9-10. Summary of lectures on rotational symmetry

\(O(3)\) group: continuous rotations of a sphere
eigenstates of \(\mathcal{H}\) for motion in central potential form basis for representation of \(O(3)\)

\[\psi_{\text{orb}}(x, y, z) \propto Y^m_l(\theta, \phi)\] for \((2l + 1)\) dimensional representation \(D^{(l)}\)

\[R_2(\alpha) Y^m_l = e^{-i m \alpha} Y^m_l\] (in natural basis)

multi-\(e^-\)

\[R_2(\alpha)|L, m_L\rangle = e^{-i m_L \alpha} |L, m_L\rangle\]

form rep = \(D^{(l)}\)

\[\chi^{(l)}(\alpha) = \frac{\sin((L + \frac{1}{2}) \alpha)}{\sin \frac{\alpha}{2}}\]

\(\chi^{(l)}(\alpha) = (-1)^l \chi(\alpha)\) improper rot

Character

\[\chi(\alpha) = \sum c_l \chi^{(l)}(\alpha)\]

\(c_l\) parity

Spin wavefunctions can also form basis

\[R_2(\alpha)|S, m_S\rangle = e^{-i m_S \alpha} |S, m_S\rangle\] (in natural basis)

if \(S = \frac{1}{2}\)

\[R(2\pi)|\frac{1}{2}, \frac{1}{2}\rangle = e^{-i \frac{1}{2} \cdot 2\pi} |\frac{1}{2}, \frac{1}{2}\rangle = (-1) |\frac{1}{2}, \frac{1}{2}\rangle\]

change sign upon rot by \(2\pi\)
Conventional point groups do not capture full symmetry properties of half-integer spin wave\(^\infty\), extra element
\[ \bar{E} = R(2\bar{u}) \quad \bar{E} \gamma_{\text{spin}} = (-1)^{2S} \gamma_{\text{spin}} \]

if Hamiltonian spherical \(\rightarrow\) lower symmetry p.i.s. \(G\), decompose \(D^{(5)}\) into irrs of double group
\[ d-G = \{ R, \bar{E}R \} \]
operation of \(G\)

in general
for given angular momentum \(J\), \((J, m_J)\) form rep. \(D^{(3)}\)

\[ \frac{2J+1}{\text{spherical}} \quad \Gamma \quad \text{irr of } D^{(3)} \text{ in point group of Hamiltonian} \]

\[ \text{Ham of lower sym.} \]

Characters
\[ Z^{(3)}(\alpha) = \frac{\sin (J + \frac{1}{2})\alpha}{\sin \frac{\alpha}{2}} \]
if environment centrosymmetric
\[ \sum \Gamma \iota \]
\[ Z^{(3)}(\bar{\alpha}) = (-1)^{\text{parity}} Z(\alpha) \]
if half-odd \(J\), use double group \(d-G\)
\[ Z(\bar{E}\alpha) = (-1)^{2S} Z(\alpha) \]
\[ Z(\bar{E}\bar{\alpha}) = (-1)^{2S} Z(\bar{\alpha}) \]
Double Groups

Example, $d-432$

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A complete listing of double groups may be found in G.F. Koster et al., Properties of the Thirty-Two Point Groups (MIT Press, Cambridge MA, 1963)