Introduction to Quantum Field Theory
Michaelmas Term 2013
Problem Set 1

Although everyone in the class is encouraged to try these problems, only the papers of students in Theoretical Physics (or any others taking the course for credit who have notified us) will be marked. Completed problem sets should be handed in by the lecture on **Thursday November 7**. Papers of Theoretical Physics students will be marked by **Juergen Rohrwild** in Theoretical Physics (Office: DWB Dalitz 9a), who will also review the problems during the **Problem Class on Tuesday November 12, in the Fisher Room, 11am-1pm**. The problem sets will also be available from http://www2.physics.ox.ac.uk/contacts/people/keranen. Any questions about the problem sets should be directed to obannon@physics.ox.ac.uk or Ville.Keranen@physics.ox.ac.uk or juergen.rohrwild@physics.ox.ac.uk.

1. Consider a one-dimensional quantum bosonic harmonic oscillator with Hamiltonian $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2$ in contact with a heat bath at temperature T.
a) Write down the expression (derived in class) for the partition function $Z = \mathrm{Tr}e^{-\beta\hat{H}}$ as a path integral over periodic functions $q(\tau)$ in imaginary time τ.
b) By expanding $q(\tau)$ in a suitable basis of functions, use the path integral to get an expression for $\langle q(\tau_1)q(\tau_2)\rangle$ at finite temperature T. Show that for large T your result agrees with what you get from classical statistical mechanics.
c) Compute the value of the partition function Z itself. Is your answer finite? If not, how can you make it so by adjusting the normalisation of the path integral? Show that the result is then the same as what you would get by computing Z as $\sum_n e^{-\beta E_n}$ where E_n is the energy of the nth eigenstate.

2. In the lectures we showed that the propagator for a free massive scalar in Euclidean field theory is

$$\Delta(x - x') = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip\cdot(x-x')}}{p^2 + m^2}. \tag{1}$$

a) Show that as $|x - x'| \to \infty$ this behaves like $e^{-m|x-x'|}$, and also compute the prefactor in front of this.
b) What is the corresponding result for Δ_F in Minkowski space? Distinguish between the answers for the space-like and time-like separations.

3. What are the momentum-space propagators of the Euclidean field theories with the following free actions? (You should be able to write these down more or less by inspection. There may be more than one propagator if there is more than one type of field.)

a) $S = \frac{1}{2}\int dxdydz \Big[(\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z^2\phi)^2\Big]$ (Note the ∂^2 in the last term.)

b) $S = \frac{1}{2}\int d^dx \Big[(\partial\phi_1)^2 + (\partial\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2) + 2\mu\phi_1\phi_2\Big]$

c) $S = \int d^Dx d\tau \Big[\frac{1}{2}(\phi^*\partial_\tau\phi - \phi\partial_\tau\phi^*) + (\nabla\phi^*)\cdot(\nabla\phi)\Big]$.

4. What are the vertices, in momentum space, corresponding to the following interaction terms in a scalar field theory? Draw the way they would appear at lowest order in a tree diagram, give the numerical factor, and label any lines as appropriate.

a) $\lambda((\phi^*)^4 + \phi^4)$ (ϕ is complex and the propagator is $\langle\phi\phi^*\rangle$)

b) λe^ϕ

c) $\lambda\phi(\partial_\mu\phi)(\partial^\mu\phi)$.

5. Consider a QFT with two real scalar fields ϕ and Φ and a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\Big((\partial\phi)^2 + m^2\phi^2 + (\partial\Phi)^2 + M^2\Phi^2 + \lambda\phi^2\Phi\Big). \tag{2}$$

Write down the Feynman rules for this theory (be careful to use a different sort of line for the propagators of different fields). Draw the tree and one loop diagrams that contribute to the correlation functions $\langle\phi\phi\rangle$, $\langle\Phi\Phi\rangle$ and $\langle\phi\phi\Phi\rangle$

Evaluate as far as you can the irreducible 1-loop diagram contributing to $\langle\phi\phi\Phi\rangle$. To simplify, you can assume that the external momenta (p_1, p_2, p_3) are all zero, but you should work in arbitrary dimension d. For what range of d is the diagram finite?

