Scale Invariance and Quantum Hydrodynamics in Expanding Strongly Interacting Fermi Gases

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Outline

• Introduction: Optically trapped Fermi gases:
  – Creating a strongly interacting Fermi gas
  – Universal energy and entropy, Quantum viscosity, KSS conjecture

• Scale Invariance in Expanding Fermi gases:
  – Defining and observing scale invariant expansion:
    “Ballistic” flow of a Hydrodynamic gas
  – Observation of conformal symmetry breaking
  – Vanishing Bulk viscosity

• Searching for Perfect fluids
  – Measuring Shear viscosity on and off resonance
  – Comparison with the KSS bound

• Future Prospects
Why Study Strongly Interacting Fermi Gases?

Strongly Interacting Fermionic Systems

- Neutron Star
- Quark Gluon Plasma
- Ultra-Cold Fermi Gas

High Temperature Superconductors
Creating a *Scale-Invariant* Strongly-Interacting Fermi gas
Creating a *Scale-Invariant* Strongly-Interacting $^6\text{Li}$ Fermi gas

Atoms precooled in a magneto-optical trap to 150 $\mu$K

$U_0/k_B = 700$ $\mu$K

$2$ MW/cm$^2$
Experimental Apparatus
Optically Trapped Fermi Gas

Our atom: Fermionic

\[^6\text{Li}\]

Magnet coils

\[|\uparrow\rangle = -\frac{1}{2}, \downarrow\rangle = -\frac{1}{2}, 0\rangle\]

|electron \( S_z \), nuclear \( I_z \)\rangle
Feshbach Resonance

Resonant Coupling between Colliding Atom Pair – Bound Molecular State

Singlet Diatomic Potential

\[ ^1\Sigma_g \]

\[ |v_1 = 38\rangle \]

SCALE INVARIANT!

Triplet Diatomic Potential

\[ ^3\Sigma_g \]

\[ ^3\Sigma_u \]

\[ ^1\Sigma_u \]

\[ ^3\Pi \]

832 G-Resonant Scattering!

\[ \sigma_{coll} = 4\pi\hbar^2 dB \]
Tunable Strong Interactions

Scattering Length

527.5 G
Zero Crossing (Ideal gas)

832.2 G
Resonance

\[ a[a_0] \]

\[ B[G] \]
**Strong Interactions:**

*Shock waves* in Fermi gases

- Trapped gas is divided into two clouds with a repulsive optical potential.
- The repulsive potential is *extinguished*, the two clouds accelerate towards each other and collide.

Really strong interactions!
**Universal Regime:** For resonant scattering, the scattering cross section is the square of the de Broglie wavelength, which is independent of the details of the collisional interactions!

Atom spacing $L$ becomes the only length scale.

**Heisenberg Uncertainty Principle:** $\Delta x \Delta p \approx \hbar$ \[ \Delta p \approx \frac{p}{L} \approx \frac{\hbar}{L} \]

Physical Properties, like Energy and Temperature have *Natural Units* determined by $L$.

Fermi Energy: $\frac{\hbar^2}{2mL^2}$
When the interparticle spacing sets the scale of energy and temperature, the pressure $p$ is a function only of density $n$ and temperature $T$:

$$p(n,T)$$

Using elementary thermodynamics, one then can show that

$$p = \frac{2}{3} \varepsilon$$

$\varepsilon = \text{energy density}$

(Ho, 2004)

This elementary result has several amazing consequences.
**Universal Gas obeys the Virial Theorem**

\[ E = \langle U \rangle + \frac{1}{2} \langle \mathbf{r} \cdot \nabla U \rangle \]

In a HO potential:  \[ E = 2 \langle U \rangle \]

Energy per particle \[ E = 3m\omega_z^2 \langle z^2 \rangle \]

For a *universal* quantum gas, the energy \( E \) is determined by the *cloud size*.
Measuring the Energy $E$ and Entropy $S$

For a *universal* quantum gas, the energy $E$ is determined by the *cloud size*.

For a *weakly interacting* quantum gas, the entropy $S$ can always be determined from the *cloud size* (textbook problem).

**Experiment**

- **Start:** 832 G
- **Universal**
- **Strongly interacting**
- **Sweep** magnetic field
- **End:** 1200 G
- **Weakly interacting**
Is the B-Field Sweep Adiabatic?

