

# Non-Equilibrium Energy Flow: From Shocks to Rarefaction Waves

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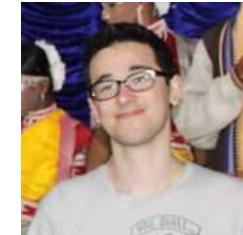
“Workshop on Non-Equilibrium Physics and Holography”

St John's, Oxford

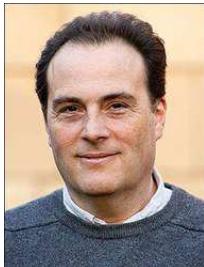
15<sup>th</sup> July 2016

# Strings, Cosmology & Condensed Matter

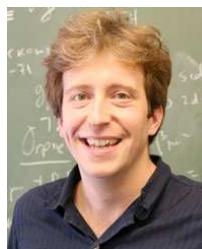
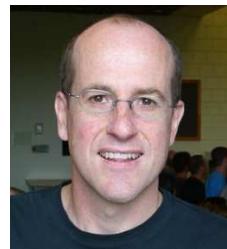
Benjamin Doyon   Koenraad Schalm   Andy Lucas



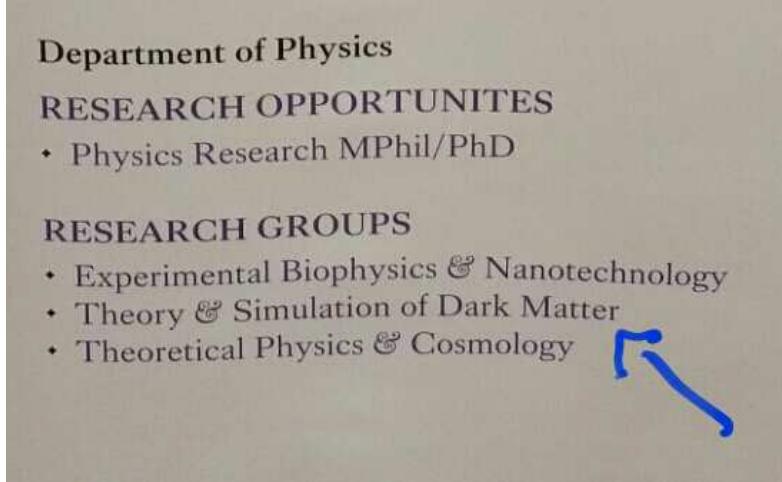
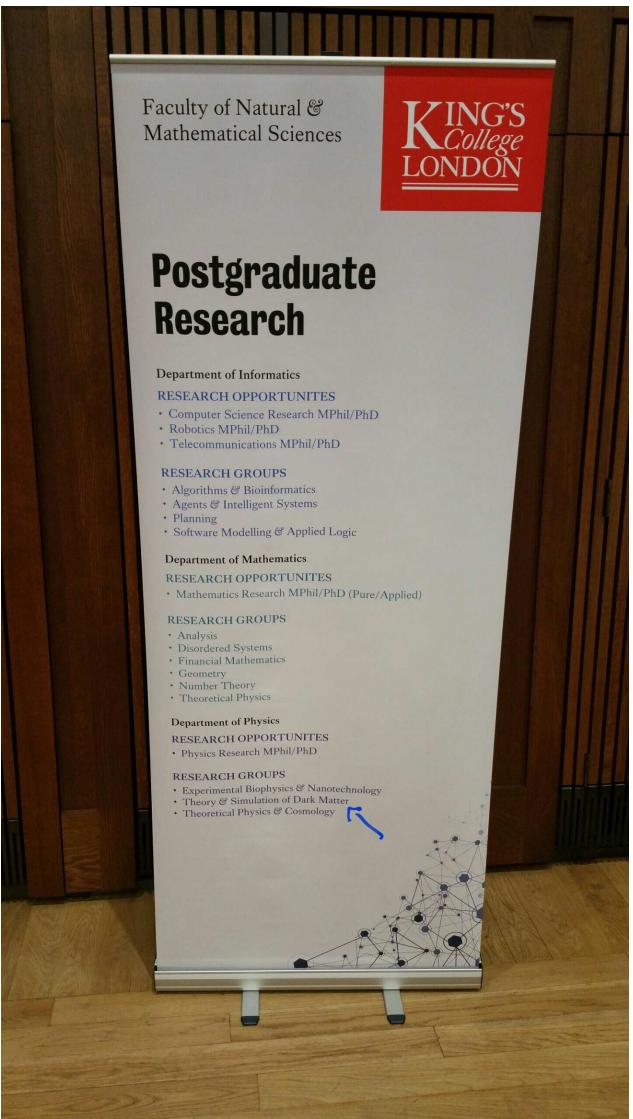
Ben Simons   Julian Sonner



Jerome Gauntlett   Toby Wiseman



King's, Leiden, Harvard, Cambridge, Imperial



# Outline

- Motivation from condensed matter
- Gauge-gravity duality
- Far from equilibrium dynamics
- Energy flow and Non-Equilibrium Steady States (NESS)
- Einstein equations, hydrodynamics, transport
- Current status and future developments

MJB, Benjamin Doyon, Andrew Lucas, Koenraad Schalm  
“*Energy flow in quantum critical systems far from equilibrium*”  
Nature Physics **11**, 509 (2015)

Andrew Lucas, Koenraad Schalm, Benjamin Doyon, MJB  
“*Shock waves, rarefaction waves and non-equilibrium steady states  
in quantum critical systems*”; Phys. Rev. D **94**, 025004 (2016)

# Progress in AdS/CMT

## Transport Coefficients

Viscosity, Conductivity, Hydrodynamics, Bose–Hubbard, Graphene

## Strange Metals

Non-Fermi liquids, instabilities, cuprates

## Holographic Duals

Superfluids, Fermi liquid,  $O(N)$ , Luttinger liquid

## Quantum Information

Entanglement entropy

## Non-Equilibrium

Quenches, Steady States, Turbulence, Thermalization

# Utility of Gauge-Gravity Duality

Quantum dynamics

Classical Einstein equations

Finite temperature

Black holes

**Real time** approach to finite temperature quantum dynamics in interacting systems, with the possibility of anchoring to  $1+1$  and generalizing to higher dimensions

Non-Equilibrium

Beyond linear response

Organizing principles out of equilibrium

# Quantum Quenches

## Simple protocol

Parameter in  $H$  abruptly changed

$$H(g) \rightarrow H(g')$$

System prepared in state  $|\Psi_g\rangle$  but time evolves under  $H(g')$

## Quantum quench to a CFT

Calabrese & Cardy, PRL **96**, 136801 (2006)

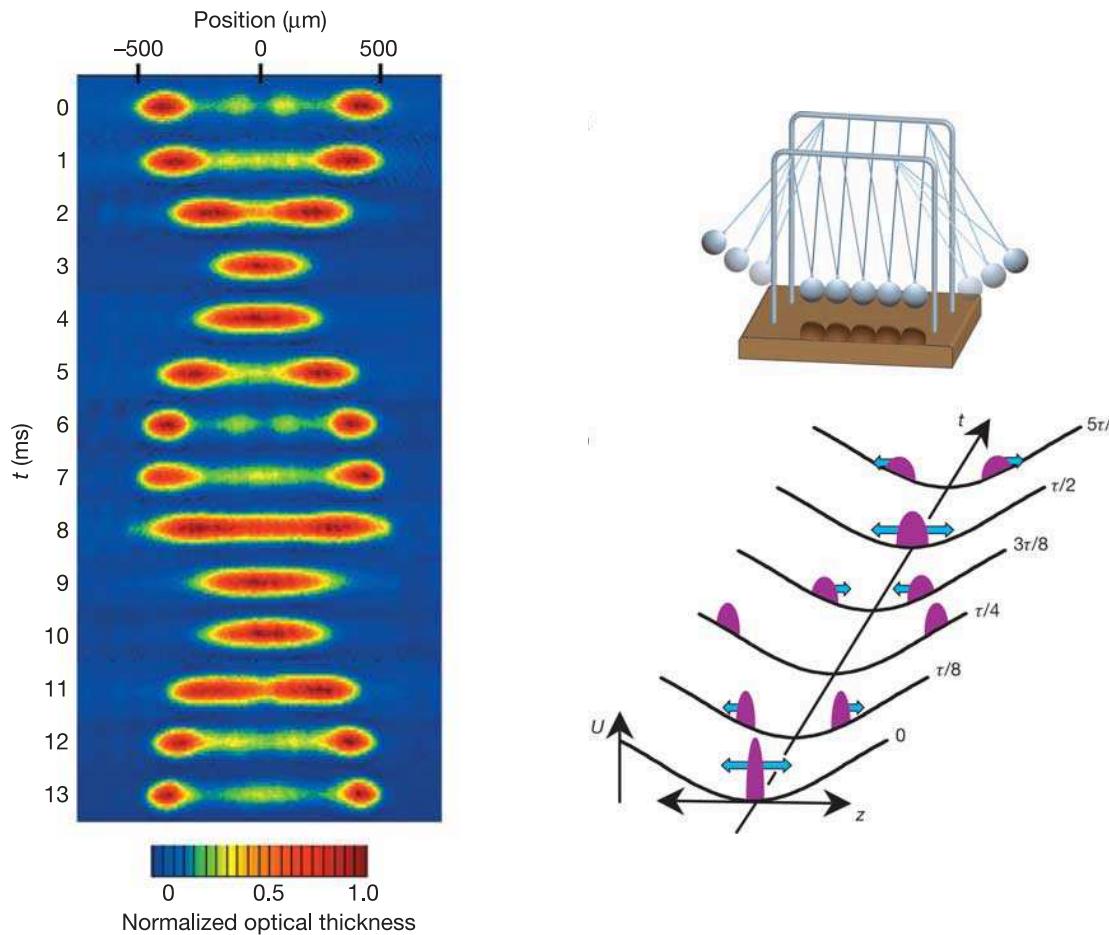
Spin chains, BCS, AdS/CFT ...

Thermalization

# Experiment

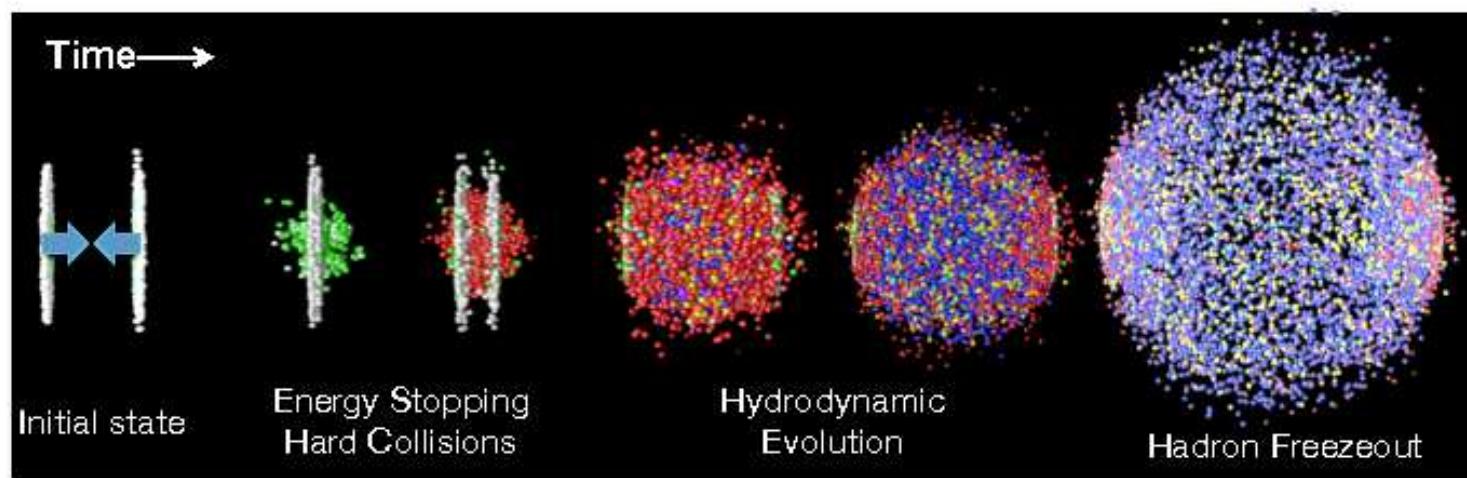
Weiss *et al* “A quantum Newton’s cradle”, Nature 440, 900 (2006)

## Non-Equilibrium 1D Bose Gas



Integrability and Conservation Laws

# Heavy Ion Collisions



Heavy Ions: Results from the Large Hadron Collider, arXiv:1201.4264

# Non-Equilibrium High Energy Physics

## Thermalization in Strongly Coupled Gauge Theories

Chesler & Yaffe (2009), de Boer & Keski Vakkuri (2011),

Buchel, Lehner, & Myers (2012),

Craps, Lindgren, Taliotis, Vanhoof, & Zhang (2014) ...

