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Keble College - Michaelmas 2014

CP3&4: Mathematical methods I&II
Tutorial 7 - Advanced problems

Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.
Leave these at Keble lodge by 5pm on Monday of 7th week.
Look at the 'class problems' in preparation for the tutorial session.
Suggested reading: RHB, all lecturers' problem sets, and past examination papers.

Goals

- Learn to apply various mathematical methods studied during term to more advanced problems.

Problems

While previous problem sets have attempted to walk you through a single topic, here we aim foremost to practice the art of problem solving — how to spot what the question wants without hints, how to spot short-cuts and simplify, how to avoid dead ends, and how to work confidently yet carefully in the face of multiple unknowns. Answering exam questions efficiently (maximising marks awarded per unit time) is a different skill to preparing good tutorial work (maximising and explaining understanding, and serving as a source for revision). Practice good exam technique, answer the questions directly, concisely, yet don't miss anything out. It should also show you how far you have come since the beginning of term, serve as a bit of revision, and be a little bit fun. Good luck.

1. Prove that

$$\sum_{r=1}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \sin(2r\theta) = 2^n \sin(n\theta)\cos^n(\theta)\,.$$

[Hint: express the left side as $\Im\left(\sum \begin{pmatrix} n \\ r \end{pmatrix} \mathrm{e}^{\mathrm{i}2r\theta}\right)$ and use the binomial theorem.]

2. Show that the complex polynomial equation with the four roots $z = (\pm\sqrt{3}\pm\mathrm{i})/2$ is $z^4 - z^2 + 1 = 0$.

3. Find the roots of the equation $(z-1)^n + (z+1)^n = 0$. Hence or otherwise solve the equation $x^3 + 15x^2 + 15x + 1 = 0$.

4. Find the stationary points of $z(x, y) = x^2 - y^2$ and determine the shapes of the contour lines. Sketch $z(x, y)$ in 3-dimensions and draw a contour map.

5. Find the eigenvalues and normalised eigenvectors of the Hermitian matrix

$$\mathbf{H} = \begin{pmatrix} 10 & 3\mathrm{i} \\ -3\mathrm{i} & 2 \end{pmatrix}.$$

[1] These problems were compiled by Prof. D. Jaksch based on problem sets by Prof. G.G. Ross, Prof. J. Yeomans and Prof. N. Harnew, and past Oxford Prelims exam questions.

Hence construct a unitary matrix $\mathbf{U}$ such that $\mathbf{U}^\dagger \mathbf{H} \mathbf{U} = \mathbf{D}$ where $\mathbf{D}$ is the real diagonal matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix},$$

that has the eigenvalues of $\mathbf{H}$ as its diagonal elements.

[Hint: Avoid a common mistake. Remember that the normalisation condition for a vector $\mathbf{v}$ that has complex elements is $\mathbf{v}^\dagger \mathbf{v} = 1$ (as opposed to $\mathbf{v}^T \mathbf{v} = 1$).]

6. Show that the quadratic surface $5x^2 + 11y^2 + 5z^2 - 10yz + 2xz - 10xy = 4$ is an ellipsoid with semi-axes of lengths 2, 1 and 1/2. Find the direction of its longest axis.

