Universality far from equilibrium

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Content

• Thermalization dynamics in isolated quantum systems

• Universality far from equilibrium: Nonthermal fixed points in early-universe cosmology, relativistic heavy-ion collisions, and ultracold quantum gases

• Infrared cascade and dynamics of Bose condensation

• Some puzzles and challenges
Universality in thermal equilibrium

Example: Schematic phase diagram of

- strong interaction matter (QCD)

\[ (\hbar = c = k_B = 1) \]

\[ T \]

Temperature [GeV]

\[ \mu \]

Quark chemical potential [GeV]

\[ \langle \bar{\psi}\psi \rangle \neq 0 \]

\[ \langle \bar{\psi}\psi \rangle \neq 0 \]

- typical liquid-gas system

Universal critical exponents: e.g.

\[ \text{order parameter } \sim (T_c - T)^\beta \]

Ising universality class \((d=3)\)

→ Fine-tuning of critical parameter(s) to observe universal behavior
(relevant operators for renormalization group fixed points)
Universality far from equilibrium

Example: Schematic thermalization for isolated quantum systems

→ No fine-tuning to observe far-from-equilibrium universal behavior (irrelevant operators for nonthermal renormalization group fixed points)
Extreme conditions

Dimensionless combination of

coupling strength

\[ \frac{\lambda \cdot \langle \varphi^2 \rangle}{Q^2} \sim 1 \]

field² expectation value (vacuum/thermal equilibrium/nonequilibrium)

Characteristic energy/momentum²

→ Extreme conditions can enhance the loss of details about microscopic properties (coupling strengths, initial conditions, …)
Isolated quantum systems in extreme conditions

Early-universe inflaton dynamics

*Preheating after inflation (~10^{16} GeV)*

Relativistic heavy-ion collision experiments

*Quark-Gluon Plasma (~100 MeV ~10^{12} K)*

Table-top experiments with ultracold atoms

*Strong quenches at nanokelvins*
Evolution of the Universe

- Inflation
- Quantum fluctuations
- Afterglow Light Pattern 400,000 yrs.
- Dark Ages
- Development of Galaxies, Planets, etc.
- Dark Energy Accelerated Expansion
- 1st Stars about 400 million yrs.
- Big Bang Expansion 13.7 billion years

WMAP Science Team
`Preheating` from parametric resonance instability

- All particles/radiation diluted away after inflation → `vacuum-like`
- Particle production after inflation \( Kofman, Linde, Starobinsky, PRL \) 73 (1994) 3195

\[
V(\phi, \chi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \mu^2 \chi^2 + \frac{\lambda \phi}{4!} \phi^4 + \frac{g}{2} \phi^2 \chi^2
\]

\[
\phi \gg m/\sqrt{\lambda}, \quad \phi \sim 0.3 M_P, \quad \lambda_\phi \lesssim 10^{-12}, \quad g^2 \lesssim \lambda
\]

Massless reheating: \( m = \mu = 0 \)

\[
M_P \equiv (8\pi G)^{-1/2} \approx 2.435 \cdot 10^{18}\text{GeV}
\]

\[
\frac{d^2 \chi_k}{dt^2} + (k^2 + g\phi^2(t)) \chi_k = 0 \quad \Rightarrow \quad \chi_k \sim Q(t) \sim e^{\gamma_Q t/2}
\]

Classical oscillator analogue:

\[
\begin{align*}
\omega(t) &\leftrightarrow \phi(t), \\
\chi(t) &\leftrightarrow \chi_{k=0}(t)
\end{align*}
\]

\[
\ddot{x} + \omega^2(t)x = 0
\]
Macroscopic fields, condensates and fluctuations