## Quantum Quenches

Aparício & López (2011), Albash & Johnson (2011), Basu & Das (2012),

Das, Galante, & Myers (2014) ...

## Hydrodynamics

Minwalla, Bhattacharyya, Hubeny, Kovtun, Rangamani ...

# Non-Equilibrium AdS/CMT

## Current Noise

Sonner and Green, “*Hawking Radiation and Nonequilibrium Quantum Critical Current Noise*”, PRL **109**, 091601 (2013)

## Hawking Radiation

## Quenches in Holographic Superfluids

MJB, Gauntlett, Simons, Sonner & Wiseman, “*Holographic Superfluids and the Dynamics of Symmetry Breaking*” PRL (2013)

## Quasi-Normal-Modes

Amado, Kaminski, Landsteiner (09); Murata, Kinoshita, Tanahashi (10);  
Witczak-Krempa, Sørensen, Sachdev (13)

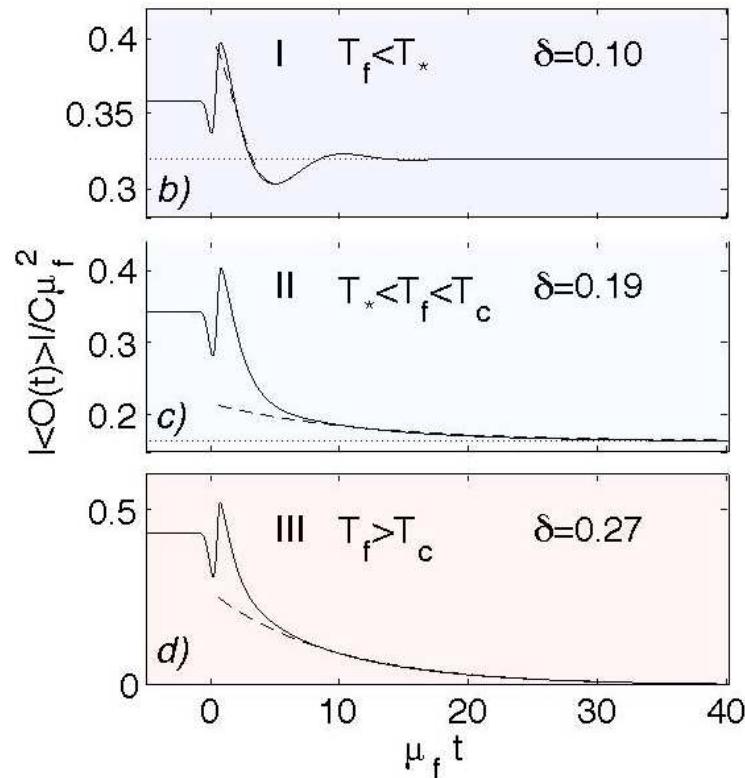
## Superfluid Turbulence

Chesler, Liu and Adams, “*Holographic Vortex Liquids and Superfluid Turbulence*”, Science **341**, 368 (2013)

## Fractal Horizons

# Three Dynamical Regimes

Bhaseen, Gauntlett, Simons, Sonner & Wiseman, “*Holographic Superfluids and the Dynamics of Symmetry Breaking*” PRL (2013)



(I) Damped Oscillatory to SC

$$\langle \mathcal{O}(t) \rangle \sim a + b e^{-kt} \cos[l(t - t_0)]$$

(II) Over Damped to SC

$$\langle \mathcal{O}(t) \rangle \sim a + b e^{-kt}$$

(III) Over Damped to N

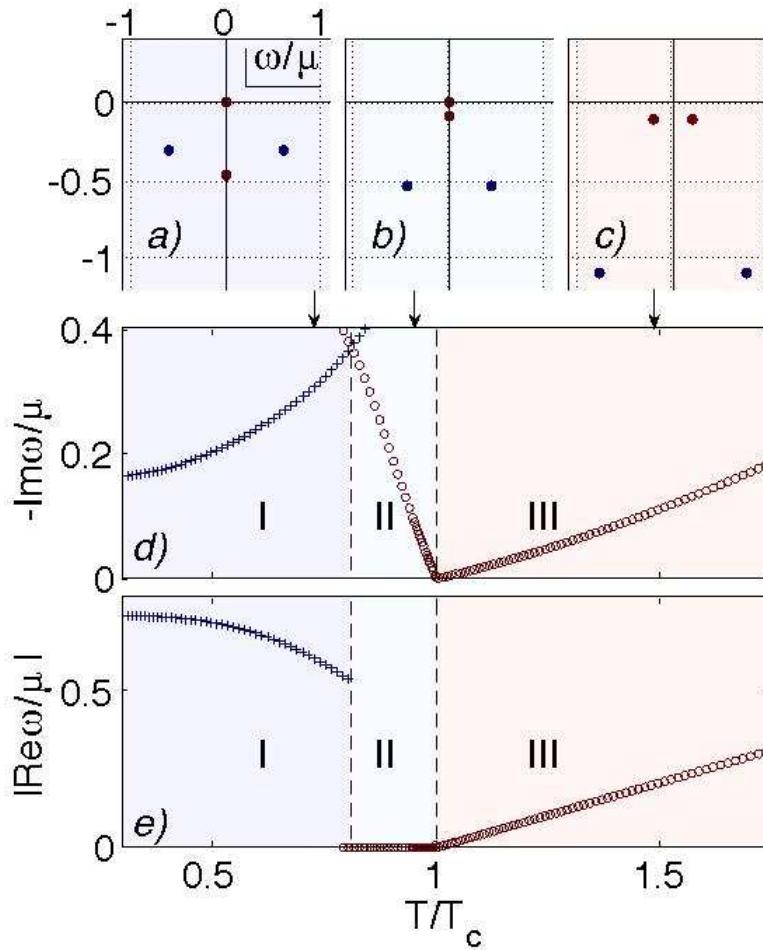
$$\langle \mathcal{O}(t) \rangle \sim b e^{-kt}$$

Asymptotics described by black hole quasi normal modes

Far from equilibrium → close to equilibrium

# Quasi Normal Modes

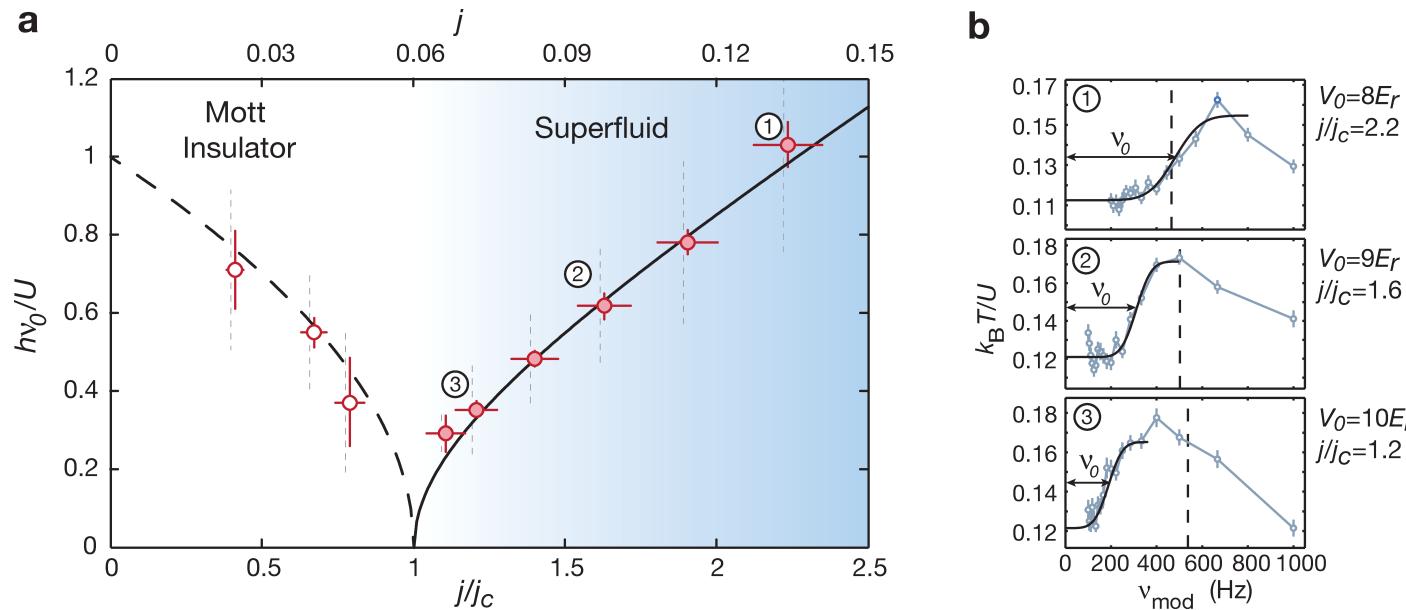
$$|\langle \mathcal{O}(t) \rangle| \simeq |\langle \mathcal{O} \rangle_f + \mathcal{A}e^{-i\omega t}|$$



Three regimes and an emergent  $T_*$

# Cold Atom Experiments

Endres *et al*, “The ‘Higgs’ Amplitude Mode at the Two-Dimensional Superfluid-Mott Insulator Transition”, Nature **487**, 454 (2012)

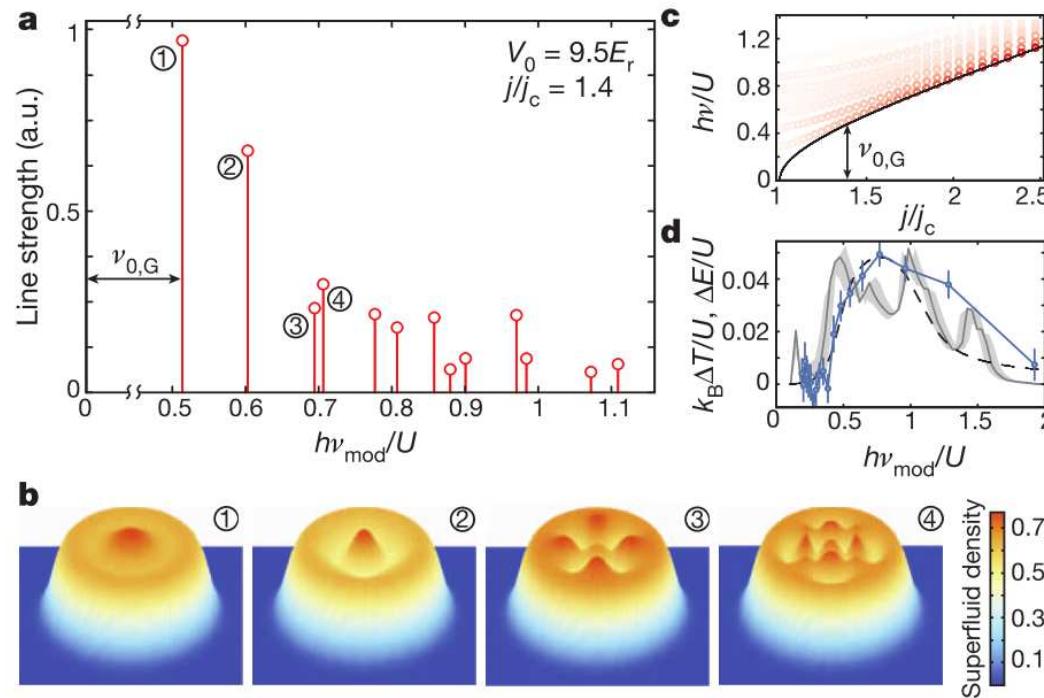


Time-dependent experiments can probe excitations

What is the pole structure in other correlated systems?

