2 - The photon propagator

Path integral formalism - reminder

$$Z = \int D\varphi e^{(i/\hbar)\int d^4x \mathcal{L}(\varphi)}$$

- Classical limit $\hbar \to 0$ stationary phase approximation

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} - \frac{\delta \mathcal{L}}{\delta \varphi} = 0$$ Euler-Lagrange classical field equation

Path integral formalism

$$Z = \int D\varphi e^{(i/\hbar)\int d^4x \mathcal{L}(\varphi)}$$

- $\mathcal{L}(\varphi) = \frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2]$

$$Z = \int D\varphi e^{i\int d^4x[-\frac{1}{2}\varphi(\partial^2+m^2)\varphi + J\varphi]}$$

Generating Functional

Klein-Gordon propagator

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[-\widetilde{\varphi}(k)(k^2+m^2)\widetilde{\varphi}(-k) + \widetilde{J}(k)\widetilde{\varphi}(-k) + \widetilde{J}(-k)\widetilde{\varphi}(k)\Big], \quad (8.7)$$

$$\widetilde{\chi}(k) = \widetilde{\varphi}(k) - \frac{\widetilde{J}(k)}{k^2+m^2} \qquad \mathcal{D}\varphi = \mathcal{D}\chi$$

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[\frac{\widetilde{J}(k)\widetilde{J}(-k)}{k^2+m^2} - \widetilde{\chi}(k)(k^2+m^2)\widetilde{\chi}(-k)\right]$$

$$Z_0(0) = \langle 0|0\rangle_{J=0} = \int D\chi \exp\left[-\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\widetilde{\chi}(k)\left(k^2+m^2\right)\widetilde{\chi}(-k)\right] = 1 \quad \text{(no interactions)}$$

$$\begin{aligned} Z_0(J) &= \exp\left[\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\frac{\widetilde{J}(k)\widetilde{J}(-k)}{k^2+m^2-i\epsilon}\right] \\ &= \exp\left[\frac{i}{2}\int d^4x\, d^4x'\, J(x)\Delta(x-x')J(x')\right]. \qquad (8.10) \end{aligned}$$

$$\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon}$$

Feynman propagator

Photon propagator

Path integral formalism

$$Z_0(J) = \int \mathcal{D}A \, e^{iS_0} \, ,$$

$$S_0 = \int d^4x \left[-\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \right] \, .$$

$$\widetilde{A}_\mu(k) = \int d^4x \, e^{-ik.x} A_\mu(x), \quad A_\mu(x) = \frac{1}{(2\pi)^4} \int d^4k \, e^{ik.x} \widetilde{A}_\mu(k)$$

$$S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Big[-\widetilde{A}_\mu(k) \Big(k^2 g^{\mu\nu} - k^\mu k^\nu \Big) \widetilde{A}_\nu(-k) + \widetilde{J}^\mu(k) \widetilde{A}_\mu(-k) + \widetilde{J}^\mu(-k) \widetilde{A}_\mu(k) \Big] .$$

Photon propagator

Path integral formalism

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$k^2 P^{\mu\nu}$

Projection matrix:

$$P^{\mu\nu}(k) \equiv g^{\mu\nu} - k^\mu k^\nu / k^2 .$$

$$\left(P^{\mu\nu}(k) P_\nu{}^\lambda(k) = P^{\mu\lambda}(k) \right)$$

Photon propagator

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[-\widetilde{A}_\mu(k)\Big(k^2 g^{\mu\nu} - k^\mu k^\nu\Big)\widetilde{A}_\nu(-k) + \widetilde{J}^\mu(k)\widetilde{A}_\mu(-k) + \widetilde{J}^\mu(-k)\widetilde{A}_\mu(k)\Big].$$

To complete the square need to invert $k^2 g^{\mu\nu} - k^\mu k^\nu \equiv k^2 P^{\mu\nu}$

...but $P^{\mu\nu} k_\nu = 0,$ zero eigenvalue...not invertible

Photon propagator

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[-\widetilde{A}_\mu(k)\Big(k^2 g^{\mu\nu} - k^\mu k^\nu\Big)\widetilde{A}_\nu(-k) + \widetilde{J}^\mu(k)\widetilde{A}_\mu(-k) + \widetilde{J}^\mu(-k)\widetilde{A}_\mu(k)\Big].$$

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...but $P^{\mu\nu}k_\nu = 0$, zero eigenvalue...not invertible

In fact component of $A_\mu \propto k_\mu$ doesn't appear ... sufficient to integrate DA_μ over transverse components, $\widetilde{A}_\mu$, only ...