- In a quantum theory the field amplitude corresponds to the expectation value of a (here relativistic, real) Heisenberg field operator $\Phi(t, x)$:

\[
\phi(t) = \langle \Phi(t, x) \rangle \equiv \text{Tr} \{ \rho_0 \Phi(t, x) \}
\]

time-dependent expectation value

density operator at some `initial´ time $t = 0$

- Fluctuations derive from correlation functions, e.g. spatially homogeneous:

\[
\text{`quantum-half´:} \quad \frac{f(t, p) + 1/2}{\omega(t, p)} + (2\pi)^3 \delta(p) \phi_0^2(t)
\]

\[
\equiv \frac{1}{2} \int d^3 x \ e^{-ipx} \langle \Phi(t, x)\Phi(t, 0) + \Phi(t, 0)\Phi(t, x) \rangle
\]

\[
\text{volume:} \quad (2\pi)^3 \delta(0) \rightarrow V
\]

\[
\text{dispersion:} \quad (2\pi)^3 \delta(p) \phi_0^2(t)
\]

\[
\text{condensate}^2
\]
Preheating: Insensitivity to initial condition details

Example: \( \text{`Inflaton'} \lambda \phi^4 \) theory (\( \lambda \ll 1 \)), \( \phi = \phi_0 + \delta \phi \)

1. Large initial field:

\[
\langle \phi \rangle \sim Q/\sqrt{\lambda} \\
\langle \delta \phi^2 \rangle \sim Q^2 \\
f = 0
\]

\[
\Delta t \sim Q^{-1} \log \lambda^{-1}
\]

2. High occupancy:

\[
f(Q) \sim e^{\gamma Q \Delta t}
\]

\[
\langle \delta \phi^2 \rangle \sim Q^2/\lambda
\]

\[
\rightarrow f(Q) \sim 1/\lambda \gg 1
\]

1. classical-statistical simulation (weak-coupling limit)

\[
\rightarrow \text{approach to nonthermal fixed point}
\]

Berges, Boguslavski, Schlichting, JHEP 1405 (2014) 054
Nonthermal fixed point: Insensitivity to coupling strength

E.g. scalar $N$-component $\lambda \phi^4$ quantum theory ($1/N$ to NLO 2PI):

Occupation number distribution:

Berges, Wallisch, arXiv:1607.02160

Universal scaling behavior for wide range of couplings!
Schematic behavior near nonthermal fixed point: dual cascade

Bose Condensation

Distribution function: $\log(f)$

Inversion cascade

$\alpha$

$-\beta$

Direct cascade

$\frac{1}{\lambda} \gg f \gg 1$

Quantum regime

$f \leq 1$

Momentum: $\log(p)$

$K \ll Q \ll \Lambda$
Particle versus energy transport

\[ K \ll Q \ll \Lambda \]

number conservation:  
\[ \dot{n}_Q = \dot{n}_K + \dot{n}_\Lambda \]

dominated by number transport, slightly affected by energy transport.

energy conservation:  
\[ Q\dot{n}_Q = K\dot{n}_K + \Lambda\dot{n}_\Lambda \]

\[ \dot{n}_K = \frac{\Lambda - Q}{\Lambda - K} \dot{n}_Q \approx \dot{n}_Q \]
\[ \dot{n}_\Lambda = \frac{Q - K}{\Lambda - K} \dot{n}_Q \approx \frac{Q}{\Lambda} \dot{n}_Q \]

\[ \Rightarrow \Lambda\dot{n}_\Lambda \approx Q\dot{n}_Q \]

Particles are transported towards lower scales, energy towards higher scales.
Self-similarity

\[
f(t, p) = s^{\alpha/\beta} f(s^{-1/\beta}t, sp)\]

\[
s^{-1/\beta}t = 1 \Rightarrow f(t, p) = t^\alpha f_s(t^\beta p)\]

Time-independent scaling function:

\[f_s(t^\beta p) \equiv f(1, t^\beta p)\]

Scaling exponents \(\alpha\) and \(\beta\) determine rate and direction of transport:

\[
K(t_1) = K_1 \Rightarrow K(t) = K_1 (t/t_1)^{-\beta}
\]

\[
f(t, K(t)) \sim t^\alpha
\]

e.g. \(\alpha > 0, \beta > 0\): particle transport towards lower momentum scales
Self-similar dynamics: infrared scaling

\[ f(t, p) = t^{\alpha} f_S(t^{\beta} p), \quad \alpha = 1.51 \pm 0.13, \quad \beta = 0.51 \pm 0.04 \]
Mass scale separating non-relativistic infrared regime

- **non-relativistic infrared dynamics** expected because of the generation of a **mass gap** (condensate + medium)