# Cold Atom Experiments

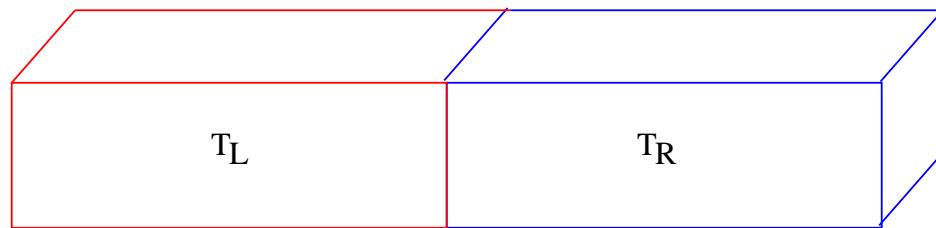
Endres *et al*, “The ‘Higgs’ Amplitude Mode at the Two-Dimensional Superfluid-Mott Insulator Transition”, Nature **487**, 454 (2012)



Excitation spectrum in a trap

# Thermalization

Condensed matter and high energy physics

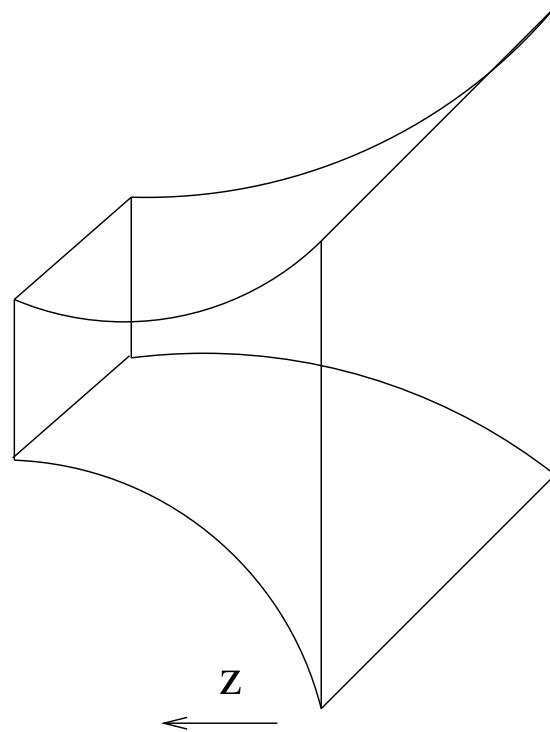


Why not connect two strongly correlated systems together  
and see what happens?

# AdS/CFT

Energy flow may be studied within pure Einstein gravity

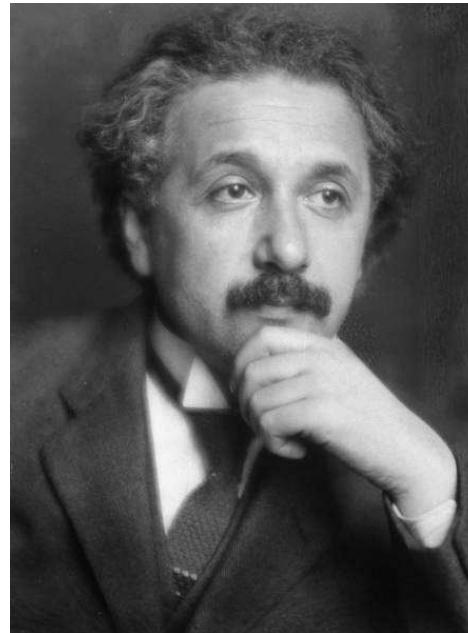
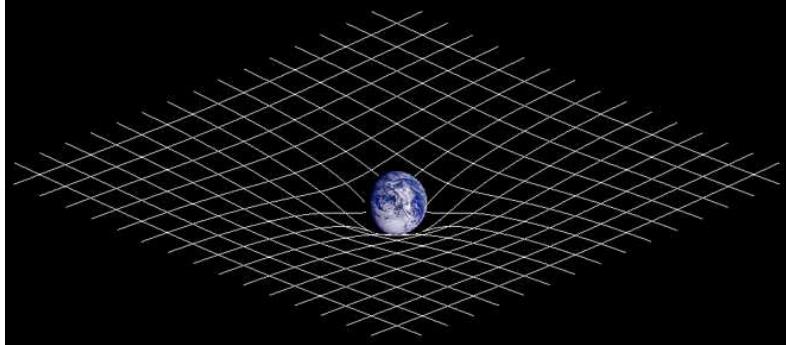
$$S = \frac{1}{16\pi G_N} \int d^{d+2}x \sqrt{-g}(R - 2\Lambda)$$



$$g_{\mu\nu} \leftrightarrow T_{\mu\nu}$$

# Einstein Centenary

## The Field Equations of General Relativity (1915)



$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

[http://en.wikipedia.org/wiki/Einstein\\_field\\_equations](http://en.wikipedia.org/wiki/Einstein_field_equations)

$g_{\mu\nu}$  metric     $R_{\mu\nu}$  Ricci curvature     $R$  scalar curvature

$\Lambda$  cosmological constant     $T_{\mu\nu}$  energy-momentum tensor

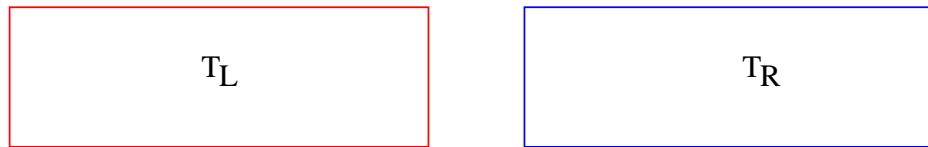
**Coupled Nonlinear PDEs**

**Schwarzchild Solution (1916)     $R_S = \frac{2MG}{c^2}$     Black Holes**

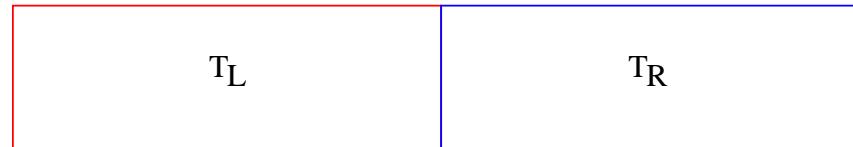
# Non-Equilibrium CFT

Bernard & Doyon, *Energy flow in non-equilibrium conformal field theory*, J. Phys. A: Math. Theor. **45**, 362001 (2012)

Two critical 1D systems (central charge  $c$ )  
at temperatures  $T_L$  &  $T_R$



Join the two systems together



Alternatively, take one critical system and impose a step profile

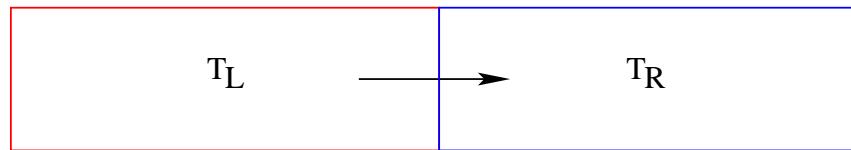
Local Quench

# Steady State Energy Flow

Bernard & Doyon, *Energy flow in non-equilibrium conformal field theory*, J. Phys. A: Math. Theor. **45** 362001 (2012)

If systems are very large ( $L \gg vt$ ) they act like heat baths

For times  $t \ll L/v$  a steady heat current flows



Non-equilibrium steady state

$$J = \frac{c\pi^2 k_B^2}{6h} (T_L^2 - T_R^2)$$

Universal result out of equilibrium

Direct way to measure central charge; velocity doesn't enter

Sotiriadis and Cardy. J. Stat. Mech. (2008) P11003

Stefan–Boltzmann

# Heuristic Interpretation of CFT Result

$$J = \sum_m \int \frac{dk}{2\pi} \hbar\omega_m(k) v_m(k) [n_m(T_L) - n_m(T_R)] \mathbb{T}_m(k)$$

$$v_m(k) = \partial\omega_m/\partial k \quad n_m(T) = \frac{1}{e^{\beta\hbar\omega_m} - 1}$$

$$J = f(T_L) - f(T_R)$$

Consider just a single mode with  $\omega = vk$  and  $\mathbb{T} = 1$

$$f(T) = \int_0^\infty \frac{dk}{2\pi} \frac{\hbar v^2 k}{e^{\beta\hbar v k} - 1} = \frac{k_B^2 T^2}{h} \int_0^\infty dx \frac{x}{e^x - 1} = \frac{k_B^2 T^2}{h} \frac{\pi^2}{6} \quad x \equiv \frac{\hbar v k}{k_B T}$$

**Velocity cancels out**

$$J = \frac{\pi^2 k_B^2}{6h} (T_L^2 - T_R^2)$$

For a 1+1 critical theory with central charge  $c$

$$J = \frac{c\pi^2 k_B^2}{6h} (T_L^2 - T_R^2)$$

# Energy Current Fluctuations

Bernard & Doyon, *Energy flow in non-equilibrium conformal field theory*, J. Phys. A: Math. Theor. **45**, 362001 (2012)

## Generating function for all moments

$$F(z) \equiv \lim_{t \rightarrow \infty} t^{-1} \ln \langle e^{z\Delta_t Q} \rangle$$

## Exact Result

$$F(z) = \frac{c\pi^2}{6h} \left( \frac{z}{\beta_l(\beta_l-z)} - \frac{z}{\beta_r(\beta_r+z)} \right)$$

$$F(z) = \frac{c\pi^2}{6h} \left[ z \left( \frac{1}{\beta_l^2} - \frac{1}{\beta_r^2} \right) + z^2 \left( \frac{1}{\beta_l^3} + \frac{1}{\beta_r^3} \right) + \dots \right]$$

$$\langle J \rangle = \frac{c\pi^2}{6h} k_B^2 (T_L^2 - T_R^2)$$

$$\langle \delta J^2 \rangle \propto \frac{c\pi^2}{6h} k_B^3 (T_L^3 + T_R^3)$$

**Poisson Process**     $\int_0^\infty e^{-\beta\epsilon} (e^{z\epsilon} - 1) d\epsilon = \frac{z}{\beta(\beta-z)}$

# Non-Equilibrium Fluctuation Relations

Bernard & Doyon, *Energy flow in non-equilibrium conformal field theory*,  
J. Phys. A: Math. Theor. **45**, 362001 (2012)

$$F(z) \equiv \lim_{t \rightarrow \infty} t^{-1} \ln \langle e^{z\Delta_t Q} \rangle = \frac{c\pi^2}{6h} \left( \frac{z}{\beta_l(\beta_l - z)} - \frac{z}{\beta_r(\beta_r + z)} \right)$$

$$F(-z) = F(z + \beta_l - \beta_r)$$

Irreversible work fluctuations in isolated driven systems

Crooks relation

$$\frac{P(W)}{\tilde{P}(-W)} = e^{\beta(W - \Delta F)}$$

Jarzynski relation

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$$

Entropy production in non-equilibrium steady states

$$\frac{P(S)}{P(-S)} = e^S$$

Esposito *et al*, “Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems”, RMP **81**, 1665 (2009)

