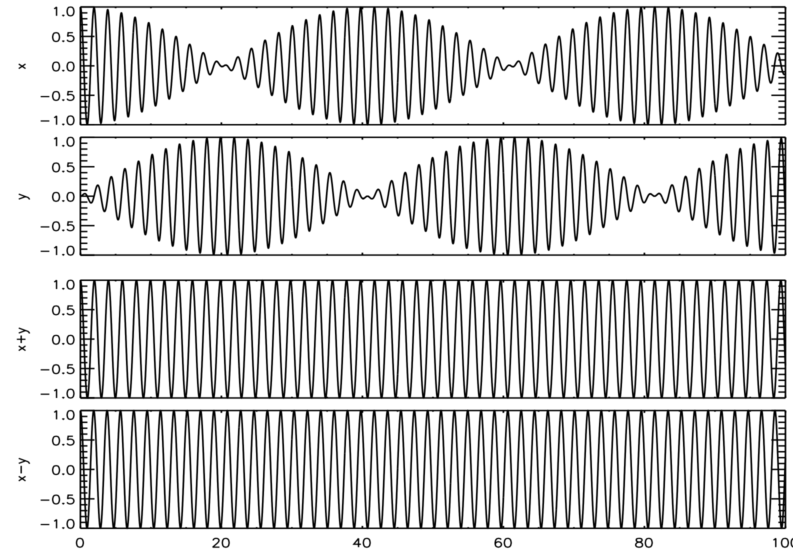


Coupled Pendulum: Different Initial Conditions

Example 3

$$x = \frac{a}{2}\left[\cos\omega_1 t + \cos\omega_2 t\right]$$

$$y = \frac{a}{2}\left[\cos\omega_1 t - \cos\omega_2 t\right]$$



Let… $S = \frac{\omega_1 + \omega_2}{2} t$ and $D = \frac{\omega_1 - \omega_2}{2} t$

$$x = \frac{a}{2}\left[\cos(S+D) + \cos(S-D)\right]$$

$$x = \frac{a}{2}\left[\cos S \cos D - \sin S \sin D + \cos S \cos D + \sin S \sin D\right]$$

$$x = a\cos S \cos D$$

$$y = \frac{a}{2}\left[\cos(S+D) - \cos(S-D)\right]$$

$$y = \frac{a}{2}\left[\cos S \cos D - \sin S \sin D - \cos S \cos D - \sin S \sin D\right]$$

$$y = -a\sin S \sin D$$

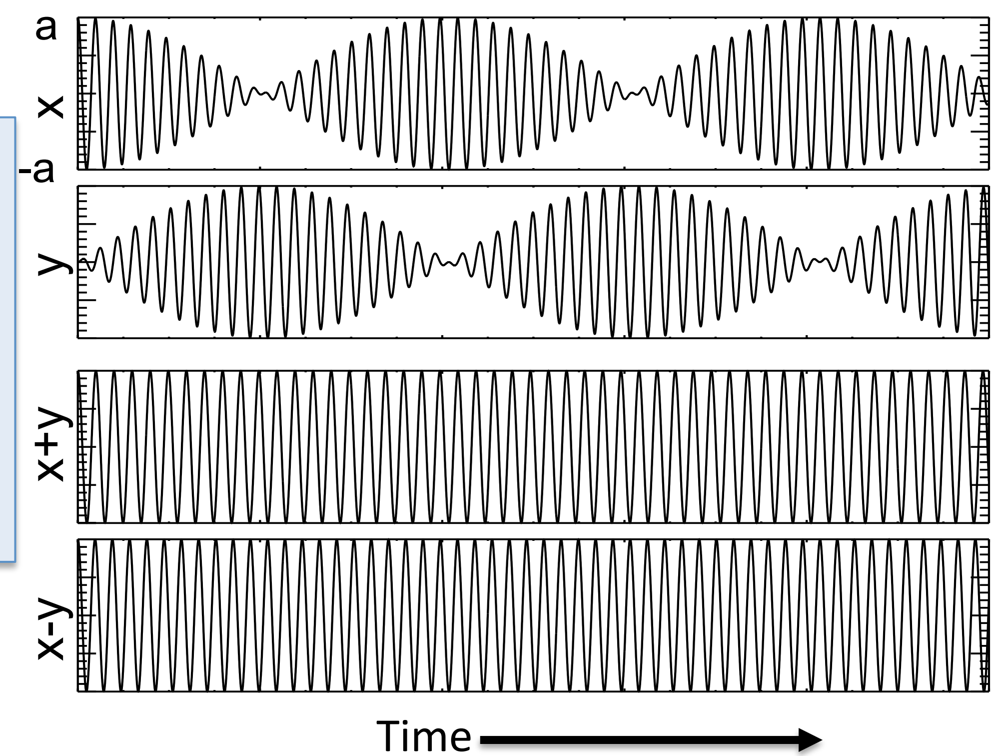
Coupled Pendulum: Different Initial Conditions