![Graph showing Heating Rate vs. Magnetic Field](image1)

![Table showing measured values](image2)

<table>
<thead>
<tr>
<th>$B_0$</th>
<th>$B_1$</th>
<th>$\Delta E$</th>
<th>$\int Q , dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>832 G</td>
<td>770 G</td>
<td>0.081(8)</td>
<td>.077</td>
</tr>
<tr>
<td>832 G</td>
<td>800 G</td>
<td>0.053(6)</td>
<td>.054</td>
</tr>
<tr>
<td>832 G</td>
<td>900 G</td>
<td>0.028(3)</td>
<td>.029</td>
</tr>
</tbody>
</table>

![Graph showing Magnetic Field and Heating Rate](image3)
Energy per particle versus Entropy per Particle

Red circles: Measured
JETLab, JLTP 2009

Solid line—from measured equation of state: Ku et al., Science, 2012
Perfect Fluidity—Viscosity

Quark-Gluon Plasma: $T = 10^{12}$ K

Both Exhibit Elliptic Flow!

Computer simulation of RHIC collision
BIG BANG

Ultra-cold Atomic Fermi gas: $T = 10^{-7}$ K
Universal Regime: Viscosity Scale

Shear forces:

\[ F = \eta \frac{A}{d} \mathbf{v} \]

Viscosity scale:

\[ \eta \approx \frac{p}{A} \]

\[ p \approx \frac{\hbar}{L} \]

\[ A \approx L^2 \]

\[ \eta \approx \frac{\hbar}{L^3} = \hbar n \]

\[ n = \text{density} \]

\[ \eta \approx \frac{\hbar}{\lambda_T^3} \propto T^{3/2} \]

High Temperature Limit

Quantum scale—requires Planck’s constant!
Quantum Viscosity

\[ \eta = \alpha_S \hbar n \]

\( n = \) density (particles/cc)

**Viscosity:**

- **Water:** \( n = 3.3 \times 10^{22} \) \( \eta = 300 \hbar n \)
- **Air:** \( n = 2.7 \times 10^{19} \) \( \eta = 6000 \hbar n \)
- **Fermi gas:** \( n = 3.0 \times 10^{13} \) \( \eta = 0.5 \hbar n \)
- **Nuclear Matter:** \( n = 3.0 \times 10^{38} \) \( \eta = ? \hbar n \)

Dimensionless shear viscosity coefficient
The Minimum Viscosity Conjecture

Resistance to flow—hydrodynamic properties

\[
\frac{\text{viscosity}}{\text{entropy}} \propto \frac{\text{surface area black hole}}{\text{surface area black hole}} \geq \frac{1}{4\pi k_B} \frac{\hbar}{M}
\]

Disorder—thermodynamic properties

Minimum defines a Perfect Fluid
Minimum Viscosity Conjecture

Experimentalist’s Approach!

Viscosity $\hbar n$ — Hydrodynamics

$$\eta \approx \frac{\hbar}{s} \frac{k_B}{n}$$

Entropy density $k_B n$ — Thermodynamics

Density cancels!

In a $^6\text{Li}$ gas we can measure $\eta$ and $s$.

Is the Expansion Scale-Invariant?
Scale Invariance in Expanding Resonant Fermi Gases

\[ \sigma_{\text{coll}} = 4\pi \hat{\lambda}^2_{dB} \]

Compressed “Balloons”

Expanded “Balloons”
Measuring the cloud in 3D

- Measure *all three* cloud radii using two cameras.
Aspect Ratio versus Expansion Time

Energy Dependent!

Average Shear Viscosity
only Fit Parameter

\[ \frac{\sigma_x}{\sigma_y} = \frac{\omega_y}{\omega_x} \frac{b_x(t)}{b_y(t)} \]
Scale-Invariance: Connecting Strongly to Weakly Interacting

- Anti-de Sitter-Conformal Field Theory Correspondence: Connects strongly interacting fields in 4-dimensions to weakly interacting gravity in 5-dimensions: Perfect fluid conjecture

- Can we connect elliptic flow of a resonant gas to the ballistic flow of an ideal gas in 3D?

For both, the pressure is 2/3 of the energy density:

\[ \Delta p \equiv p - \frac{2}{3} \varepsilon = 0 \]

Scale Invariant?