# Linear Response

Bernard & Doyon, *Energy flow in non-equilibrium conformal field theory*, J. Phys. A: Math. Theor. **45**, 362001 (2012)

$$J = \frac{c\pi^2 k_B^2}{6h} (T_L^2 - T_R^2)$$

$$T_L = T + \Delta T/2 \quad T_R = T - \Delta T/2 \quad \Delta T \equiv T_L - T_R$$

$$J = \frac{c\pi^2 k_B^2}{3h} T \Delta T \equiv g \Delta T \quad g = cg_0 \quad g_0 = \frac{\pi^2 k_B^2 T}{3h}$$

## Quantum of Thermal Conductance

$$g_0 = \frac{\pi^2 k_B^2 T}{3h} \approx (9.456 \times 10^{-13} \text{ WK}^{-2}) T$$

## Free Fermions

Fazio, Hekking and Khmelnitskii, PRL **80**, 5611 (1998)

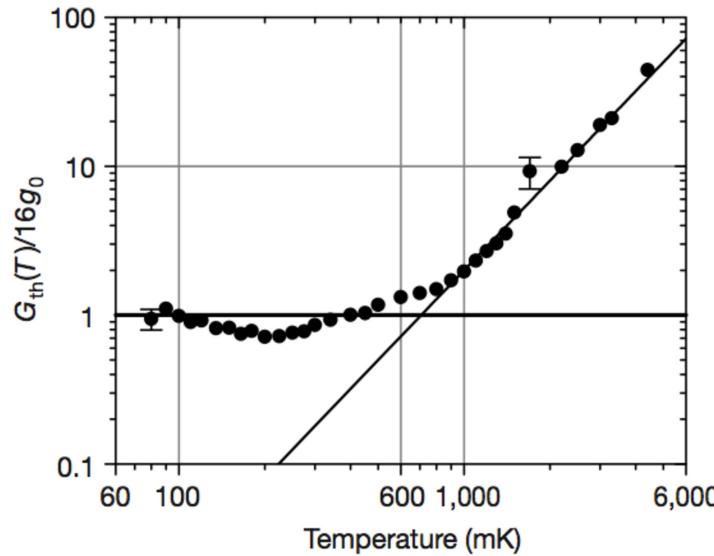
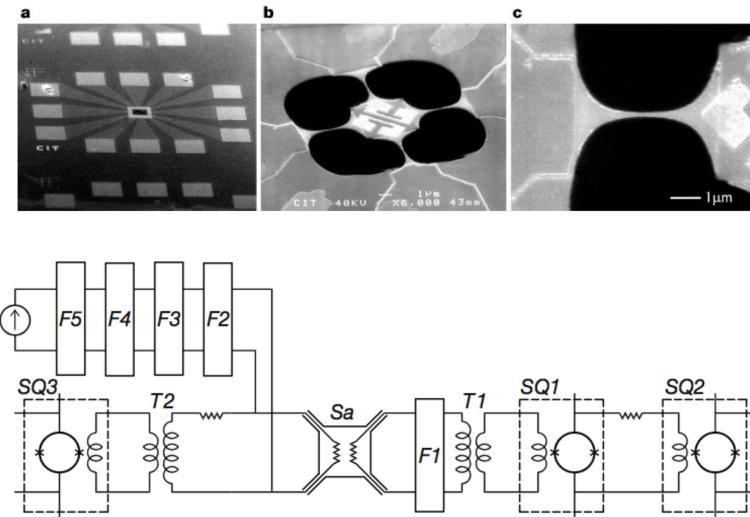
$$\text{Wiedemann-Franz} \quad \frac{\kappa}{\sigma T} = \frac{\pi^2}{3e^2} \quad \sigma_0 = \frac{e^2}{h} \quad \kappa_0 = \frac{\pi^2 k_B^2 T}{3h}$$

## Conformal Anomaly

Cappelli, Huerta and Zemba, Nucl. Phys. B **636**, 568 (2002)

# Experiment

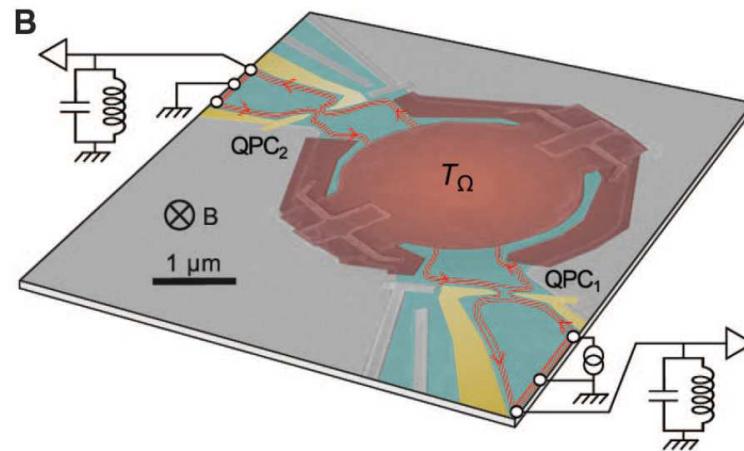
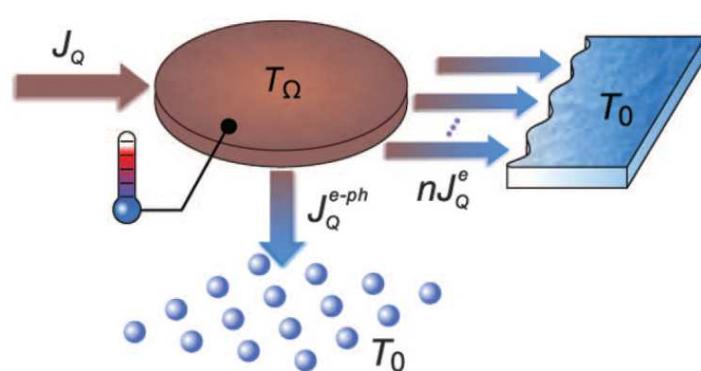
Schwab, Henriksen, Worlock and Roukes, *Measurement of the quantum of thermal conductance*, Nature **404**, 974 (2000)



Quantum of Thermal Conductance

# Experiment

S. Jezouin *et al*, “Quantum Limit of Heat Flow Across a Single Electronic Channel”, Science **342**, 601 (2013)



Electrons heated up by a known Joule power  $J_Q$

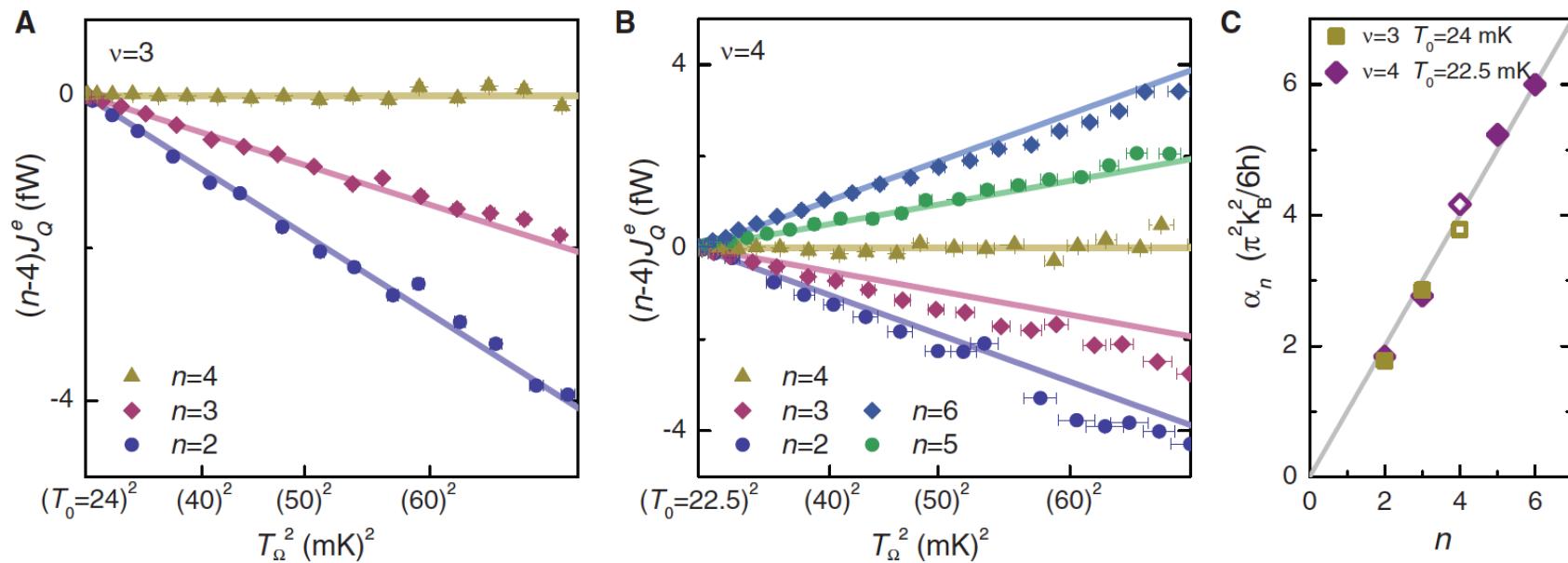
$$J_Q = nJ_Q^e(T_\Omega, T_0) + J_Q^{e-ph}(T_\Omega, T_0)$$

$J_Q^e$ : increase of power to keep  $T_\Omega$  constant when a channel is opened

$$J_Q^e(T_\Omega, T_0) = \frac{\pi^2 k_B^2}{6h} (T_\Omega^2 - T_0^2)$$

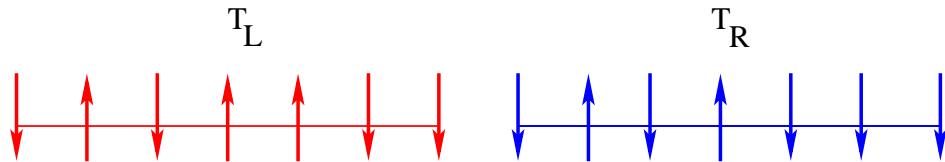
# Experiment

S. Jezouin *et al*, “Quantum Limit of Heat Flow Across a Single Electronic Channel”, Science **342**, 601 (2013)



$$\frac{J_Q^e(T_\Omega, T_0)}{T_\Omega^2 - T_0^2} = (1.06 \pm 0.07) \times \frac{\pi^2 k_B^2}{6h}$$