→ relativistic & non-relativistic field theories have same infrared scaling

*Piñeiro Orioli, Boguslavski, Berges, PRD 92 (2015) 025041*
Non-relativistic system: Dilute quantum gases

- Interacting bosons with s-wave scattering length $a$, interatomic distance $\sim n^{-1/3}$

`diluteness` parameter: $\zeta = \sqrt{n a^3}$

`inverse coherence length`: $Q = \sqrt{16\pi an}$

- E.g. Gross-Pitaevskii equation for dilute Bose gas:

$$i\partial_t \psi(t, x) = \left(-\frac{\nabla^2}{2m} + g|\psi(t, x)|^2\right)\psi(t, x)$$

$$g = 4\pi a/m \quad \quad N_{\text{total}} = \int d^3x |\psi(t, x)|^2$$
Dilute bose gas in extreme conditions

- E.g. in the mean-field approximation for a spatially homogeneous system without condensate the interaction term leads to a constant energy shift:

\[
\Delta E = 2g \langle |\psi|^2 \rangle = 2gn = 2g \int \frac{d^3p}{(2\pi)^3} f_{nr}(|p|)
\]

- For the overoccupied Bose gas with

\[
f_{nr}(Q) \sim \frac{1}{\zeta}
\]

the mean-field shift in energy is of the same order as the relevant kinetic energy \(Q^2/2m\) \textit{irrespective of the coupling} \(g\), since

\[
g \int d^3p \ f_{nr}(p) \sim g \ Q^3 \ f_{nr}(Q) \sim g \ Q^3 / \zeta \sim g \ Q^3 / mgQ \sim Q^2 / m
\]

\[
a = mg/(4\pi) \quad Q = \sqrt{16\pi an} \quad Q = 2\sqrt{mgn} \quad \zeta = \sqrt{na^3} \quad \zeta = mgQ/(16\pi^{3/2})
\]
Universal scaling form of the distribution function

\[ f_S(\xi) \simeq \frac{A(\kappa_> - \kappa_<)}{(\kappa_> - 2)(\xi/B)^{\kappa_<} + (2 - \kappa_<)(\xi/B)^{\kappa_>}} \quad , \quad \kappa_< \simeq 0.5 \quad , \quad \kappa_> \simeq 4.5 \]

\[ f_S(\xi = B) = A \quad , \quad df_S(\xi = B)/d\xi = -2A/B \]
Estimating scaling properties

\[ \frac{\partial f(t, p)}{\partial t} = C[f](t, p) \]

`collision integral`

\[ C[f](t, p) = s^{-\mu} C[f](s^{-1/\beta} t, s p) = t^{-\beta \mu} C[f_s](1, t^\beta p) \]

\[ f(t, p) = s^{\alpha/\beta} f(s^{-1/\beta} t, s p) \quad s^{-1/\beta} t = 1 \]

Use

\[ \frac{\partial}{\partial t} [t^\alpha f_S(t^\beta p)] = t^{\alpha-1} [\alpha + \beta q \cdot \nabla q] f_S(q)|_{q=t^\beta p} \]

Time-independent fixed point equation:

\[ [\alpha + \beta p \cdot \nabla p] f_S(p) = C[f_s](1, p) \]

+ scaling relation:

\[ \alpha - 1 = -\beta \mu \]

Micha, Tkachev, PRD 70 (2004) 043538
Conservation laws

\[ n = \int \frac{d^d p}{(2\pi)^d} f(t, p) = t^{\alpha-\beta d} \int \frac{d^d q}{(2\pi)^d} f_S(q) \]

\[ \Rightarrow \text{particle conservation: } \alpha = \beta d \]

\[ \epsilon = \int \frac{d^d p}{(2\pi)^d} \omega(p) f(t, p) = t^{\alpha-\beta(d+z)} \int \frac{d^d q}{(2\pi)^d} \omega(q) f_S(q) \]

\[ \Rightarrow \text{energy conservation: } \alpha = \beta(d+z) \]

\[ \omega(p) = s^{-z} \omega(sp) \]
Perturbative estimate (Gross-Pitaevskii)

\[ C^{2 \leftrightarrow 2} [f](t, \mathbf{p}) = \int d\Omega^{2 \leftrightarrow 2}(\mathbf{p}, \mathbf{l}, \mathbf{q}, \mathbf{r}) \]

\[ \times [(f_p + 1)(f_l + 1)f_q f_r - f_p f_l(f_q + 1)(f_r + 1)] \]

\[ \approx f_p \int d\Omega^{2 \leftrightarrow 2}(\mathbf{p}, \mathbf{l}, \mathbf{q}, \mathbf{r}) [(f_p + f_l)f_q f_r - f_p f_l(f_q + f_r)] \]