Elliptic Flow: Observe 2 dimensions + time
Scale Invariance: Ideal Gas

**Ideal gas:** \( \mathbf{r} = \mathbf{r}_0 + \mathbf{v}t \)

**Ballistic flow**

How does the *mean square radius* evolve in time? \( \langle r^2 \rangle = \langle x^2 + y^2 + z^2 \rangle \)

**Virial Theorem:**

\[
m \langle v^2 \rangle_0 = \langle \mathbf{r} \cdot \nabla U \rangle_0
\]

**Cloud average**

\[
\langle r^2 \rangle = \langle r^2 \rangle_0 + t^2 \langle v^2 \rangle_0
\]

**Virial Theorem:**

\[
\langle r^2 \rangle = \langle r^2 \rangle_0 + \frac{t^2}{m} \langle \mathbf{r} \cdot \nabla U \rangle_0
\]

**Ballistic Flow**
Defining Scale Invariant Flow

\[ \langle r^2 \rangle = \langle x^2 + y^2 + z^2 \rangle \]

Cloud average

\[ m \left( \langle r^2 \rangle - \langle r^2 \rangle_0 \right) = t^2 \]

\[ \langle r \cdot \nabla U \rangle_0 \]

Defines Scale Invariant Flow!

\( t = \) expansion time
Hydrodynamic gas: Elliptic flow

How does the *mean square radius* evolve in time? \( \langle r^2 \rangle = \langle x^2 + y^2 + z^2 \rangle \)
Using the hydrodynamic equations and energy conservation it is easy to show that

\[
\frac{d^2}{dt^2} \frac{m\langle r^2 \rangle}{2} = \langle r \cdot \nabla U \rangle_0 + \frac{3}{N} \int d^3r \left( \Delta p - \Delta p_0 \right) - \frac{3}{N} \int d^3r \zeta_B \nabla \cdot \mathbf{v}
\]

Initial trap potential

\[ \Delta p \equiv p - \frac{2}{3} \varepsilon \]

Conformal symmetry breaking \( \Delta p \)

Bulk viscosity
Scale Invariance!

Resonant gas  \( \Delta p \equiv p - \frac{2}{3} \varepsilon = 0 \)

The bulk viscosity is predicted to vanish so

\[
\langle r^2 \rangle = \langle r^2 \rangle_0 + \frac{t^2}{m} \langle r \cdot \nabla U \rangle_0
\]

Can we observe \textit{ballistic} flow of an \textit{elliptically} expanding gas?
Scale-invariant “Ballistic” Expansion of a Resonant Fermi gas

The aspect ratio exhibits elliptic flow, but the mean-square cloud radius expands ballistically: SCALE-ININVARIANT!

\[ \tau^2(t) = \frac{m \left( \langle r^2 \rangle - \langle r^2 \rangle_0 \right)}{\langle r \cdot \nabla U \rangle_0} \]

Elliott, Joseph, JET PRL 112, 040405 (2014)
String theory has sometimes been characterized as an elegant scale-invariant theory of everything with one minor defect: It Predicts Nothing!

JET:

“Now we have an experiment that Measures Nothing to compare to it!”
Conformal Symmetry Breaking

Observing the conformal symmetry breaking pressure:

\[
\frac{d^2}{dt^2} \frac{m \langle r^2 \rangle}{2} = \langle r \cdot \nabla U \rangle_0 + \frac{3}{N} \int d^3r \left( \Delta p - \Delta p_0 \right) - \frac{3}{N} \int d^3r \int \mathcal{B} \nabla \cdot \mathbf{v}
\]

Initial trap potential  \hspace{1cm} Conformal symmetry breaking $\Delta p$  \hspace{1cm} Bulk viscosity

\[
\Delta p \equiv p - \frac{2}{3} \varepsilon
\]
**Pressure Change $\Delta p$**

Dimensional analysis, to leading order in $1/k_F a$, requires

$$p - \frac{2}{3} \varepsilon \equiv \Delta p = n \varepsilon_F(n) \frac{f_p(\theta)}{k_F a}$$

Time dependence:

$$k_F = (3\pi^2 n)^{1/3} \propto \Gamma^{-1/3} \Gamma(t) = \text{volume scale factor}$$