# Lattice Models



Quantum Ising Model

$$H = J \sum_{\langle ij \rangle} S_i^z S_j^z + \Gamma \sum_i S_i^x$$

$$\Gamma = J/2 \quad \text{Critical} \quad c = 1/2$$

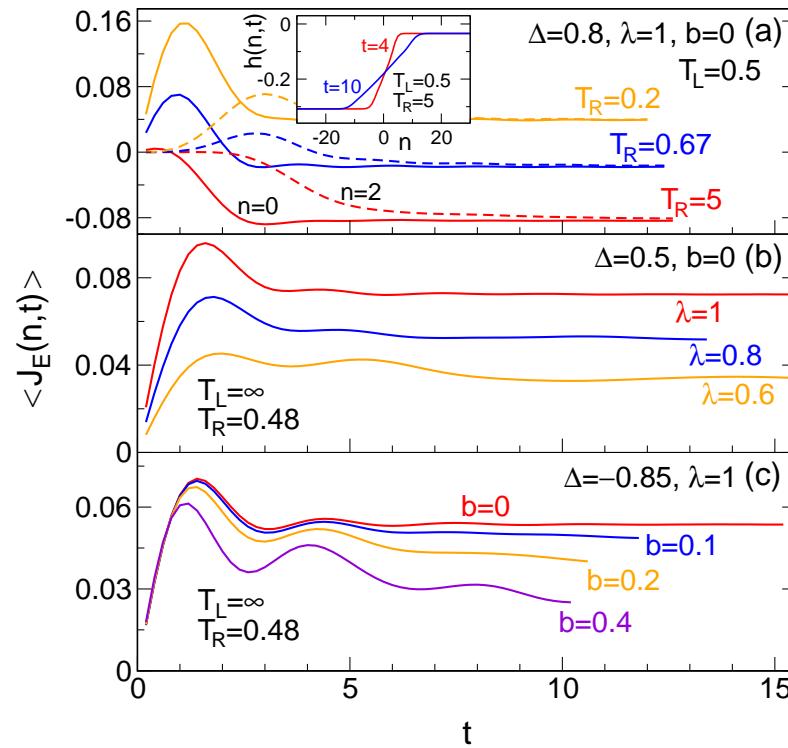
Anisotropic Heisenberg Model (XXZ)

$$H = J \sum_{\langle ij \rangle} (S_i^x S_j^x + S_i^y S_j^y + \Delta S_i^z S_j^z)$$

$$-1 < \Delta < 1 \quad \text{Critical} \quad c = 1$$

# Time-Dependent DMRG

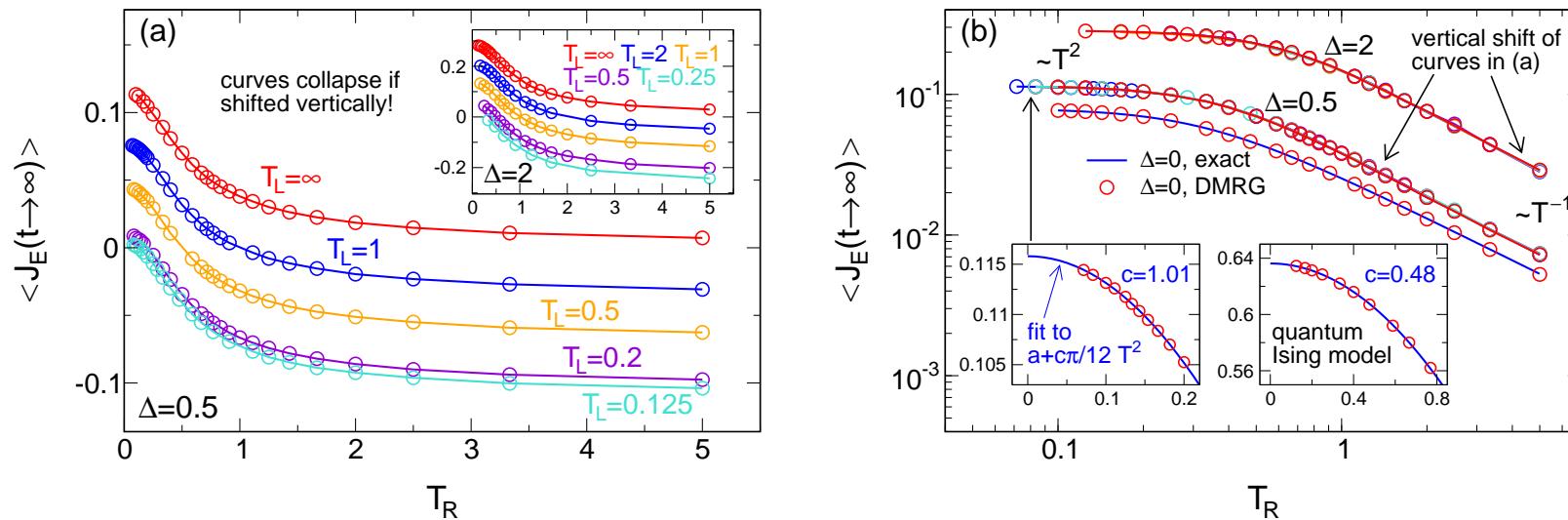
Karrasch, Ilan and Moore, *Non-equilibrium thermal transport and its relation to linear response*, Phys. Rev. B 88, 195129 (2013)



Dimerization  $J_n = \begin{cases} 1 & n \text{ odd} \\ \lambda & n \text{ even} \end{cases}$        $\Delta_n = \Delta$     Staggered  $b_n = \frac{(-1)^n b}{2}$

# Time-Dependent DMRG

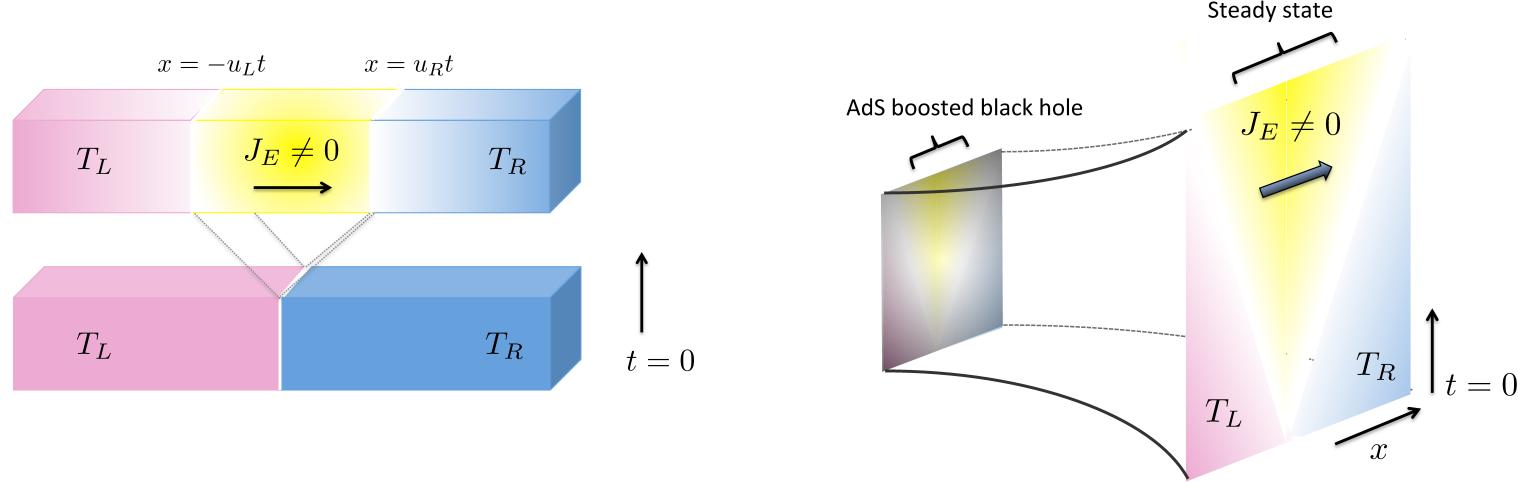
Karrasch, Ilan and Moore, *Non-equilibrium thermal transport and its relation to linear response*, Phys. Rev. B 88, 195129 (2013)



$$\lim_{t \rightarrow \infty} \langle J_E(n, t) \rangle = f(T_L) - f(T_R)$$

$$f(T) \sim \begin{cases} T^2 & T \ll 1 \\ T^{-1} & T \gg 1 \end{cases}$$

# AdS/CFT



Steady State Region

Spatially Homogeneous

# Solutions of Einstein Equations

$$S = \frac{1}{16\pi G_N} \int d^{d+2}x \sqrt{-g} (R - 2\Lambda) \quad \Lambda = -d(d+1)/2L^2$$

Unique homogeneous solution = boosted black hole

$$\begin{aligned} ds^2 = & \frac{L^2}{z^2} \left[ \frac{dz^2}{f(z)} - f(z)(dt \cosh \theta - dx \sinh \theta)^2 + \right. \\ & \left. (dx \cosh \theta - dt \sinh \theta)^2 + dy_{\perp}^2 \right] \end{aligned}$$

$$f(z) = 1 - \left( \frac{z}{z_0} \right)^{d+1} \quad z_0 = \frac{d+1}{4\pi T}$$

Fefferman–Graham Coordinates

$$\langle T_{\mu\nu} \rangle_s = \frac{L^d}{16\pi G_N} \lim_{Z \rightarrow 0} \left( \frac{d}{dZ} \right)^{d+1} \frac{Z^2}{L^2} g_{\mu\nu}(z(Z))$$

$$z(Z) = Z/R - (Z/R)^{d+2}/[2(d+1)z_0^{d+1}] \quad R = (d!)^{1/(d-1)}$$

# Boost Solution

Lorentz boosted stress tensor of a finite temperature CFT

Perfect fluid

$$\langle T^{\mu\nu} \rangle_s = a_d T^{d+1} (\eta^{\mu\nu} + (d+1) u^\mu u^\nu)$$

$$\eta^{\mu\nu} = \text{diag}(-1, 1, \dots, 1), \quad u^\mu = (\cosh \theta, \sinh \theta, 0, \dots, 0)$$

$$\langle T^{tx} \rangle_s = \frac{1}{2} a_d T^{d+1} (d+1) \sinh 2\theta$$

$$a_d = (4\pi/(d+1))^{d+1} L^d / 16\pi G_N$$

One spatial dimension

$$a_1 = \frac{L\pi}{4G_N} \quad c = \frac{3L}{2G_N}$$

$$T_L = T e^\theta$$

$$T_R = T e^{-\theta}$$

$$\langle T_{tx} \rangle = \frac{c\pi^2 k_B^2}{6h} (T_L^2 - T_R^2)$$

Run past a thermal state with temperature

$$T = \sqrt{T_L T_R}$$

# Steady State Density Matrix

$$\langle \mathcal{O} \dots \rangle = \frac{\text{Tr}(\rho_s \mathcal{O} \dots)}{\text{Tr}(\rho_s)}$$

$$\rho_s = e^{-\beta E \cosh \theta + \beta P_x \sinh \theta}$$

$$\beta = \sqrt{\beta_L \beta_R} \quad e^{2\theta} = \frac{\beta_R}{\beta_L}$$

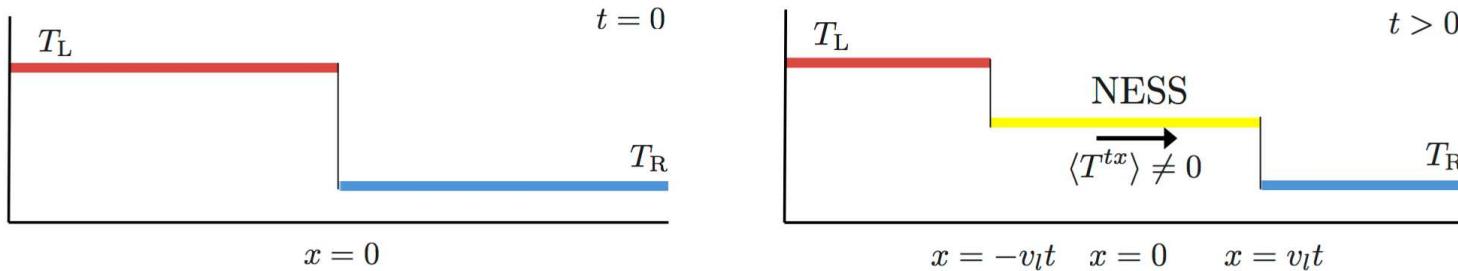
Lorentz boosted thermal density matrix

Describes all the cumulants of the energy transfer process

The non-equilibrium steady state (NESS)  
is a Lorentz boosted thermal state

# One Dimension

Time evolution shows light-cone propagation



Conservation of  $T^{\mu\nu}$  and tracelessness in  $d = 1$  imply that the dynamics can be factorized into left- and right-moving components

$$\langle T^{tx}(x, t) \rangle = F(x - t) - F(x + t), \quad \langle T^{tt}(x, t) \rangle = F(x - t) + F(x + t)$$