Using:

\[ \int d\Omega^{2 \leftrightarrow 2}(\mathbf{p}, \mathbf{l}, \mathbf{q}, \mathbf{r}) \sim g^2 \int \frac{d^d l}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \]

\[ \times (2\pi)^{d+1} \delta^{(d)}(\mathbf{p} + \mathbf{l} - \mathbf{q} - \mathbf{r}) \delta(\omega_p + \omega_l - \omega_q - \omega_r) \]

gives scaling relation:

\[ \alpha - 1 = -\beta [3d - (d + 2) - 3\alpha/\beta] \]

\[ \alpha = \beta d \]

\[ \Rightarrow \quad \text{nonrel. particle transport:} \quad \alpha = -\frac{d}{2}, \quad \beta = -\frac{1}{2} \]

Negative perturbative exponents do not account for inverse particle cascade!
Beyond perturbation theory: large-$N$ expansion to NLO

\[ \int d\Omega^{2\leftrightarrow 2}(p, l, q, r) \rightarrow \int d\Omega^{\text{NLO}}[f](t, p, l, q, r) \sim \int \frac{d^d l}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{d^d r}{(2\pi)^d} \times (2\pi)^{d+1} \delta^{(d)}(p + l - q - r) \delta(\omega_p + \omega_l - \omega_q - \omega_r) \times g_{\text{eff}}^2[f](t, p, q), \]

FIG. 11. Illustration of different scattering channels. The vertex correction at NLO may be viewed as an effective interaction, which involves the exchange of an intermediate particle.

Vertex correction (NLO $1/N$)

\[ g_{\text{eff}}^2(t, p, q) \equiv \frac{g^2}{|1 + \Pi_{\text{nr}}^R(t, \omega_p - \omega_q, p - q)|^2} \]

one-loop retarded self-energy:

\[ \Pi_{\text{nr}}^R(t, \omega, p) = g \int \frac{d^d q}{(2\pi)^d} f(t, p - q) \]
\[ \times \left[ \frac{1}{\omega_q - \omega_{p-q} - \omega - i\epsilon} + \frac{1}{\omega_q - \omega_{p-q} + \omega + i\epsilon} \right] \]

scaling behavior:

\[ \Pi_{\text{nr}}^R(t, \omega_p, p) = s^{\alpha/\beta - d + 2} \Pi_{\text{nr}}^R(s^{-1/\beta}t, \omega s p, s p) \]

\[ \alpha/\beta \geq d \text{, } \Pi_{\text{nr}}^R(t, \omega_p, p) \gg 1 \text{ in the infrared:} \]

\[ \Rightarrow \quad g_{\text{eff}}^2(t, p, q, r) = s^{-2(\alpha/\beta - d + 2)} g_{\text{eff}}^2(s^{-1/\beta}t, s p, s q, s r) \]
Scaling solution at NLO $1/N$

\[
C_{nr}^{\text{NLO}}[f](t, \mathbf{p}) = s^{-(2-\alpha/\beta)} C_{nr}^{\text{NLO}}[f](s^{-1/\beta} t, s \mathbf{p})
= t^{\alpha-2\beta} C_{nr}^{\text{NLO}}[f_S](1, t^{\beta} \mathbf{p})
\]

gives scaling relation: \( \alpha - 1 = \alpha - 2\beta \)

\[\Rightarrow \text{nonrel. transport: } \beta = \frac{1}{2} \text{ of } \left\{ \begin{array}{l}
\text{particles: } \alpha = d/2 \\
\text{energy: } \alpha = (d+2)/2 
\end{array} \right. \]

Positive nonperturbative exponents can describe inverse cascade!