[Hint: You might find this question quite tricky, but nevertheless it is very instructive. First write the equation in matrix form $\mathbf{r}^T \mathbf{O} \mathbf{r} = r^2$, with $\mathbf{r} = (x, y, z)^T$. $\mathbf{O}$ is a real symmetric matrix, and is thus diagonalisable by a unitary $\mathbf{O} = \mathbf{U}\mathbf{D}\mathbf{U}^\dagger$. Here $\mathbf{U} = (\mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_3)$ has i-th column $\mathbf{u}_i$ given by an eigenvector of $\mathbf{O}$ and $\mathbf{D}$ is diagonal with the i-th element equal to the corresponding eigenvector λ_i. Hence we obtain $\mathbf{r}'^T \mathbf{D} \mathbf{r}'$ with the coordinate vector in the basis that diagonalises $\mathbf{O}$ given by $\mathbf{r}' = (x', y', z')^T = \mathbf{U}^\dagger \mathbf{r}$. In other words, changing the basis by rotation gives us an ellipse whose axes lie along the axes of the coordinate system. Because the transformation is a rotation, lengths don't change, so the length of the longest axis in this basis is the same as the length of the longest axis in the original basis. Inspection reveals that the axis lengths are given by $r/\sqrt{\lambda_i}$, so we need to find the eigenvalues of $\mathbf{O}$, with the smallest corresponding to the longest axis length. If we find the i-th eigenvalue to be smallest then this axis lies along $\hat{\mathbf{e}}_i$ in the new basis, where $\hat{\mathbf{e}}_i$ is the vector with all zeros except for the i-th element, which is unity. Therefore in the old basis it must lie along $\mathbf{U}^\dagger \hat{\mathbf{e}}_i = \mathbf{u}_i$. So we just need to calculate this eigenvector to find the length. If you need a good reference for this question, see Riley, Hobson and Bence.]

7. [From Prelims 1999] Leibnitz theorem and McLaurin series.

Show that the function $y(x) = \cos(a \arccos(x))$ fulfils the ODE

$$(1 - x^2) y''(x) - x y'(x) + a^2 y(x) = 0,$$

where a is a constant. Use Leibnitz' theorem to differentiate this ODE n times and then put $x = 0$ to show that for $n \geq 0$

$$y^{(n+2)}(0) = (n^2 - a^2) y^{(n)}(0),$$

where $y^{(n)}(0)$ is the n^{th} derivative of $y(x)$ evaluated at $x = 0$.

Use this result to obtain a terminating power series expansion for $y(x) = \cos(3 \arccos(x))$ in terms of x. Verify that your solution solves the above ODE.

8. Find a continuous solution with continuous first derivative of the system

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \sin(x) + f(x),$$

subject to $y(-\pi/2) = y(\pi) = 0$, where

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & x > 0 \end{cases}.$$

[Hint: Calculate a general solution for each of the cases $x \leq 0$ and $x > 0$ and then obtain relations between your four arbitrary constants by making the solutions agree at $x = 0$.]

9. [From Prelims 1997] A variable z may be expressed either as a function of (u, v) or of (x, y), where $u = x^2 + y^2$, $v = 2xy$.

(a) Find
$$\left.\frac{\partial z}{\partial x}\right|_y \quad \text{in terms of} \quad \left.\frac{\partial z}{\partial u}\right|_v \quad \text{and} \quad \left.\frac{\partial z}{\partial v}\right|_u .$$

(b) Find
$$\left.\frac{\partial z}{\partial u}\right|_v \quad \text{in terms of} \quad \left.\frac{\partial z}{\partial x}\right|_y \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_x .$$

(c) Express
$$\left.\frac{\partial z}{\partial u}\right|_v - \left.\frac{\partial z}{\partial v}\right|_u \quad \text{in terms of} \quad \left.\frac{\partial z}{\partial x}\right|_y \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_x .$$

(d) Check your expressions by seeing if they hold for the specific case $z = u + v$.

10. A mass m is constrained to move in a straight line and is attached to a spring of strength $\lambda^2 m$ and a dashpot that produces a retarding force $-\alpha m v$, where v is the velocity of the mass. Find the steady state displacement of the mass when an amplitude-modulated periodic force $Am\cos(pt)\sin(\omega t)$ with $p \ll \omega$ and $\alpha \ll \omega$ is applied to it, where all parameters given above are real.