Sharp “shock waves” moving at the speed of light

Initial energy density on the left and the right yields

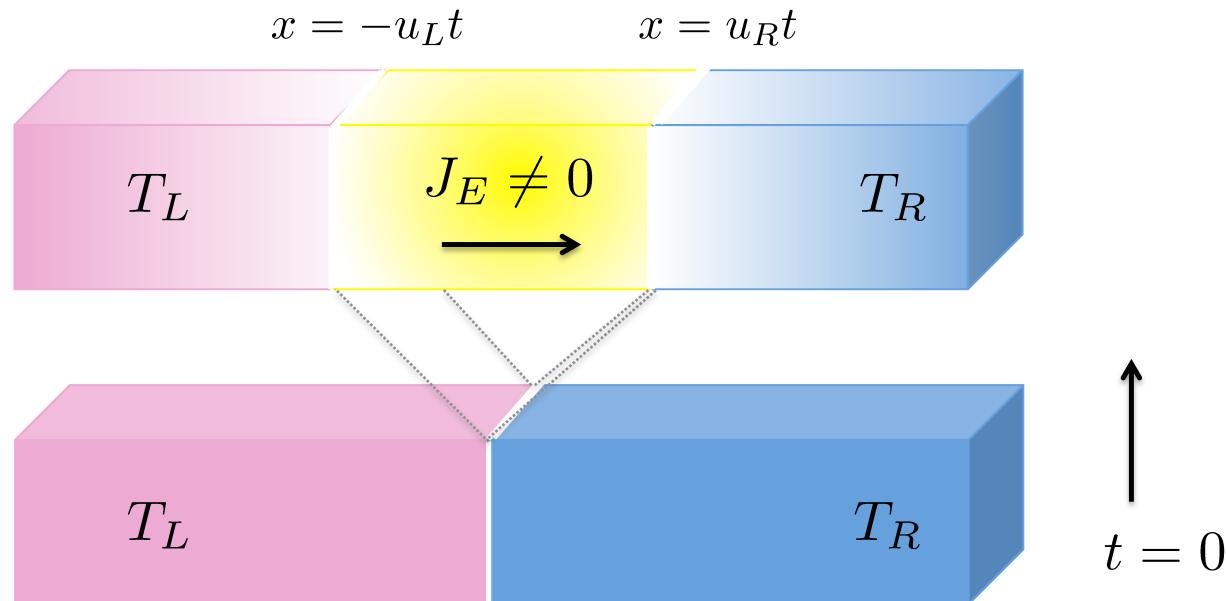
$$F(x) = \frac{c\pi}{12} [T_L^2 \Theta(-x) + T_R^2 \Theta(x)]$$

Steady state energy current  $\langle T^{tx} \rangle_s = \frac{c\pi}{12} (T_L^2 - T_R^2)$

Full evolution can also be obtained from Einstein equations

# Higher Dimensions

## Idealized Shock Waves



Energy-Momentum conservation across the shocks

# Shock Solutions

## Rankine–Hugoniot

Energy-Momentum conservation  $\partial_\mu T^{\mu\nu} = 0$  across shock

$$\langle T^{tx} \rangle_s = a_d \left( \frac{T_L^{d+1} - T_R^{d+1}}{u_L + u_R} \right)$$

Invoking boosted steady state gives  $u_{L,R}$  in terms of  $T_{L,R}$ :

$$u_L = \frac{1}{d} \sqrt{\frac{\chi+d}{\chi+d^{-1}}}$$

$$u_R = \sqrt{\frac{\chi+d^{-1}}{\chi+d}}$$

$$\chi \equiv (T_L/T_R)^{(d+1)/2}$$

Steady state region is a boosted thermal state with

$$T = \sqrt{T_L T_R}$$

Boost velocity  $(\chi - 1)/\sqrt{(\chi + d)(\chi + d^{-1})}$  Agrees with  $d = 1$

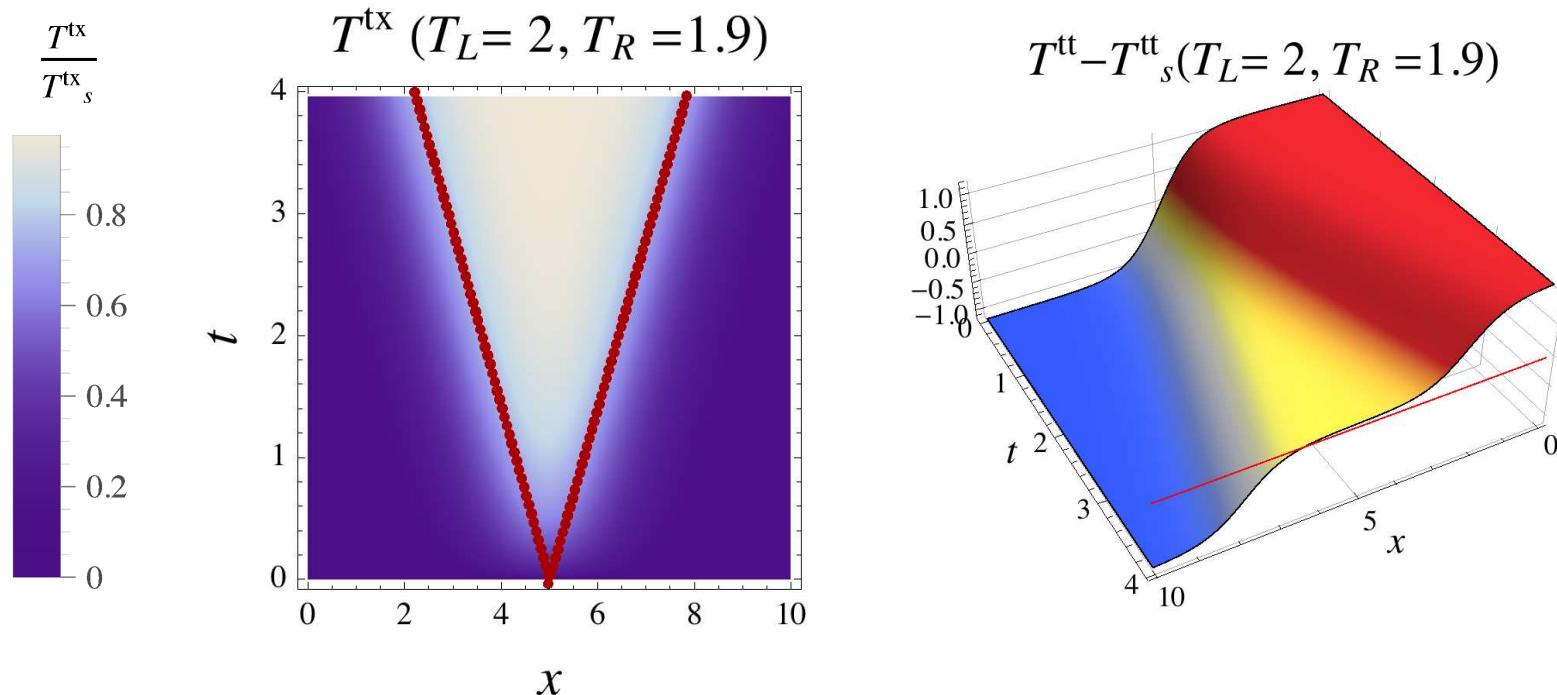
**Shock waves are non-linear generalizations of sound waves**

EM conservation:  $u_L u_R = c_s^2$ , where  $c_s = v/\sqrt{d}$  is speed of sound

$c_s < u_R < v$      $c_s < u_L < c_s^2/v$     reinstated microscopic velocity  $v$