NLO result in good agreement with full numerical simulation

Piñeiro Orioli, Boguslavski, Berges, PRD 92 (2015) 025041
Self-similar dynamics from classical-statistical simulations

\[ f(t, p) = t^\alpha f_S(t^\beta p) , \quad \alpha = 1.66 \pm 0.12 , \quad \beta = 0.55 \pm 0.03 \]
Condensation far from equilibrium

\[ F(t, t', x - x') = \frac{1}{2} \langle \psi(t, x) \psi^*(t', x') + \psi(t', x') \psi^*(t, x) \rangle \]

\[ f(t, p) + (2\pi)^3 \delta^3(p) |\psi_0|^2(t) \equiv \int d^3x \ e^{-i px} \ F(t, t, x) \]

volume: \((2\pi)^3 \delta(0) \rightarrow V\)
Condensation time

\[
\frac{N_0(t)}{N_{\text{total}}} = \frac{|\psi_0|^2(t)}{\int d^3p/(2\pi)^3 f(t, p) + |\psi_0|^2(t)} , \quad V^{-1}F(t, t, p = 0) \sim t^\alpha
\]

\[\Rightarrow \quad t_f \simeq t_0 \left( \frac{|\psi_0|^2(t_f)}{f(t_0, 0)} \right)^{1/\alpha} V^{1/\alpha}\]

Analytic estimates agree well with simulations!
Thermalization dynamics in relativistic heavy-ion collisions

\[ x^+ = 0 \]
\[ \eta = \text{const} \]
\[ \tau = \text{const} \]

\[ \tau = \sqrt{t^2 - z^2} \]
\[ x^\pm = \frac{(t \pm z)}{\sqrt{2}} \]
\[ \eta = \text{atanh}(z/t) \]
Heavy-ion collisions in the high-energy limit

**Large** initial gauge fields: \( \langle A \rangle \sim Q_s / g \)

CGC: Lappi, McLerran, Dusling, Gelis, Venugopalan, Epelbaum…

**Small** initial (vacuum) fluctuations: \( \langle \delta A^2 \rangle \sim Q_s^2 \)

\[ \rightarrow \text{ plasma instabilities} \]

\( Q_s \tau \sim 1 \)

\( 1 < Q_s \tau < \log^2(\alpha_s^{-1}) \)

\( Q_s \tau \sim \log^2(\alpha_s^{-1}) \)

JB, Schenke, Schlichting, Venugopalan, NPA931 (2014) 348 for initial spectrum from Epelbaum, Gelis, PRD88 (2013) 085015. **Plasma instabilities from wide range of initial conditions:**

Mrowczynski; Rebhan, Romatschke, Strickland; Arnold, Moore, Yaffe; Bödecker; Attems, … Romatschke, Venugopalan; J.B., Scheffler, Schlichting, Sexty; Fukushima, Gelis …
Overoccupied non-Abelian plasma

- To discuss attractor: Initial overoccupation described by family of distributions at $\tau_0$ (read-out in Coulomb gauge)

$$f(p_T, p_z, \tau_0) = \frac{n_0}{2g^2} \Theta \left( Q_s - \sqrt{p_T^2 + (\xi_0 p_z)^2} \right)$$

- occupancy parameter

- anisotropy parameter
  (controls “prolateness” or “oblateness” of initial momentum distribution)

Nonthermal fixed point

Evolution in the `anisotropy-occupancy plane´

`Bottom-up´* scaling emerges as a consequence of the fixed point!

*Baier et al, PLB 502 (2001) 51
Self-similar evolution

Scaling exponents: \( \alpha = -2/3 \), \( \beta = 0 \), \( \gamma = 1/3 \)

and scaling distribution function \( f_S \):

\[
f(p_T, p_z, \tau) = (Q\tau)^\alpha f_S\left((Q\tau)^\beta p_T, (Q\tau)^\gamma p_z\right)
\]

stationary fixed-point distribution
Comparing gauge and scalar field theories with longitudinal expansion

- For gauge & scalar fields: Inertial range of *thermal-like transverse spectrum* ~$1/p_T$ even as longitudinal distribution is being `squeezed’

- Strongly enhanced infrared regime for scalars: *inverse particle cascade leading to Bose condensation*, ~$1/p^5$ as in isotropic *superfluid turbulence*

- At latest available times for scalars a flat distribution for $p_T \gtrsim Q$ emerges