Show that for $\omega = \lambda$ the displacement of the amplitude-modulated wave is approximately given by
$$-A\frac{\cos(\omega t)\sin(pt+\phi)}{\sqrt{4\omega^2 p^2 + \alpha^2\omega^2}} ,$$
where A is a constant and ϕ is the principal branch solution to the equation
$$\phi = \arctan(\alpha/2p) .$$

Solution: Your solution to this question may need some extra lines of work that do not fit onto the dots below, which, however, should still provide useful guidance. Using Newton's law for the position of the mass x we find
$$\ddot{x} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = A\cos(pt)\sin(\omega t) .$$

We rearrange this equation by dividing through m. We then rewrite the right hand side as
$$A\cos(pt)\sin(\omega t) = \frac{A}{2}[\sin((\omega+p)t) + \ldots\ldots\ldots\ldots\ldots] .$$

We do not need to work out the solution to the homogeneous equation because we are only interested in the .... and thus only consider the particular solution (the complimentary function will eventually decay in time, which can be verified — physically, any evolution due to the initial conditions will decay due to damping, only motion caused by the driving will remain at long times).

There are many ways to proceed. We could use the trial function $x = B\sin((\omega+p)t) + C\cos((\omega+p)t) + D\sin((\omega-p)t) + E\cos((\omega-p)t)$ or the trial function $x = Be^{i(\omega+p)t} + Ce^{-i(\omega+p)t} + De^{i(\omega-p)t} + Ee^{-i(\omega-p)t}$. You are at risk of dying from boredom with these two approaches. A better idea is to consider the equation
$$\ddot{z} + \alpha\dot{z} + \lambda^2 z = \frac{A}{2}[e^{i(\omega+p)t} + e^{i(\omega-p)t}] .$$

By linearity, taking the real part, we see that if z is a solution to the above equation then $x = \Im\{z\}$ is a solution to the original equation. Thus we focus on solving the equation for z using the trial function $z = C_+ e^{i(\omega+p)t} + C_- e^{i(\omega-p)t}$, or using linearity to trial the two terms separately. This is half the number of terms we previously had and is just about bearable.

Doing this we find

$$C_{\pm} = \frac{A}{2(\lambda^2 - (\omega \pm p)^2 + \mathrm{i}.......(\omega \pm p))} .$$

We can put this into exponential form

$$C_{\pm} = r_{\pm} \mathrm{e}^{\mathrm{i}\phi_{\pm}} ,$$

where

$$r_{\pm} = \frac{A}{2\sqrt{(\lambda^2 - (\omega \pm p)^2)^2 + \alpha^2(\omega \pm p)^2}} ,$$

and $\phi_{\pm}$ satisfies $\phi_{\pm} = \arctan(..................)$. Crucially though, we have not yet decided which branch of arctan $\phi_{\pm}$ belongs to. Nevertheless, with this still to be decided, we may write the solution to the complex equation as

$$z = \sum_{\pm} r_{\pm}................................,$$

and thus for the original real equation

$$x = \sum_{\pm} r_{\pm}.................................$$

We now set $\lambda = \omega$ and only keep the lowest order terms in p/ω and α/ω. This leads to equal amplitudes for both terms

$$r_{\pm} \approx \frac{A}{2}(4\omega^2 p^2 + ...)^{-1/2} = r .$$

However, the phases are different, $\phi_{\pm} = \arctan(\pm\alpha/2p)$. In particular, since the exact phases will decide how the two terms interfere, we must further know which branch of arctan each belongs to. This can be deduced by looking at the original epxression for $C_{\pm}$ within the new approximations

$$C_{\pm} \approx \frac{A}{2}(\mp 2\omega p - \mathrm{i}........)^{-1} .$$

By looking at the signs of the real and imaginary part of the inverse we see that for $+/-$ we must take the secondary/principal branch of arctan. Thus defining ϕ is the principal branch solution to the equation $\phi = \arctan(\alpha/2p)$ we may write $\phi_+ = \phi + \pi$ and $\phi_- = -\phi$. Inputting this into our previous expression for x we get

$$x = r(............................... + ...) ,$$

which can be rearranged to the form given in the question. You won't get a much more complicated question than that! It really shows the usefulness of complex numbers (imagine trying to answer the question without using them). The final solution can perhaps be obtained more simply by setting $\lambda = \omega$ and assuming small p/ω and α/ω at the very beginning before defining amplitudes $r_{\pm}$ and phases $\phi_{\pm}$. The complexity involving branches of arctan can be simplified then before it arises.