# Numerics I

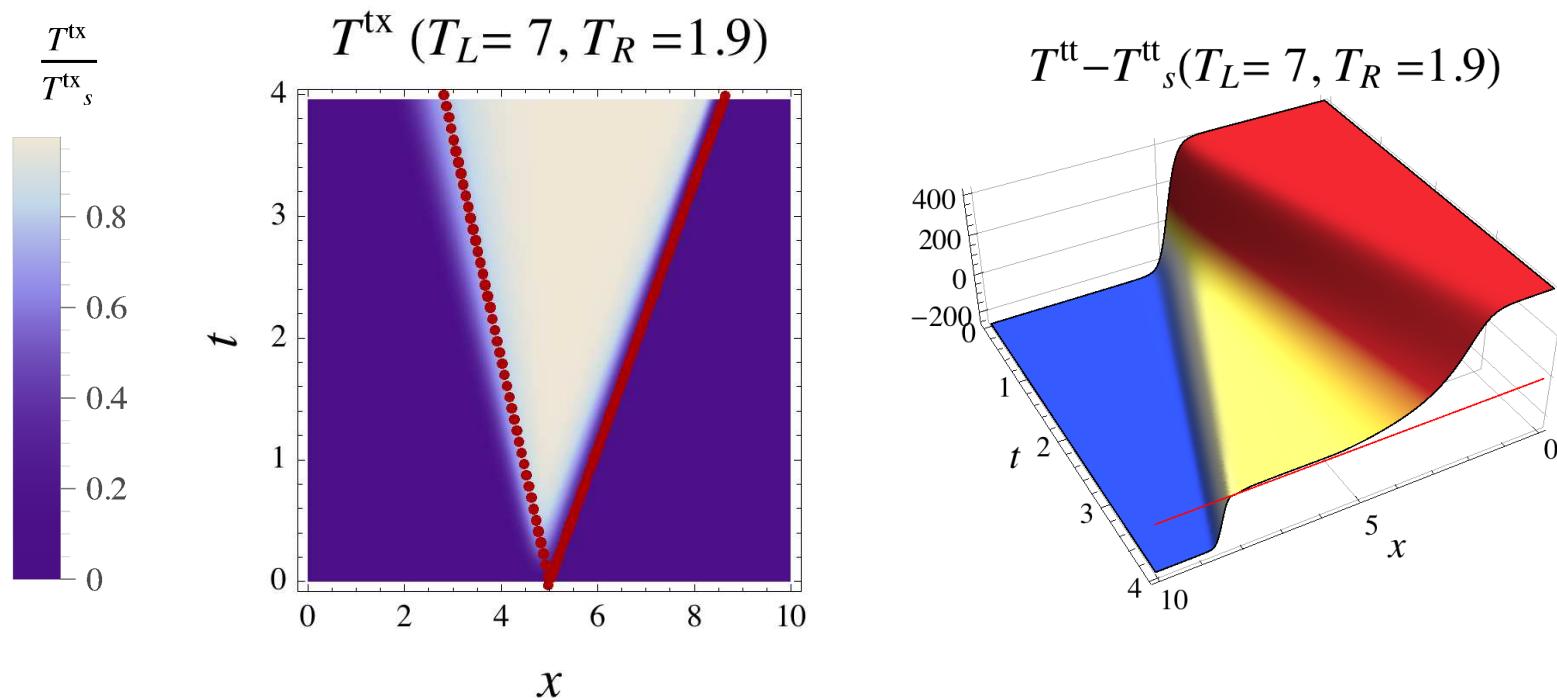
$$\partial_\mu T^{\mu\nu} = 0$$



Excellent agreement with predictions

# Numerics II

$$\partial_\mu T^{\mu\nu} = 0$$

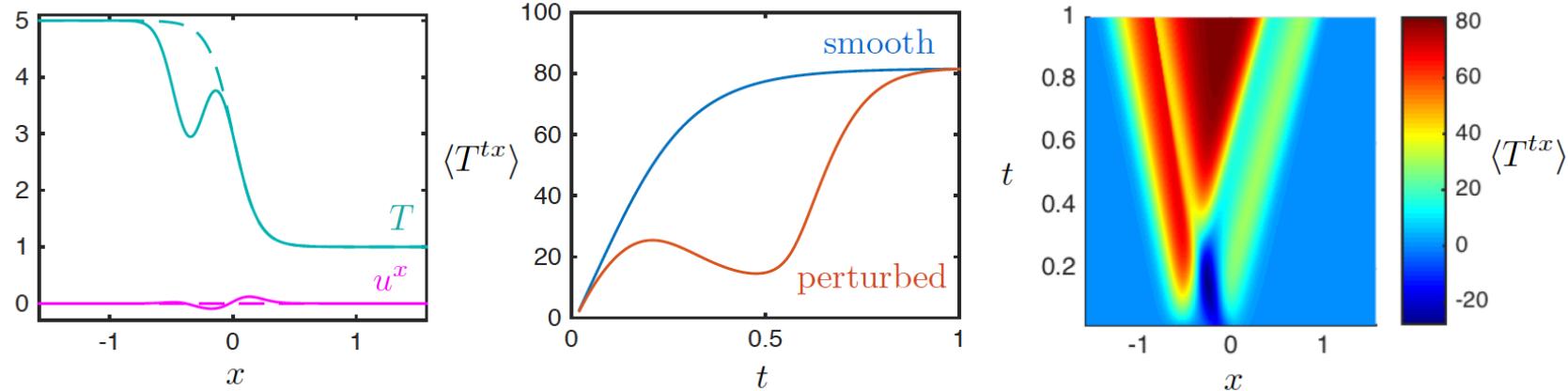


Excellent agreement far from equilibrium

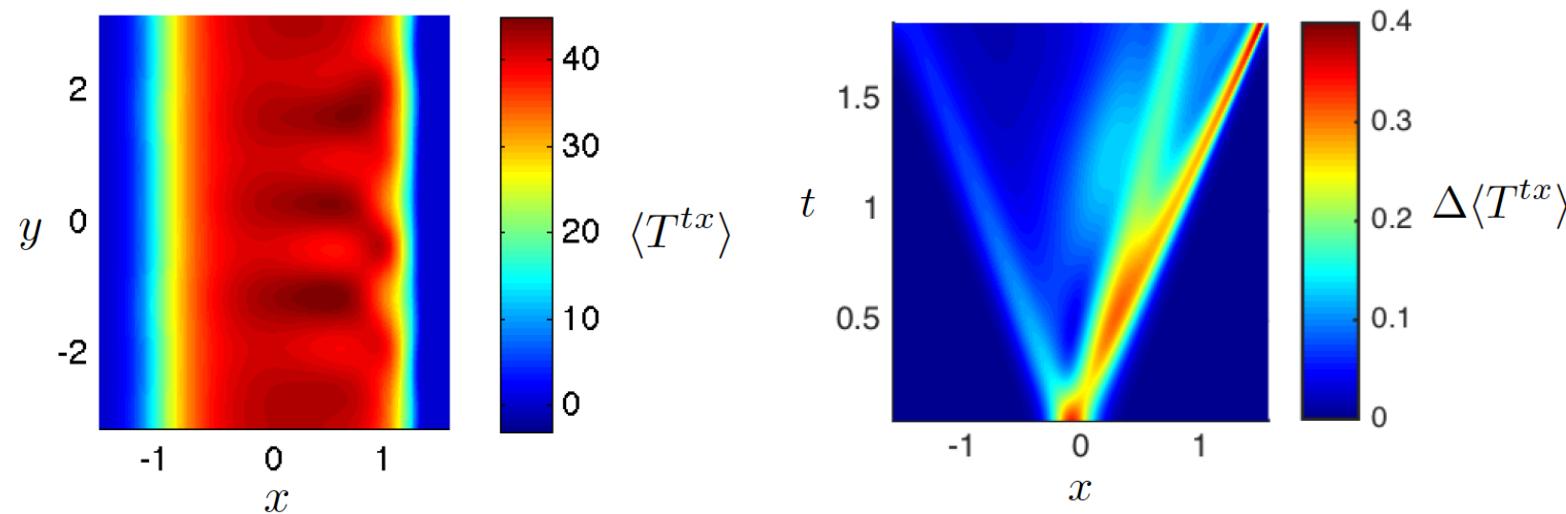
Asymmetry in propagation speeds

# Perturbations

Longitudinal perturbations ( $d = 2$ )

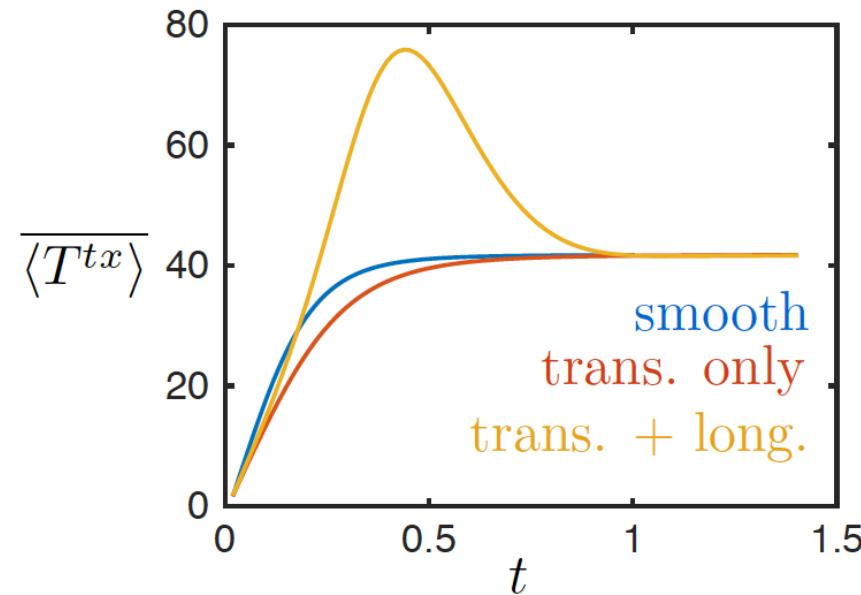


Transverse perturbations ( $d = 2$ )



# Perturbations

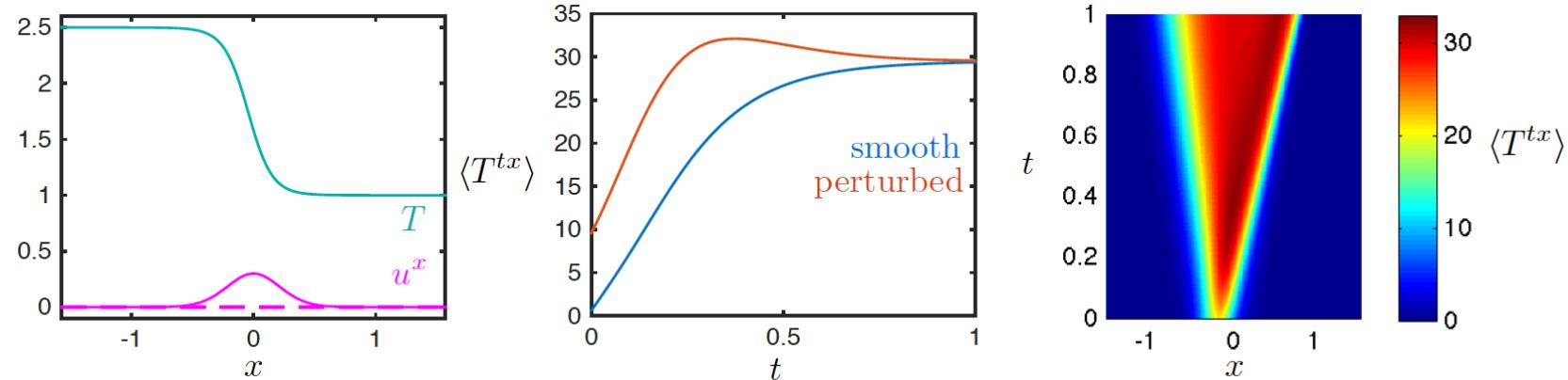
Longitudinal & transverse perturbations ( $d = 2$ )



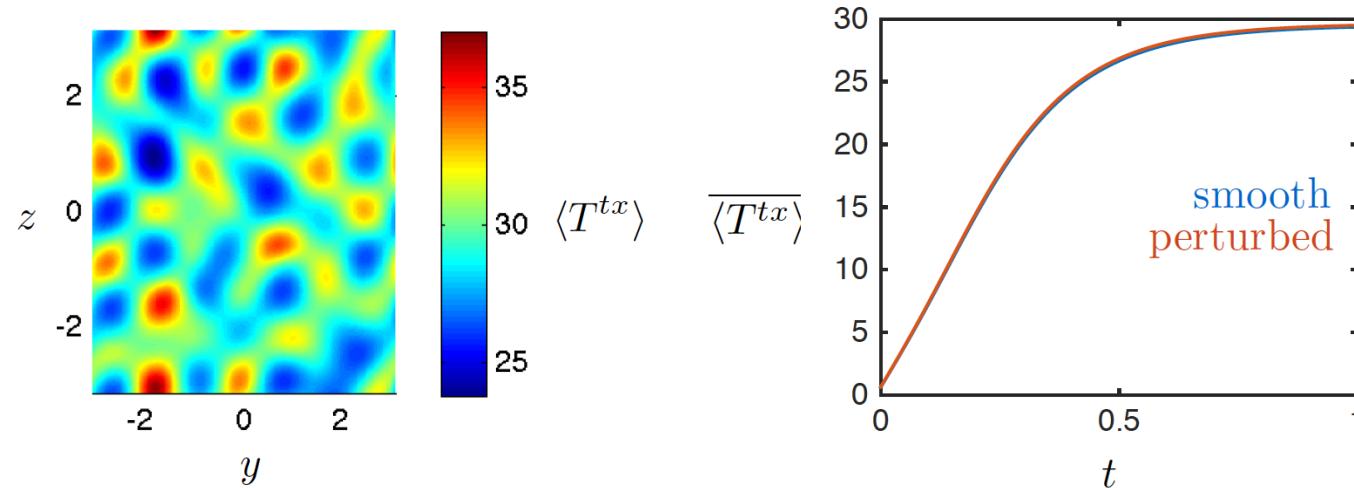
Spatially averaged profiles approach homogeneous results

# Perturbations

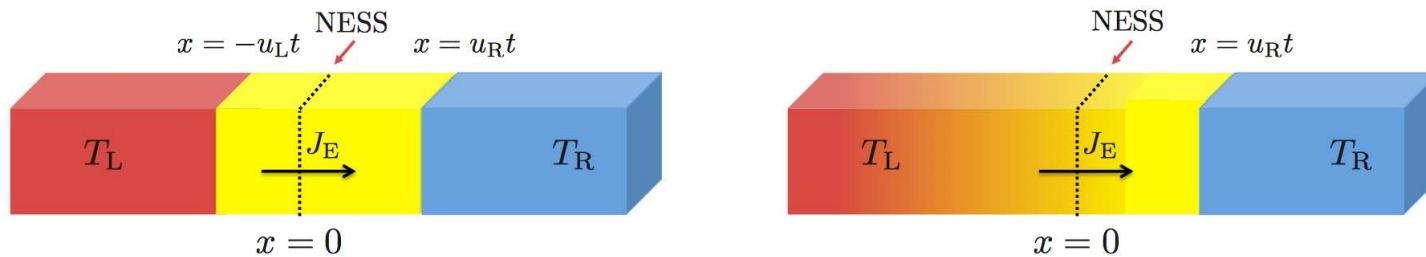
Longitudinal perturbations ( $d = 3$ )



Transverse perturbations ( $d = 3$ )



