Introduction to Quantum Field Theory
Michaelmas Term 2013
Problem Set 2

Although everyone in the class is encouraged to try these problems, only the papers of students in Theoretical Physics (or any others taking the course for credit who have notified us) will be marked. Completed problem sets should be handed in by the lecture on **Tuesday December 3**. Papers of Theoretical Physics students will be marked by **Juergen Rohrwild** in Theoretical Physics (Office: DWB Dalitz 9a), who will also review the problems during the **Problem Class on Thursday December 5, in the Fisher Room, 12-2pm**. The problem sets will also be available from http://www2.physics.ox.ac.uk/contacts/people/keranen. Any questions about the problem sets should be directed to obannon@physics.ox.ac.uk or Ville.Keranen@physics.ox.ac.uk or juergen.rohrwild@physics.ox.ac.uk.

1. Consider a theory with two scalar fields $\phi^{(1)}$ and $\phi^{(2)}$ and a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{2}((\partial\phi^{(j)})^2 + m_0^2(\phi^{(j)})^2) + \frac{\lambda_0}{4!}((\phi^{(1)})^4 + (\phi^{(2)})^4) + \frac{\sigma_0}{4}(\phi^{(1)})^2(\phi^{(2)})^2. \qquad (1)$$

Assume that the bare mass m_0 is adjusted so that both fields have zero renormalized mass.

a) Calculate, within any suitable renormalization scheme, the renormalized couplings λ and σ to 1-loop order in λ_0 and σ_0.

b) Work out the beta-functions

$$\beta_\lambda = \mu\frac{\partial\lambda}{\partial\mu}\Big|_{\lambda_0,\sigma_0}, \quad \beta_\sigma = \mu\frac{\partial\sigma}{\partial\mu}\Big|_{\lambda_0,\sigma_0}, \qquad (2)$$

to 1-loop order in $d = 4-\epsilon$ dimensions.

c) Find the simultaneous zeroes of the beta-functions and analyse their IR stability in $4-\epsilon$ dimensions. Sketch the RG flows in the (λ,σ) plane.

2. In QCD, the renoralization group functions have the form $\beta(g) = -bg^3 + O(g^4)$ and $\gamma(g) = cg^2 + O(g^3)$, where b and c are positive constants.

a) QCD is believed to exhibit dynamical mass generation: there is no mass scale in the Lagrangian, nevertheless the physical particles have mass. By observing that the mass of, say, the proton must have the form $M = \mu f(g)$, but that M cannot depend on the renormalization scale, deduce how M must depend on g for small g.

b) What is the asymptotic behavior of $\Gamma^{(2)}(p)$ in this theory for $p \to \infty$? You will need to use the full solution of the Callan-Symanzik equation.

3. This final problem is a summary, covering most of the key concepts from the second half of the course in a sample theory: the theory of a massive real scalar field ϕ with bare interaction $\frac{1}{6}\lambda_0\phi^3$.

Strictly speaking, this theory is unphysical. Classically, the potential of this theory is unbounded from below, so the energy can become arbitrarily negative. The quantum theory thus has no ground state. This problem is invisible in perturbation theory. This question focuses on the perturbative expansion.

a) What is the critical dimension d_c in which this theory is exactly renormalizable?
b) In d_c, which of the $\Gamma^{(N)}$ contain primitive divergences, and how should these be made finite? (Note that $\Gamma^{(1)}$ and related "tadpole" diagrams can be removed by a constant shift of the field.)
c) In the massless theory, work out, in a suitable renormalization scheme, the renormalised coupling constant λ at one loop in $d = d_c - \epsilon$. (This involves computing two one-loop diagrams, one for $\Gamma^{(3)}$, and also the field renormalization from $\Gamma^{(2)}$ which in this theory has a contribution at one loop order.)
d) Work out the beta-function for λ at one loop.
e) By considering the same theory in Minkowski space, draw the diagrams contributing to the $\mathcal{T}$-matrix for $\phi\phi \to \phi\phi$ scattering to lowest non-trivial order in λ, and hence calculate the differential cross-section for this process in the center-of-mass frame, showing explicitly how it depends on energy E and scattering angle θ.
f) Is the interaction between the particles attractive or repulsive?