Assuming the temperature drops adiabatically, the reduced temperature $\theta$ is time-independent:

$$\theta \equiv \frac{k_B T}{\varepsilon_F} \propto \frac{T}{k_F^2}$$

$$\frac{1}{N} \int d^3 r \Delta p = \frac{C}{3} \frac{\langle \mathbf{r} \cdot \nabla U \rangle_0}{k_{FI} a} \Gamma^{-1/3}(t)$$

$$\frac{1}{k_{FI} a} = \pm 0.6$$

Changes sign with the scattering length.
Breaking Scale Invariance

\[ \frac{\langle r^2 \rangle - \langle r^2 \rangle_0}{\langle r^2 \rangle_0} \]

\[ \frac{\langle r^2 \rangle - \langle r^2 \rangle_0}{\langle r^2 \rangle_0} \]

C = 0.21

\[ \Delta p > 0 \]
\[ \Delta p < 0 \]

\[ \Delta p = 0 \]

Elliott, Joseph, JET PRL 112, 040405 (2014)
Bulk Viscosity at Resonance

Measuring the bulk viscosity:

\[
\frac{d^2}{dt^2} \frac{m\langle r^2 \rangle}{2} = \langle \mathbf{r} \cdot \nabla U \rangle_0 + \frac{3}{N} \int d^3r (\Delta p - \Delta p_0) - \frac{3}{N} \int d^3r \zeta_B \nabla \cdot \mathbf{v}
\]

Initial trap potential \hspace{1cm} \text{Conformal symmetry breaking } \Delta p = 0 \hspace{1cm} \text{Bulk viscosity}

Evaluate last term in scaling approximation:

\( \Gamma(t) = \text{volume scale factor} \)

\( \nabla \cdot \mathbf{v} = \dot{\Gamma} / \Gamma \)
Bulk Viscosity

Dimensional analysis, to leading order in 1/k_F a, with ζ_B ≥ 0 requires

$$\zeta_B \equiv \hbar n \frac{f_B(\theta)}{(k_F a)^2} \equiv \alpha_B \hbar n$$

Time dependence:

$$k_F = (3\pi^2 n)^{1/3} \propto \Gamma^{-1/3} \quad \Gamma(t) = \text{volume scale factor}$$

Assuming the temperature drops adiabatically, the reduced temperature θ is time-independent:

$$\theta \equiv \frac{k_B T}{\epsilon_F} \propto \frac{T}{k_F^2}$$

$$\frac{1}{N} \int d^3r \zeta_B \equiv \hbar \overline{\alpha}_B(0) \Gamma^{2/3}(t)$$
Bulk Viscosity at Resonance

- Shear viscosity: $\eta_S = \alpha_S \hbar n$
- Bulk viscosity: $\zeta_B = \alpha_B \hbar n$

Elliott, Joseph, JET
PRL 112, 040405 (2014)

The bulk viscosity vanishes! $\overline{\alpha}_B(0) < 0.04$
Measuring the Shear Viscosity

From the Navier-Stokes and continuity equations, it is easy to show that a single component fluid obeys:

\[ \frac{d^2}{dt^2} \frac{m\langle x_i^2 \rangle}{2} = m\langle v_i^2 \rangle + \frac{1}{N} \int d^3 r \, p - \langle x_i \partial_i U \rangle - \hbar \langle \alpha_S \sigma_{ii} + \alpha_B \nabla \cdot \mathbf{v} \rangle \]

**Pressure**

\[ \frac{3}{N} \int d^3 r \, p_0 = \langle \mathbf{r} \cdot \nabla U \rangle_0 \equiv \tilde{E} \]

**Equilibrium**

Measured from the cloud profile and trap parameters

Need to find the **time-dependent** volume integral of the pressure:
Energy Conservation

For a temporally constant potential energy $U$, the internal energy change during expansion is:

$$dE_{\text{int}} = dQ - pdV$$

$$\dot{E}_{\text{int}} = \dot{Q} - p\dot{V}$$

The local volume dilates at a rate:

$$\dot{V} = d^3r \nabla \cdot v$$

$$E_{\text{int}} = \int d^3r \varepsilon$$

energy density

$$\frac{d}{dt} \int d^3r \varepsilon = \dot{Q} - \int d^3r (\nabla \cdot v)p$$

$$\frac{2}{3} \varepsilon = p - \Delta p$$

$\Delta p = 0$ for resonantly interacting gas

Easy to solve in scaling approximation:

$$\nabla \cdot v = \dot{\Gamma} / \Gamma$$

$\Gamma(t) =$ volume scale factor
Scaling Approximation

\[ n(x, y, z, t) = \frac{n_0(x/b_x, y/b_y, z/b_z)}{\Gamma} \]

\[ \Gamma = b_x b_y b_z \]