... equivalent to $k^\mu \widetilde{A}_\mu = 0$... Lorentz gauge $\partial^\mu \widetilde{A}_\mu = 0$

$$P^{\mu\nu}(k) P_\nu{}^\lambda(k) = P^{\mu\lambda}(k)$$

identity matrix in $\widetilde{A}_\mu$ subspace

$$\left(k^2 P^{\mu\nu}(k)\right)^{-1} = \frac{P^{\mu\nu}}{k^2 - i\varepsilon}$$

$$k^2 P^{\mu\nu}$$

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[-\widetilde{A}_\mu(k)\Big(k^2 g^{\mu\nu} - k^\mu k^\nu\Big)\widetilde{A}_\nu(-k) + \widetilde{J}^\mu(k)\widetilde{A}_\mu(-k) + \widetilde{J}^\mu(-k)\widetilde{A}_\mu(k)\Big].$$

$$\widetilde{\chi}_\mu(k) = \widetilde{A}_\mu(k) + \frac{P_\mu^\alpha J_\alpha(k)}{k^2 - i\varepsilon}$$

$$k^2 P^{\mu\nu}$$

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[-\widetilde{A}_\mu(k)\Big(k^2 g^{\mu\nu} - k^\mu k^\nu\Big)\widetilde{A}_\nu(-k) + \widetilde{J}^\mu(k)\widetilde{A}_\mu(-k) + \widetilde{J}^\mu(-k)\widetilde{A}_\mu(k)\Big].$$

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$$Z_0(0) = \langle 0|0\rangle_{J=0} = \int D\widetilde{\chi}\exp\left[-\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\widetilde{\chi}_\mu(k)k^2P^{\mu\nu}\widetilde{\chi}_\nu(-k)\right] = 1$$

$$Z_0(J) = \exp\left[\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\,\widetilde{J}_\mu(k)\,\frac{P^{\mu\nu}(k)}{k^2 - i\epsilon}\,\widetilde{J}_\nu(-k)\right]$$

$$= \exp\left[\frac{i}{2}\int d^4x\, d^4y\, J_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right],$$

$$\Delta^{\mu\nu}(x-y) = \int\frac{d^4k}{(2\pi)^4}\,e^{ik(x-y)}\,\frac{P^{\mu\nu}(k)}{k^2 - i\epsilon}$$

Photon propagator
In the Lorenz gauge

The $i\varepsilon$ term

Fields must match onto free fields at t=±∞

$$\langle f|S|i\rangle = \int d\phi\, e^{iS(\phi)} \langle f|\phi(t=+\infty)\rangle\langle\phi(t=-\infty)|i\rangle$$ (time ordering)

$$\begin{aligned} |q',t'\rangle &= e^{iHt'}|q'\rangle \\ &= \sum_{n=0}^{\infty} e^{iHt'}|n\rangle\langle n|q'\rangle \\ &= \sum_{n=0}^{\infty} \psi_n^*(q')e^{iE_nt'}|n\rangle\ , \end{aligned}$$

$$\psi_n(q) = \langle q|n\rangle$$

$$H \to (1-i\varepsilon)H$$

$$Lim_{t'\to-\infty}|q',t'\rangle = \psi_0^*(q')|0\rangle$$ etc.

i.e. picks out ground state in initial and final states

Fixing the gauge

Want to be able to determine the propagator is various gauges …

.. instructive to see how it is done in canonical formalism

Klein-Gordon propagator

Want to solve :

$$(-\partial_\mu \partial^\mu + m^2)\psi = -V\psi$$

Solution :

$$\psi(x) = \phi(x) + \int d^4x' \Delta_F(x' - x) V(x') \psi(x')$$

where

$$(\partial_\mu \partial^\mu - m^2)\phi(x) = 0$$

and

$$(\partial_\mu \partial^\mu - m^2)\Delta_F(x' - x) = \delta^4(x' - x)$$

Reminder: Klein-Gordon propagator

Want to solve : $$(-\partial_\mu \partial^\mu + m^2)\psi = -V\psi$$

Solution : $$\psi(x) = \phi(x) + \int d^4x' \Delta_F(x'-x)V(x')\psi(x')$$

where $$(\partial_\mu \partial^\mu - m^2)\phi(x) = 0$$

and $$(\partial_\mu \partial^\mu - m^2)\Delta_F(x'-x) = \delta^4(x'-x)$$

Solve for propagator in momentum space by taking Fourier transform

$$\frac{1}{(2\pi)^2}\int e^{-ip.(x'-x)} \quad (\partial_\mu \partial^\mu \cdot \quad m^2)\Delta_F(x'-x)d^4(x'-x) = \frac{1}{(2\pi)^2}\int e^{-ip.(x'-x)}\delta^4(x'-x)d^4(x'-x)$$