6. *Noether's Theorem* states that any continuous symmetry of the equations of motion of a physical system implies a conservation law, and vice-versa. To be concrete, consider a scalar field $\phi(x)$ over $\mathbb{R}^{D,1}$ with action $S = \int dt\, d^Dx\, \mathcal{L}(\phi, \partial_\mu\phi)$. A continuous transformation is a transformation continuously connected to the identity, *i.e.* admitting the infinitesimal form

$$\phi(x) \to \phi(x) + \delta\phi(x), \tag{3}$$

where $\delta\phi(x) \ll \phi(x)$, and $\delta\phi(x) \to 0$ is the identity transformation. Under the transformation in eq. (3), the Lagrangian changes as $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$. For eq. (3) to be a symmetry, $\delta\mathcal{L}$ must be a total derivative: $\delta\mathcal{L} = \partial_\mu K^\mu$ for some K^μ. If so, then the action will change only by boundary terms (integrate $\partial_\mu K^\mu$ by parts), which do not affect the derivation of the Euler-Lagrange equations of motion. In other words, if eq. (3) produces $\delta\mathcal{L} = \partial_\mu K^\mu$, then eq. (3) is a symmetry of the system.

The proof of Noether's theorem is simple. Under the transformation in eq. (3),

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right)\partial_\mu\delta\phi && \text{(4a)} \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi\right) + \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right)\right]\delta\phi && \text{(4b)} \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi\right), && \text{(4c)} \end{aligned}$$

where in the second equality we used the chain rule and in the final equality we used the equations of motion. If eq. (3) is a symmetry, then on the left-hand-side of eq. (4) we can take $\delta\mathcal{L} = \partial_\mu K^\mu$. We can then define a current

$$j^\mu \equiv \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi - K^\mu, \tag{5}$$

that is conserved: $\partial_\mu j^\mu = 0$. This completes the proof of Noether's theorem.

a) Show that the time component of the current, j^t, integrated over all space, does not change in time: $\frac{d}{dt}\int d^Dx\, j^t = 0$ (assuming that the spatial current $\vec{j}$ goes to zero at spatial infinity). The quantity $\int d^Dx\, j^t$ is thus called a conserved charge.

b) The Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ is clearly invariant ($\delta\mathcal{L} = 0$) under continuous shifts of the value of the field: $\phi(x) \to \phi(x) + \alpha$ with constant α. Derive the Noether current associated with this symmetry.

c) Consider a complex scalar field Φ with $\mathcal{L} = \partial_\mu\Phi\,\partial^\mu\Phi^* + V(\Phi^*\Phi)$. Clearly $\mathcal{L}$ is invariant ($\delta\mathcal{L} = 0$) under continuous shifts of the phase of the field: $\Phi(x) \to e^{i\alpha}\Phi(x)$ with constant α, or in infinitesimal form, $\Phi(x) \to \Phi(x) + i\alpha\Phi(x)$. Derive the Noether current associated with this symmetry. The conserved charge is particle number.

d) Consider a real scalar field with $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)$. Perform a spacetime translation: $\phi(x) \to \phi(x + a)$, with constant a^μ, or in infinitesimal form, $\phi(x) \to \phi(x) + a^\mu\partial_\mu\phi(x)$. Show that K^μ is non-zero in this case (that is, $\delta\mathcal{L} = \partial_\mu K^\mu \neq 0$). Derive the Noether current associated with this symmetry. Show that the Noether current can be written as $j^\mu = -a_\nu T^{\nu\mu}$. Here $T^{\nu\mu}$ is called the stress-energy tensor or energy-momentum tensor. What is the conserved charge(s)?