Volume scale factor

Velocity field is linear in the spatial coordinates

\[ v_i = x_i \frac{\dot{b}_i}{b_i} \]

Velocity field is linear in the spatial coordinates

\[ \langle x_i^2 \rangle = \langle x_i^2 \rangle_0 b_i^2(t) \]

\[ \frac{\overline{\omega_i^2}}{\sigma_{ii}} = \frac{\langle r \cdot \nabla U \rangle_0}{3m \langle x_i^2 \rangle_0} \]

\[ \sigma_{ii} = 2 \frac{\dot{b}_i}{b_i} - 2 \frac{\dot{\Gamma}}{3 \Gamma} \]

Cloud-averaged shear viscosity coefficient

Pressure correction factors

\[ \ddot{\dot{b}}_i = \frac{\overline{\omega_i^2}}{\Gamma^{2/3}} \frac{b_i}{b_i} \left[ 1 + C_Q(t) + C_{\Delta p}(t) \right] - \frac{\hbar \alpha_s \sigma_{ii}}{m \langle x_i^2 \rangle_0 b_i} - \omega_{im\alpha}^2 b_i \]
Pressure Correction Factors

Velocity field is linear in the spatial coordinates:

\[
\mathbf{v}_i = x_i \frac{\dot{b}_i}{b_i} \quad \nabla \cdot \mathbf{v} = \frac{\dot{\Gamma}}{\Gamma}
\]

Heating rate per particle:

\[
\dot{\dot{Q}} \frac{N}{\dot{h}} = \frac{h}{2} \left( \bar{\alpha}_S \sum_i \sigma_{ii}^2 + 2 \bar{\alpha}_B (\nabla \cdot \mathbf{v})^2 \right)
\]

Conformal symmetry breaking pressure:

\[
\frac{1}{N} \int d^3 \mathbf{r} \Delta p = \frac{C}{3} \frac{\langle \mathbf{r} \cdot \nabla U \rangle_0}{k_{FI} a} \Gamma^{-1/3} (t)
\]

Heating correction factor:

\[
\dot{C}_Q (t) = \frac{2 \dot{Q}(t)}{N} \frac{\Gamma^{2/3} (t)}{\langle \mathbf{r} \cdot \nabla U \rangle_0}
\]

Pressure correction factor:

\[
C_{\Delta p} (t) = -\frac{C}{k_{FI} a} \left( \Gamma^{1/3} (t) - 1 \right)
\]
Cloud-Averaged Viscosity

Shear Viscosity: \( \eta_S \equiv \alpha_S \bar{\eta} n \)

Cloud-averaged shear viscosity coefficient:

\[
\bar{\alpha}_S \equiv \frac{1}{N\bar{\eta}} \int d^3 r \, \eta_S(r) = \int d^3 r \, \frac{n(r)}{N} \alpha_S[\theta(r)] = \langle \alpha_S \rangle_0
\]

*Temporally constant in the adiabatic approximation = Trap average.

Volume integrated KSS bound:

\[
\bar{\alpha}_S \equiv \frac{1}{N\bar{\eta}} \int d^3 r \, \eta_S(r) \geq \frac{1}{N} \int d^3 r \, \frac{s(r)}{4\pi k_B} = \frac{1}{4\pi} \frac{S / N}{k_B}
\]
Shear Viscosity at Resonance versus Reduced Temperature

Transition to Superfluid

*EoS from Ku et al., Science, 2012

Reduced temperature at the trap center
Shear Viscosity: Universal Scaling

\[ \eta = \alpha_s \hbar n \]

- Red: For trap that is 50 times deeper
- Blue: New Data

\[ \eta = 0.282 (mk_B T)^{3/2} / \hbar^2 \]

Bluhm, Schaefer, PRL 116, 115301 (2016)

JETLab, Science 331, 58 (2011)
Shear Viscosity/Entropy versus Entropy: Resonance

Transition to Superfluid

Can we do better?

Volume Integrated KSS Limit

\[ \frac{\langle \alpha_S \rangle_0}{S / k_B} \]

\[ \frac{1}{4\pi} \]

\[ S / k_B \]
Shear Viscosity/Entropy versus Entropy: Perfect Fluid?

\[ \langle \alpha_S \rangle_0 \]
\[ \frac{S}{k_B} \]

\[ \frac{1}{k_F a} = 0 \]
\[ \frac{1}{k_F a} = 0.25 \]

KSS limit
Problems with Viscosity Measurement

• We have measured the “Cloud-Averaged” shear viscosity.