$$\Rightarrow -(p^2 + m^2)\tilde{\Delta}_F(p) = \frac{1}{(2\pi)^2}$$

$$\tilde{\Delta}_F(p) = -\frac{1}{(2\pi)^2}\frac{1}{p^2+m^2-i\varepsilon}, \qquad \Delta_F(x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.x}\frac{1}{p^2+m^2-i\varepsilon}$$

The photon propagator

- The propagators determined by terms quadratic in the fields, using the Euler Lagrange equations.

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial^\mu A_\mu) = j^\nu \equiv \left(g^{\nu\lambda} \partial^2 - \partial^\nu \partial^\lambda \right) A_\lambda$$

Gauge ambiguity

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha \qquad \partial^\mu A_\mu \rightarrow \partial^\mu A_\mu + \partial^2 \alpha$$

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Choose as

$$-\frac{1}{\xi}\partial^\mu A_\mu$$

(gauge fixing)

Gauge ambiguity

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha \qquad\qquad \partial^\mu A_\mu \rightarrow \partial^\mu A_\mu + \partial^2 \alpha$$

i.e. with suitable "gauge" choice of α ("ξ" gauge) want to solve

$$\partial_\mu \partial^\mu A^\nu - (1-\frac{1}{\xi})\partial^\nu (\partial_\mu A^\mu) \equiv (g^{\nu\lambda}\partial^2 - (1-\frac{1}{\xi})\partial^\nu \partial^\lambda) A_\lambda = j^\nu$$

Now invertible … in momentum space the photon propagator is

$$-i\left(g^{\mu\nu} p^2 - (1-\frac{1}{\xi}) p^\mu p^\nu \right)^{-1} = \frac{i}{p^2}\left(-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2} \right)$$

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i.e. with suitable "gauge" choice of α ("ξ" gauge) want to solve

$$\partial_\mu \partial^\mu A^\nu - (1 - \frac{1}{\xi}) \partial^\nu (\partial_\mu A^\mu) \equiv (g^{\nu\lambda} \partial^2 - (1 - \frac{1}{\xi}) \partial^\nu \partial^\lambda) A_\lambda = j^\nu$$

We have used choice of a specific gauge transformation to modify the equation of motion. The question is how do you modify the Lagrangian to get this equation of motion? Will need to add a ``gauge fixing" term such that it is no longer gauge invariant and gives this equation of motion....what is this term?

Fixing the gauge – path integral formalism

$$I \equiv \int DA e^{iS(A)}$$ path integral

Suppose under $A \to A_g$, $S(A) = S(A_g)$ and $DA = DA_g$

Transformations form a group, under g, g', $A_g \to (A_g)_{g'} = A_{gg'}$

Want to rewrite in the form $I = \int dg \, J$ with J independent of g

Redundant integration

Volume of the group

Fixing the gauge – path integral formalism

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Suppose under $A \to A_g,\ S(A) = S(A_g)$ and $DA = DA_g$

Transformations form a group, under g, g', $A_g \to (A_g)_{g'} = A_{gg'}$

Redundant integration

Want to rewrite in the form $I = \left(\int dg\right) J$ with J independent of g

so that remaining path integral has no redundant variables

c.f. $I = \int dx\, dy\, e^{iS(x,y)} \equiv \int dx\, dy\, e^{iS(x^2+y^2)}$

$I = \left(\int d\theta\right) J = (2\pi) J$ where $J = \int dr\, r\, e^{iS(r)}$

volume of group of rotations in 2 dimensions

Faddeev, Popov gauge fixing

will be gauge fixing term

Define $$1 = \Delta(A) \int Dg\,\delta[f(A_g)]$$

Faddeev Popov determinant

$$[\Delta(A_{g'})]^{-1} = \int Dg\,\delta[f(A_{g'g})] = \int Dg''\,\delta[f(A_{g''})] = [\Delta(A)]^{-1}$$