# Rarefaction Waves



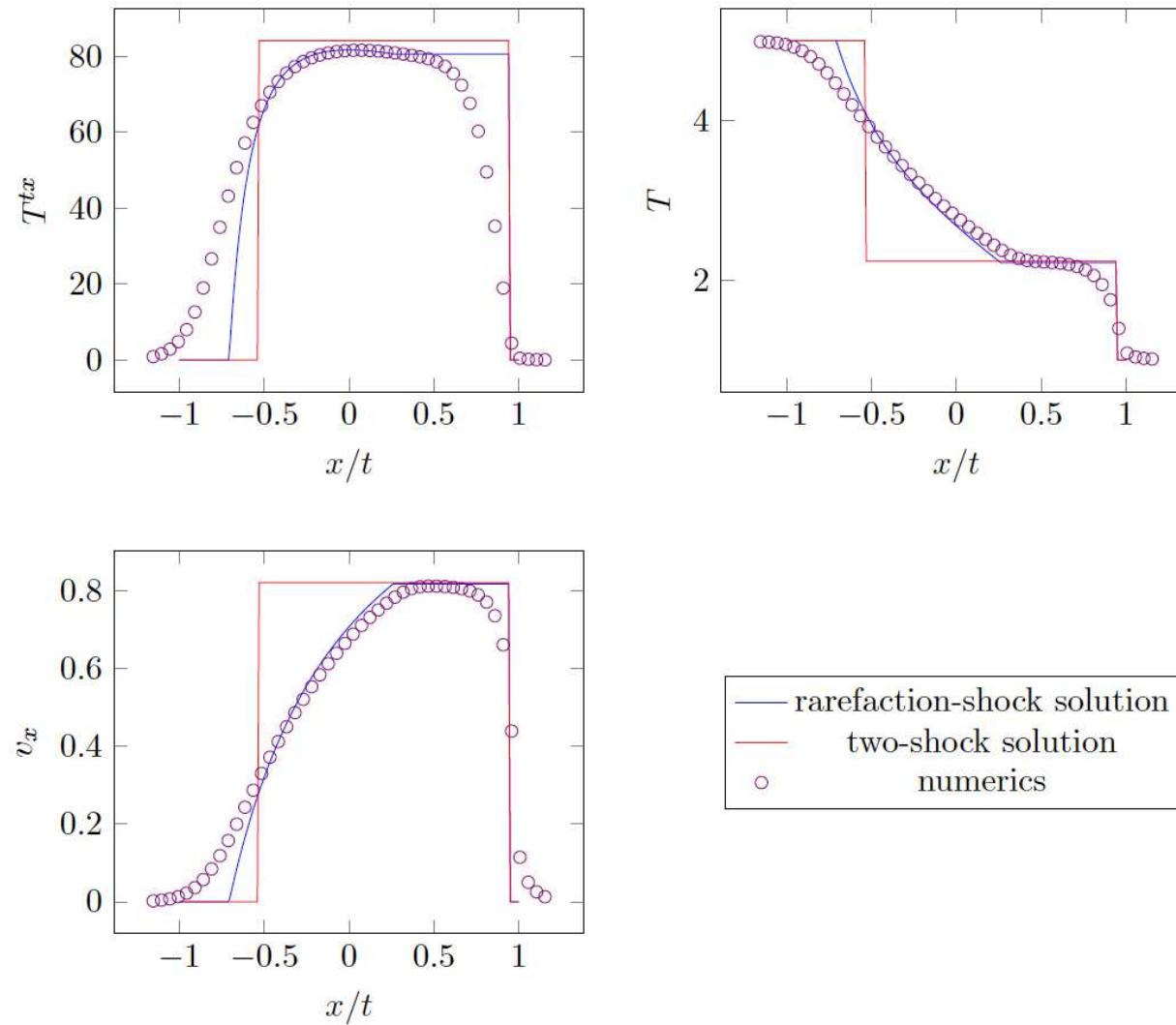
Left-moving idealized sharp shock wave needs to be treated as a smoothly varying rarefaction wave

Lucas, Schalm, Doyon and MJB; arXiv:1512.09037

PRD **94**, 025004 (2016)

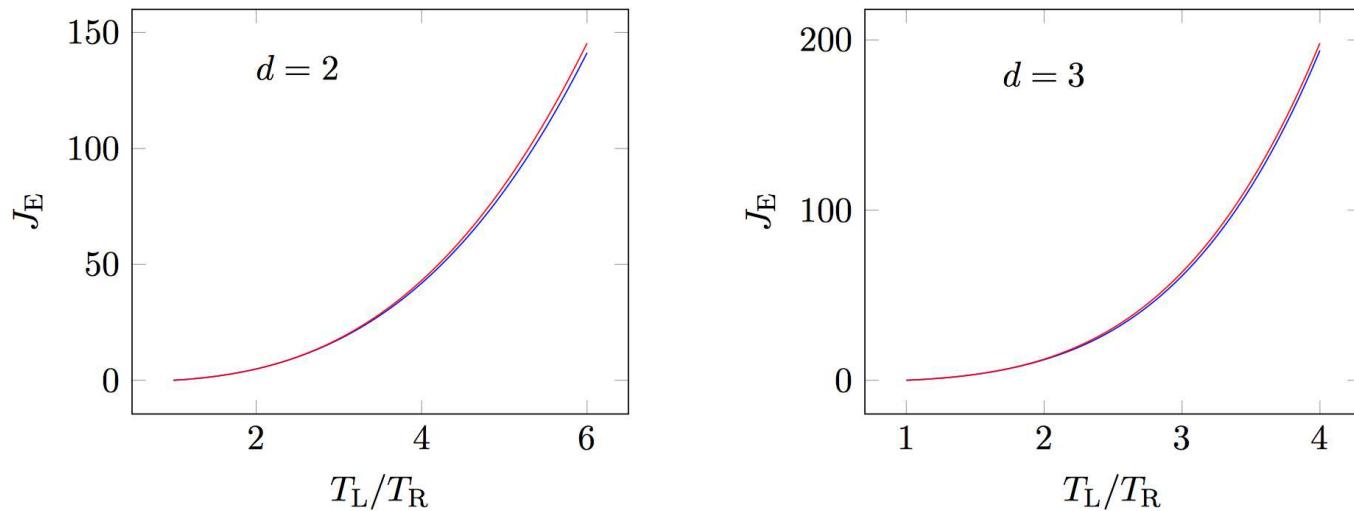
Spillane and Herzog; arXiv:1512.09071

# Comparison



Rarefaction-shock provides a better fit to simulation data

# Comparison



Rarefaction-shock (blue)    two-shock (red)

Close agreement for energy transport across the interface

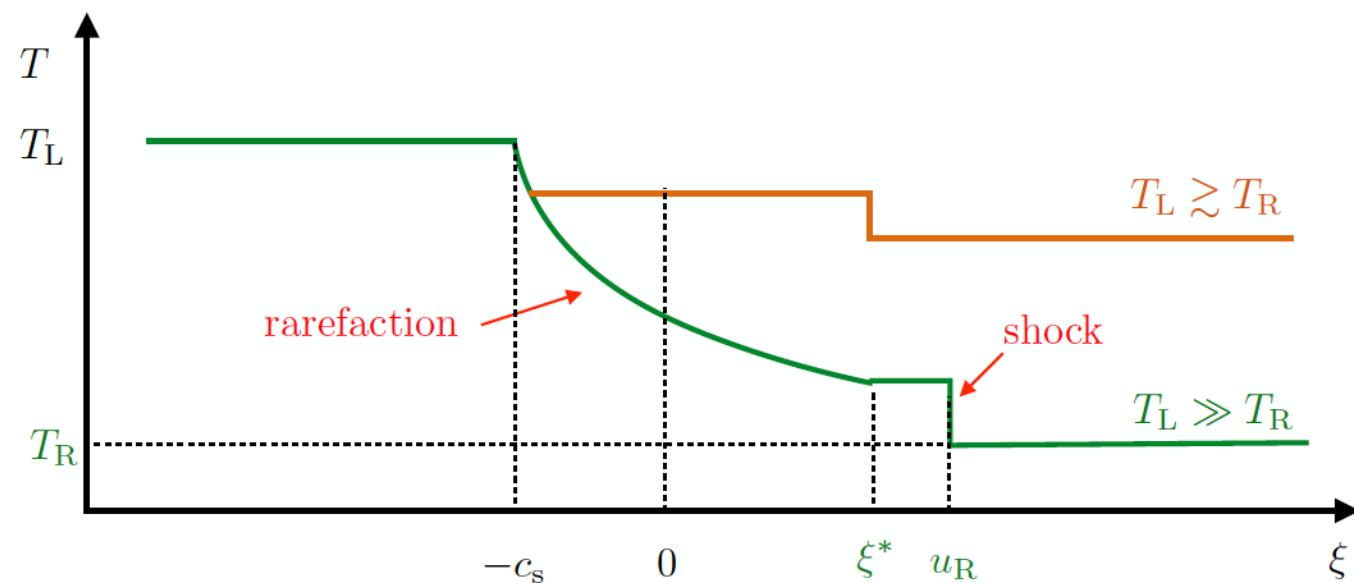
Rarefaction-shock provides a better fit to simulation data

New regime for  $T_L > \Gamma T_R$

$$\Gamma = 3.459 \ (d = 2) \quad \Gamma = 2.132 \ (d = 3)$$

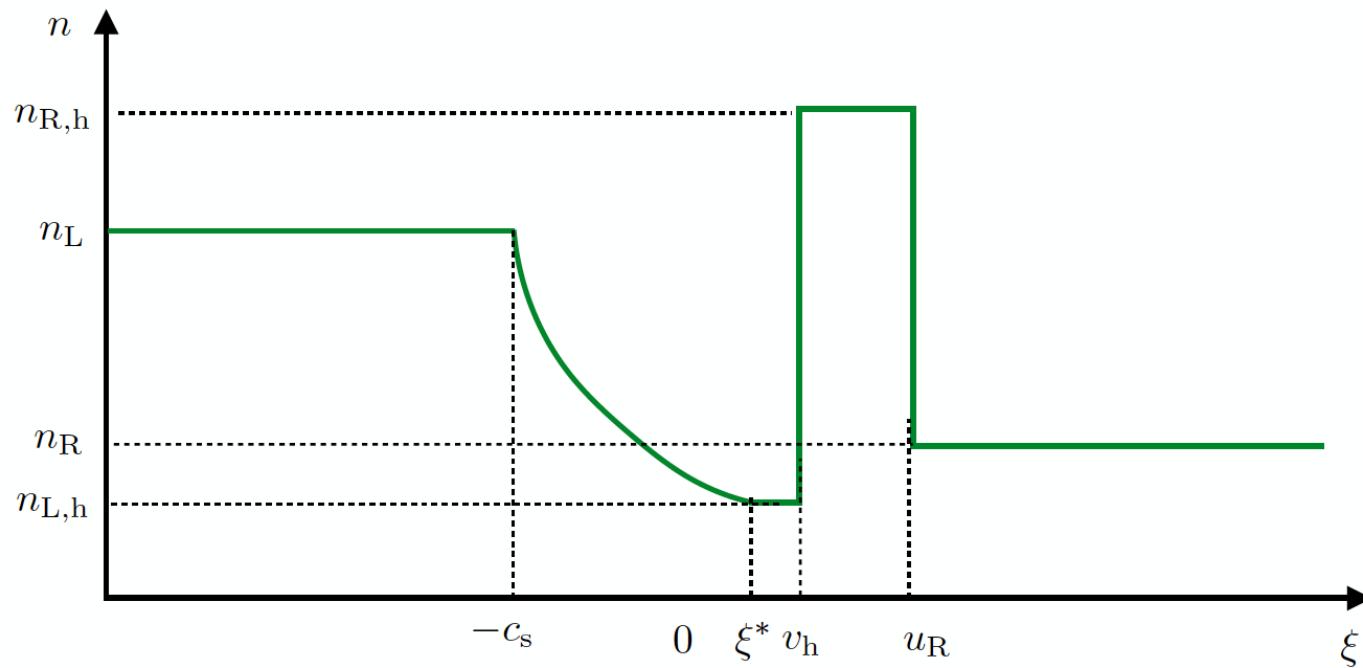
Readily generalized to other equations of state

# Regimes



$$\xi = \frac{x}{t}$$

# Charged Fluid



$$\xi = \frac{x}{t}$$

Charge “surfs” on the energy flow



<http://traveller.easyjet.com/emagazine/2471/june-2016/>

# Conclusions

Average energy flow in arbitrary dimension

Lorentz boosted thermal state

Energy current fluctuations

Exact generating function of fluctuations

Work in Progress

Other types of charge noise      Non-Lorentz invariant situations

Unitary Fermi gas      Fluctuation theorems

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S. Sachdev, A. Starinets

100 years after the advent of General Relativity  
Einstein still has an enormous amount to teach us