Example 3

$$T = 2\pi/\omega$$

Envelope has period $T_{env} = \dfrac{4\pi}{\omega_1 - \omega_2}$

In this example both of the normal modes are excited



‘Beats’ – energy is being transferred between pendula

Coupled Pendulum: Energy

Calculate total energy of the system ***U = KE + PE = T + V***

Kinetic Energy $T =$

Potential Energy

$V_{spring} =$

$V_{gravity} =$

Coupled Pendulum: Energy

Can also calculate Potential Energy using

$$F_x = -\frac{\partial V}{\partial x} \text{ and } F_y = -\frac{\partial V}{\partial y}$$

$$F_x = -\frac{\partial V}{\partial x} = m\ddot{x} = -mg\frac{x}{l} + k(y-x)$$

$$\Rightarrow V(x,y) = mg\frac{x^2}{2l} + \frac{1}{2}kx^2 - kxy + f(y) + C$$

$$F_y = -\frac{\partial V}{\partial y} = m\ddot{y} = -mg\frac{y}{l} - k(y-x)$$

$$\Rightarrow V(x,y) = mg\frac{y^2}{2l} + \frac{1}{2}ky^2 - kxy + f(x) + C$$

Neglecting the constant, C, which is an arbitrary offset

Coupled Pendulum: Energy

This is a bit unwieldy.
Why don't we go back to the normal coordinates and see what it looks like?

$$q_1 = \frac{1}{\sqrt{2}}(x+y) \qquad \omega_1^2 = \frac{g}{l} \qquad\qquad q_2 = \frac{1}{\sqrt{2}}(x-y) \qquad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

The cross-term in V has now disappeared

$$U = \underbrace{\qquad}_{\text{Energy in mode 1}} + \underbrace{\qquad}_{\text{Energy in mode 2}}$$

Energy in mode 1

Energy in mode 2

Total energy in the system = sum of energies in each mode

Solving with matrix method

$$m\ddot{x} = -mg\frac{x}{l} + k(y-x)$$

$$m\ddot{y} = -mg\frac{y}{l} - k(y-x)$$

Expecting an oscillatory solution, so let's try substituting one in, making use of complex notation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$$

X & Y are complex constants

We obtain:

eigenvector equation

With –w being the eigenvalues

Solving with matrix method

We have an homogeneous matrix equation of the sort $A\Psi = 0$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

eigenvector equation

The non-trivial solution requires the matrix is singular, i.e. has no inverse

$$\Rightarrow \quad \det[A] = 0$$

So here:

$$= 0$$

Solving with matrix method

$$= 0$$

eigenvalue equation $\left(-\omega^2 + \frac{g}{l} + \frac{k}{m}\right) = \pm\frac{k}{m}$ →

$$\omega_1^2 = \frac{g}{l}$$

$$\omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

Substitute back into eigenvector equation to learn

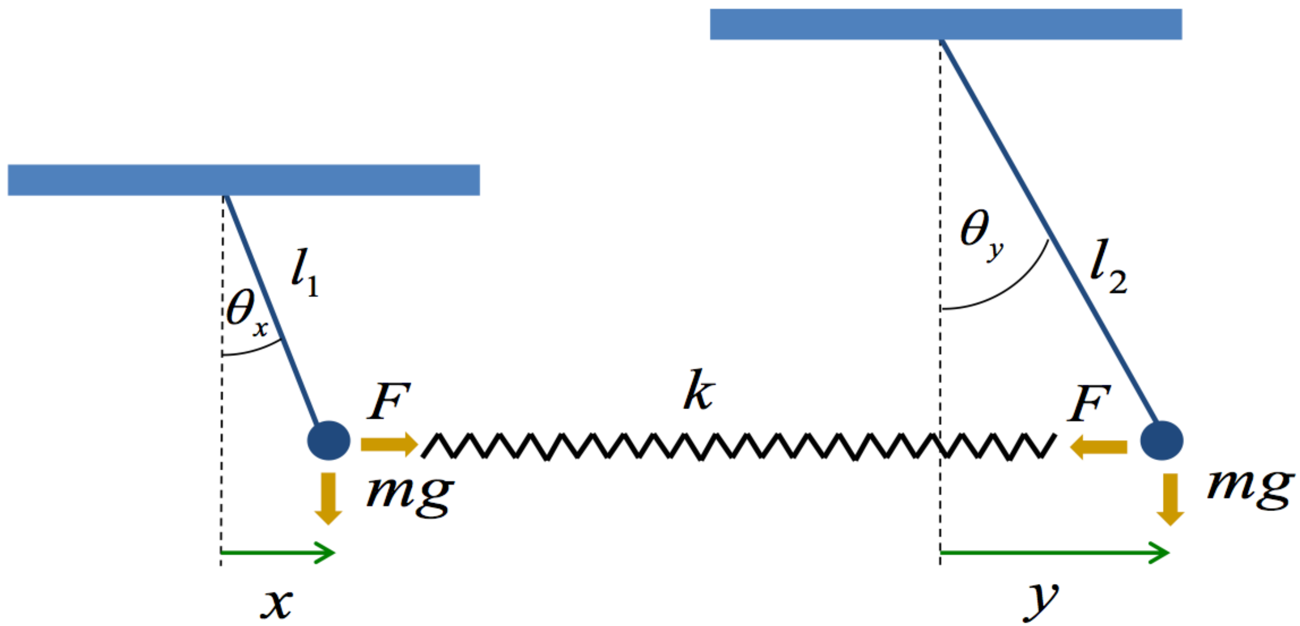
- when $\omega=\omega_1$ then X=Y, call it $A_1 e^{i\phi_1}$ $\Rightarrow$
- when $\omega=\omega_2$ then X=-Y, call it $A_2 e^{i\phi_2}$ $\Rightarrow$

Same normal modes & frequencies as before!