$Dg'' = Dg$

$i.e.\ \Delta(\mathrm{A}) = \Delta(\mathrm{A}_g)$

Faddeev, Popov gauge fixing

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Define $$1 = \Delta(A) \int Dg\,\delta[f(A_g)]$$

Faddeev Popov determinant

$$[\Delta(A_{g'})]^{-1} = \int Dg\,\delta[f(A_{g'g})] = \int Dg''\delta[f(A_{g''})] = [\Delta(A)]^{-1}$$

$Dg'' = Dg$

$$i.e.\ \ \Delta(\mathrm{A}) = \Delta(\mathrm{A}_g)$$

$I \equiv \int DA\, e^{iS(A)}$

$$I = \int DA e^{iS(A)}$$
$$= \int DA e^{iS(A)} \Delta(A) \int Dg\,\delta[f(A_g)]$$
$$= \int Dg \int DA e^{iS(A)} \Delta(A)\delta[f(A_g)] \;= \left(\int Dg\right) \int DA e^{iS(A)} \Delta(A)\delta[f(A)]$$

No redundant variables

changing $A \to A_{g^{-1}}$ and noting $DA, S(A), \Delta(A)$ are invariant

Fixing the electromagnetic gauge

$$I = \left(\int Dg\right) \int DA e^{iS(A)} \Delta(A) \delta[f(A)]$$

$$A_g = A_\mu - \partial_\mu \Lambda$$

$\Lambda \equiv g$

Choose $f(A) = \partial_\mu A^\mu - \sigma(x)$

Fixing the electromagnetic gauge

$$I = \left(\int Dg\right) \int DA e^{iS(A)} \Delta(A) \delta[f(A)]$$

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$\Lambda \equiv g$

Choose $f(A) = \partial_\mu A^\mu - \sigma(x)$

$$\Delta(A)^{-1} = \int Dg\, \delta\left[f(A_g)\right] = \int D\Lambda\, \delta(\partial A - \partial^2 \Lambda - \sigma) \text{ "="} \int D\Lambda\, \delta(\partial^2 \Lambda)$$

i.e. Δ(A) is const …absorb in normalisation

Fixing the electromagnetic gauge

$$I = \left(\int Dg\right) \int DA e^{iS(A)} \Delta(A) \delta[f(A)]$$

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i.e. Δ(A) is const …absorb in normalisation

$I \equiv \int DA \, e^{iS(A)}$

Since I is independent of f, we can integrate I with an arbitrary functional of σ

$$Z = \int D\sigma e^{-(i/2\xi)\int d^4x \sigma(x)^2} \int DA e^{iS(A)} \delta(\partial A - \sigma)$$

$$= \int DA e^{iS(A) - (i/2\xi)\int d^4x (\partial A)^2}$$

$$Z = \int D\sigma e^{-(i/2\xi)\int d^4x\sigma(x)^2} \int DAe^{iS(A)}\delta(\partial A - \sigma)$$

$$= \int DAe^{iS(A)-(i/2\xi)\int d^4x(\partial A)^2}$$

$$S_{eff}(A) = S(A) - \frac{1}{2\xi}\int d^4x(\partial A)^2$$

$$= \int d^4x \left\{\frac{1}{2}A_\mu\left[\partial^2 g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]A_\nu + A_\mu J^\mu\right\}$$

R_ξ gauge

i.e.

$$L_{gauge\ fixing} = -\frac{1}{2}\xi^{-1}\partial^\mu A_\mu \partial^\nu A_\nu$$

R_ξ gauge

Propagator:

$$-i\left(g^{\mu\nu}p^2 - (1-\frac{1}{\xi})p^\mu p^\nu\right)^{-1} = \frac{i}{p^2}\left(-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)$$

Propagator:

$$-i\left(g^{\mu\nu}p^2-(1-\frac{1}{\xi})p^\mu p^\nu\right)^{-1}=\frac{i}{p^2}\left(-g_{\mu\nu}+(1-\xi)\frac{p_\mu p_\nu}{p^2}\right)$$

$\xi=1$: $$i\Pi^{\mu\nu}(p)=\frac{-ig^{\mu\nu}}{p^2+i\varepsilon}$$ 't Hooft - Feynman gauge

$\xi=0$: $$i\Pi^{\mu\nu}(p)=-i\frac{g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}}{p^2+i\varepsilon}$$ Lorentz gauge