Universality far from equilibrium

- Same *universal exponents and scaling function* in $1/p_T$ inertial range

\[ \alpha = -\frac{2}{3} \quad , \quad \beta = 0 \quad , \quad \gamma = \frac{1}{3} \]

\[ t^{-\alpha} f(t, p_T = Q/2, p_z) \quad \text{universal scaling function:} \]

\[ f_g \quad \text{gluon distribution:} \]

\[ f_\phi \quad \text{scalar } \lambda \phi^4: \]

\[ t^\gamma p_z \quad \text{rescaled momentum:} \]

→ Remarkably large universality class far from equilibrium!
Some puzzles and challenges

- The scaling solution seems well understood for the gauge theory (BMSS) – but the corresponding kinetic theory arguments fail for the scalar theory

Consider the Boltzmann equation

$$
\left[ \partial_\tau - \frac{p_z}{\tau} \partial_{p_z} \right] f(p_T, p_Z, \tau) = C[f](p_T, p_Z, \tau)
$$

with a self-similar evolution

$$
f(p_T, p_Z, \tau) = (Q\tau)^\alpha f_S((Q\tau)^\beta p_T, (Q\tau)^\gamma p_Z)
$$

→ **Non-thermal fixed point solution** \( (f \gg 1) \)

$$
\left[ \alpha + \beta p_T \partial_{p_T} + (\gamma - 1) p_Z \partial_{p_Z} \right] f_S(p_T, p_Z) = Q^{-1} C[f_S](p_T, p_Z)
$$

→ **Scaling exponents determined by scaling relations for**

- Small angle elastic scattering \( (2\alpha - 2\beta + \gamma = -1) \) for gauge theory
- Energy conservation \( (\alpha - 3\beta - \gamma = -1) \)
- Particle number conservation \( (\alpha - 2\beta - \gamma = -1) \)

→ \( \alpha = -2/3, \beta = 0, \gamma = 1/3 \) in excellent agreement with lattice data!

**However:** No dominance for small angle scattering in scalar theory!

**Much more general principle underlying nonthermal fixed point?**
In the weak-coupling limit, kinetic theory is expected to have an overlapping range of validity with classical-statistical simulations:

\[ f(p) \sim 1/g^2 \quad (\text{\textquoteleft overoccupied\textquoteright}) \]

\[ 1/g^2 > f(p) > 1 \quad (\text{classical particle}) \]

\[ f(p) < 1 \quad (\text{quantum}) \]

**classical-statistical lattice gauge theory**

**kinetic theory**

However, kinetic theory cannot reproduce important quantities such as \( P_L/P_T \) characterizing isotropization of the longitudinally expanding plasma:

- Scaling behavior of \( P_L/P_T \) the same for scalar and gauge field simulations!

- In scalar theory behavior of \( P_L/P_T \) known to arise from infrared contributions (Bose condensation)

\[ \rightarrow \text{Nonperturbative despite (weak) coupling parameter} \]
- Perturbative estimates extrapolated beyond the weak-coupling regime suggest the absence of transient universal scaling behavior


- By now, detailed understanding of the dynamics of Bose condensation in scalar quantum field theory (NLO-1/N)/vertex-resummed kinetic equation

However, no such understanding in gauge theories – despite indications for infrared contributions to gauge invariant quantities \((\mathcal{P}_L / \mathcal{P}_T)\)

\(\rightarrow\) Holography / functional renormalization group / solving QFTs by ultracold quantum gas measurements / …
Universality far from equilibrium

- **Nonthermal fixed point**: $f(t,p) \sim e^{\lambda t} \rightarrow$ overoccupation

- **Extremal conditions, e.g. large fields**: $f(t,p) \sim e^{\lambda t} \rightarrow$ overoccupation

- **Nonthermal fixed point**: $f(t,p) \sim e^{\lambda t} \rightarrow$ overoccupation

- **Self-similar evolution/condensation**: $f(t,p) \sim t^{\alpha} f_S(t^\beta p)$

- **Initial conditions**

- **Far from equilibrium**

- **Close to equilibrium**

- **Universal relaxation**: $f(t,p) - f_T(t) \sim e^{-\gamma t}$

- **Thermal equilibrium**