Unequal coupled pendula

- Up until now we have only considered the case where the 2 pendula were of the same length
- Now we will find the equations of motion for pendula of unequal length

Unequal coupled pendula



Equations of motion:

Unequal coupled pendula

Attack problem with matrix method:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Try $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{Re}\begin{pmatrix} X \\ Y \end{pmatrix} e^{i\omega t}$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so we must find solutions of

$$= 0$$

Unequal coupled pendula

Requiring

$$= 0$$

yields $\left(-\omega^2+\beta_1^2+\frac{k}{m}\right)\left(-\omega^2+\beta_2^2+\frac{k}{m}\right)-\left(\frac{k}{m}\right)^2=0 \qquad {\beta_{1,2}}^2=\frac{g}{l_{1,2}}$

Expanding this and then solving for ω^2 gives

$${\omega_{1,2}}^2 =$$

Sanity check: $l_1 = l_2 = l \Rightarrow {\beta_1}^2 = {\beta_2}^2 = \frac{g}{l}$ and ${\omega_{1,2}}^2$ reduce to equal length solutions

Unequal coupled pendula

Substitute $\omega_{1,2}{}^2 = \frac{1}{2}\left[(\beta_1{}^2 + \beta_2{}^2) + \frac{2k}{m} \pm \sqrt{(\beta_1{}^2 - \beta_2{}^2)^2 + \left(\frac{2k}{m}\right)^2}\right]$ with $\beta_{1,2}{}^2 = \frac{g}{l_{1,2}}$

into

$$\begin{pmatrix} -\omega_{1,2}{}^2 + \frac{g}{l_1} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\omega_{1,2}{}^2 + \frac{g}{l_2} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

to yield

In the case $l_1 = l_2$ then $\beta_1{}^2 = \beta_2{}^2$ and one recovers the same length pendulum solutions $X/Y = +1$ and -1. It is also interesting to note that one can show

$$\left(\frac{Y}{X}\right)_1 = -1 \Big/ \left(\frac{Y}{X}\right)_2$$

and so we define

Unequal coupled pendula: a specific solution

General solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ r \end{pmatrix} A_1 \cos(\omega_1 t + \phi_1) + \begin{pmatrix} -r \\ 1 \end{pmatrix} A_2 \cos(\omega_2 t + \phi_2)$$

Now consider the initial conditions

$$x = a;\ y = 0;\ \dot{x} = \dot{y} = 0$$

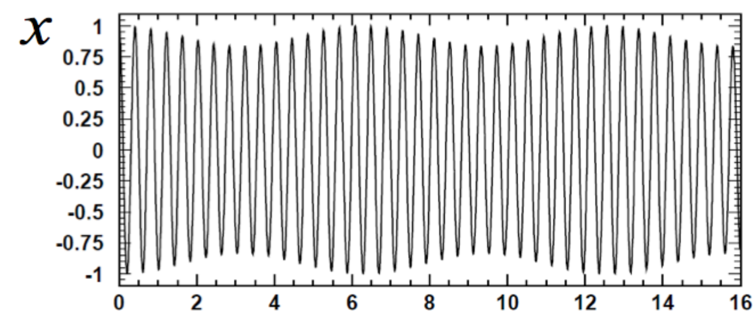
$$\Rightarrow A_1 = a/(1+r^2);\ A_2 = -ra/(1+r^2);\ \phi_1 = \phi_2 = 0$$

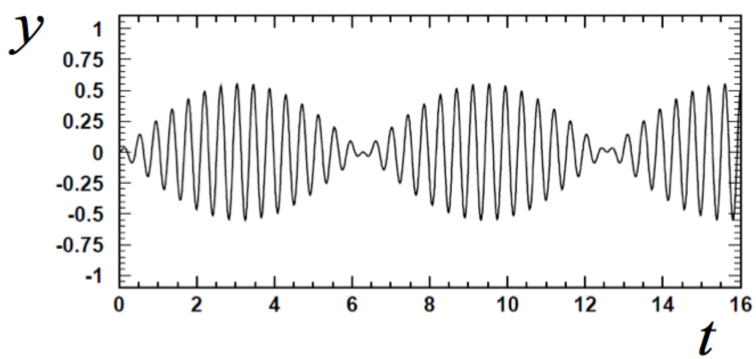
Hence

$$x(t) = a\left[\cos\omega_1 t + r^2 \cos\omega_2 t\right]/(1+r^2)$$

$$y(t) = ar\left[\cos\omega_1 t - \cos\omega_2 t\right]/(1+r^2)$$

which can be written





'Beats' solution as before, but now with *r*< *1* there is incomplete transfer of energy between pendula