Class problems

11. Find all the roots, real and complex, of the equation $z^3 - 1 = 0$. If ω is any one of the complex roots, prove that $1 + \omega + \omega^2 = 0$. Find the sums of the following series:

$$S_1 = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots ; \qquad S_2 = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots ; \qquad S_3 = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots$$

[Hint: Note that $S_1 + S_2 + S_3 = \mathrm{e}^x$ and calculate $\mathrm{e}^{\omega x}$ and $e^{\omega^2 x}$.]

12. The relativistic expression for the energy of a particle of mass m is

$$E = \frac{mc^2}{(1 - v^2/c^2)^{1/2}}$$

where v is the particle velocity and c the speed of light. Expand this up to and including $O(v^4/c^4)$ and identify the terms you obtain.

13. Evaluating derivatives numerically: Use Taylor's theorem to show that when h is small

(a) $f'(a) = (f(a+h) - f(a-h))/(2h)$ with an error $O(h^2 f'''(a))$.

(b) $f''(a) = (f(a+h) - 2f(a) + f(a-h))/h^2$ with an error $O(h^2 f''''(a))$.

Taking $f(x) = \sin(x)$, $a = \pi/6$, and $h = \pi/180$ find from (a) and (b) the approximate values of $f'(a)$ and $f''(a)$ and compare them to exact values.

These finite-difference formulae are often used to calculate derivatives numerically. How would you construct a more precise finite-difference approximation to $f'(a)$?

14. Surfaces: (a) Find the stationary points of $z(x,y)$ and determine the shapes of the contour lines close to these points. (b) Sketch $z(x,y)$ in 3-dimensions and (c) draw a contour map of the surface, where

(i) $z = (4 - x^2 - y^2)^{1/2}$,

(ii) $z = 1 - 2(x^2 + y^2)$,

(iii) $z = xy$,

15. Prove using vector methods that:

(i) The diagonals of a parallelogram bisect each other.

(ii) The diagonals of a rhombus are perpendicular to each other.

(iii) Two lines, drawn from the end-points of the line of diameter of a circle to a common end-point on the circumference, intersect at a right angle.

16. Find the eigenvalues and a set of normalized eigenvectors of the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Verify that its eigenvectors are mutually orthogonal. Would you have expected this?

17. Consider N identical atoms of mass m whose motion is restricted to one dimension, arranged in a line with nearest-neighbours couple via a spring with sping constant κ.

(a) Show that the Newtonian equations of motion may be written in matrix form

$$\ddot{\mathbf{x}} = -\mathbf{M}\mathbf{x},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_N)^{\mathrm{T}}$ is a vector giving the displacements of the atoms from their equilibrium positions, and the dynamical matrix is

$$\mathbf{M} = \frac{\kappa}{m} \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

(b) Now (as it is symmetric) consider some unitary matrix $\mathbf{U}$, which diagonalises $\mathbf{M}$, such that $\mathbf{U}^\dagger\mathbf{M}\mathbf{U} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix with non-negative diagonal elements ω_n^2. Show that the equations of motion may then be rewritten in terms of N independent harmonic oscillators as

$$\ddot{\mathbf{x}}' = -\mathbf{D}\mathbf{x}',$$

where $\mathbf{x}' = \mathbf{U}^\dagger\mathbf{x}$ and $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_N)^{\mathrm{T}}$.

18. Solve the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(3x^2 + 2xy + y^2)\sin(x) - (6x + 2y)\cos(x)}{(2x + 2y)\cos(x)}\,.$$

19. Find the general solution $x(t)$ to

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = F_0\cos(\omega t + \phi),$$

for all possible values of γ, ω_0, ω and F_0. Interpret your solution, including the special cases.