• Integration “Volumes” for entropy and viscosity may not be the same.

• What can we say about the “Local” shear viscosity?
  – Inverting cloud-averaged data
  – Local ratio of shear viscosity to entropy density
  – Comparison with predictions
Cloud-Averaged Shear Viscosity versus Reduced Temperature

*EoS from Ku et al., Science, 2012

Superfluid Transition

Reduced temperature at the trap center
Obtaining Local Viscosity from Cloud-Averaged Viscosity Data

\[
\langle \alpha_s \rangle_0 = \frac{1}{N} \int_0^\infty d^3r \alpha_s[\theta(r)]n(r) \rightarrow \infty \quad \text{A problem!}
\]

\[ \propto T^{3/2} \quad \text{as } r \rightarrow \infty \]

Cutoff radius: \[
\langle \alpha_s \rangle_0 = \frac{1}{N} \int_0^{R_C} d^3r \alpha_s[\theta(r)]n(r)
\]

Choose \( R_C \) to agree with high temperature data:

\[
R_C = 0.98 \langle r^2 \rangle^{1/2}
\]

Now we can estimate \( \alpha_s(\theta) \) by image processing methods!
Local Shear Viscosity versus Reduced Temperature

Structure appears at low temperature

High Temperature Limit
\[ \alpha_S = 2.77 \theta^{3/2} \]

Bruun, Smith
Phys. Rev. A 75
043612 (2007)
Cloud-Averaged Shear Viscosity versus Reduced Temperature

Shear viscosity \( (\eta_n) \)

Superfluid Transition

Integrated local viscosity

\[ \langle \alpha_s \rangle \]

Joseph, Elliott, JET
PRL 115, 020401 (2015)

*EoS from Ku et al., Science, 2012

Reduced temperature at the trap center
Local Shear Viscosity
(Comparison to Theory)

Kinetic theory
\[ \alpha_S = 2.77 \theta^{3/2} \]

Enss et al., Annals 2011
(Diagrammatic Kubo formula)

Wlazłowski et al., PRL 2012 (Monte Carlo)

Guo et al., PRL 2011

"Measured"
Ratio of the Local Shear Viscosity to the Entropy Density

*EoS from Ku et al., Science, 2012*
Summary: Image Processing

**Quote** on extrapolating QMC data in a recent viscosity theory paper:

“As a result the integral equation [3] belongs to a class of numerically ill-posed problems. Therefore, the use of special techniques is warranted in order to extract numerically stable results.”

**JET:**

“Now we have numerically ill-posed measurements to compare to numerically ill-posed predictions!”
Summary

• Testing “string” theory
  – Scale invariant hydrodynamics and thermodynamics

• Scale invariance in expanding Fermi gases:
  – “Ballistic” flow of resonant, hydrodynamic gas
  – Bulk viscosity very small compared to shear viscosity

• Perfect fluidity and shear viscosity:
  – Need for direct measurement of local shear viscosity
  – Need for non-relativistic conformal field theory or a trapped “relativistic” gas
1D Flow in a Rectangular Pipe

Use a micro-mirror array to create a *four sheet* repulsive optical potential:

Camera
December 2010 Celebration!

Elliptic Flow Birthday Cake!
**Bulk Viscosity** $\zeta_B$

*Near unitarity, the bulk viscosity generally takes the form:*

\[
\zeta_B = \frac{f_B(\theta)}{(k_F a)^2} \hbar n \equiv \alpha_B \hbar n
\]

$\theta = $ reduced temperature,

$a = $ s-wave scattering length,

$k_F = $ local Fermi wave vector

The trap-average gives:

\[
\bar{\alpha}_B(t) = \bar{\alpha}_B(0) \Gamma^{2/3}(t)
\]

**High T:** Dusling and Schaefer point out that the bulk viscosity must be second order in the fugacity $z \approx n\lambda_T^3 / 2$

\[
\zeta_B = \frac{1}{24\pi\sqrt{2}} \frac{\lambda_T^2}{a^2 \lambda_T^3} \hbar z^2
\]

\[
\overline{\alpha}_B(0) = \frac{9}{32} \frac{1}{(k_F a)^2} \left( \frac{E_F}{E} \right)^4 \equiv c_B \left( \frac{E_F}{E} \right)^4
\]