VORTEX NUCLEATION IN A ROTATING OPTICAL LATTICE OF ULTRACOLD ATOMS

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This thesis describes the design and implementation of a two-dimensional rotating optical lattice for ultracold atoms.

Rotating optical lattices for ultracold atoms have generated a great deal of theoretical interest due to the close analogy between electrons under a magnetic field and rotating neutral atoms. However, such is the difficulty associated with creating rotating lattice potentials which do not cause significant heating of the atoms there has been only one previous experimental realisation of a practical device, [Phys. Rev. Lett. 97, 240402 (2006)]. There S. Tung et al. reported that heating due to rotation limited the optical lattice to depths of less than 30% of the condensate’s chemical potential.

We describe a new method for generating a rotating optical lattice, based on acousto-optic deflection and forming the lattice in the focal plane of a high numerical aperture lens. The intrinsically smooth rotation and excellent degree of control over the rotation frequency enabled a new breed of experiments to be performed, allowing the weak-lattice regime to be surpassed for the first time. In particular we report on the first experiments with a rotating optical lattice where lattice depths were reached such that a two-dimensional array of weakly-linked condensates was created, realising a bosonic Josephson-junction array under an effective magnetic field. We observe a new method of vortex nucleation, whereby rotation-induced phase differences between neighbouring condensates are converted to vortices upon ramping down the lattice.

The arrangement for generating a rotating optical lattice also allowed the first realisation of a two-dimensional optical lattice with variable periodicity, known as an accordion lattice. Calibration of the accordion lattice is described, together with a discussion on the crossover from quantum to classical dynamics in large periodicity optical lattices.
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CHAPTER 1

Introduction

The realization of Bose-Einstein condensation in weakly interacting dilute alkali gases in 1995 [1, 2] presented a unique opportunity to study quantum mechanics on a macroscopic scale. As with the advent of other macroscopic quantum phenomena before it (superfluidity, superconductivity, the laser), the experimental work led to the Nobel prize being awarded to Eric Cornell, Carl Wiemann and Wolfgang Ketterle. Two fields of ultracold atom research which have since generated an array of striking experimental results and intense theoretical interest are ultracold atoms in optical lattices and the rotation of quantum gases.

Optical lattices, formed by standing wave intensity patterns of laser light, provide a periodic array of microtraps for neutral atoms. While initial experiments on Bose-Einstein condensates (BECs) focussed on exploring the matter-wave properties of a condensate, a 1998 paper by Jaksch and co-workers [3] and subsequent experimental work in Munich [4] on the superfluid-Mott insulator transition of ultracold atoms in an optical lattice realised a new regime in ultracold atomic research. This work represented a paradigm shift whereby a weakly interacting BEC, well described by mean-field theories developed by Bogoliubov and others in the fifties, could be used to explore strongly interacting many-body systems of interest.
to the condensed matter physicist in a new environment offering unrivalled experimental control. This clean access to many-body Hamiltonians is of fundamental importance in studying the dynamic behaviour of a range of condensed matter systems, including Hubbard and spin models, the Berezinskii-Kosterlitz-Thouless (BKT) crossover in two-dimensional gases [5], disordered systems (recently Anderson localization has been directly observed in cold atoms experiments [6, 7]) and high temperature fermionic superfluidity (with its important analogy to high temperature superconductivity in cuprates) [8]. It is the application of quantum degenerate Bose and Fermi gases to the problems of many-body physics which represents one of the most important avenues of research in the ultracold atom community today. Such experiments can be considered to be direct quantum simulators, essentially a very specific form of a quantum computer, implementing model many-body Hamiltonians. The beauty of ultracold atom experiments is that the long coherence times achievable and the unparalleled control of system parameters allow access to parameter regimes beyond that available to solid state experiments. It should also be stressed that the realization of many-body toy models in a cold atom context is of fundamental interest in itself.

Similarly rotating quantum gases have provided a plethora of beautiful experimental results, from the first nucleation of a quantized vortex in a weakly interacting condensate [9, 10], to the generation of large vortex lattices in Bose gases [11, 12], and rotating Fermi gases at the BCS-BEC crossover [13]. While the analogy between cold atoms in an optical lattice and electrons in a solid is conceptually clear a perhaps less obvious result is that rotating quantum gases are also predicted to enter the regime of strongly-interacting physics. Some of the most exotic strongly-correlated phenomena in condensed matter physics occur when systems are subjected to a large magnetic field, for example the fractional
quantum Hall effect [14]. The close analogy between the physics of rapidly rotating neutral atoms and electrons under a magnetic field has led to considerable theoretical and experiment interest in the possibility of achieving strongly-correlated quantum Hall states in a rapidly rotating atomic gas [15, 16].

It is in this context that ultracold quantum gases in rotating optical lattices find themselves at the intersection of two fields which have produced an impressive body of experimental and theoretical work. Rotating optical lattices for cold atoms have generated a large amount of interest in their own right, which can broadly be divided into three main areas:

1. Weak rotating optical lattices have been used to pin vortices [17] and theoretical work has shown that rich vortex lattice structures, including lattices of doubly-quantized vortices are predicted to emerge [18, 19];

2. Stronger optical lattice potentials with a large number of atoms per site (100-1000) realise the physics of Josephson-junction arrays (JJA) under magnetic fields [20, 21];

3. Arguably the most interesting regime is for a dilute gas in the tight binding regime in a rotating lattice where fractional quantum Hall physics is predicted to occur [22, 23, 24].

In all three regimes rich structures emerge that depend on the relative density of vortices and lattice sites.

This thesis is concerned with the creation and implementation of dynamic optical lattice potentials for cold atoms, with the motivation of facilitating cold atom experiments which can realise new quantum systems of interest. In particular we describe a novel realization of a rotating optical lattice based on acousto-optic
deflection which enabled a bosonic Josephson-junction array under an effective magnetic field to be created for the first time. In the only previous experiment to have employed a rotating optical lattice, Tung et al. [17] reported that heating due to mechanical instabilities and aberrations limited the rotating lattice to depths less than 30% of the condensate’s chemical potential. Our work represents an important step forward from a weak rotating optical lattice perturbing a BEC to a deep optical lattice forming an array of weakly-linked condensates with rotation-induced phase differences between neighbouring lattice sites.

In addition the first two-dimensional optical lattice with a variable lattice constant (an accordion lattice) is demonstrated. Having fast and flexible control over the lattice periodicity allows the experimentalist to rapidly scan large parameter spaces, a useful feature for potentially any optical lattice experiment, especially those concerned with superlattices (two superimposed optical lattices leading to a more complicated periodic structure).

1.1 Thesis overview

This thesis is structured as follows:

- Chapter 2 gives a succinct overview of the experimental apparatus and techniques used to cool $^87$Rb to quantum degeneracy, reproducibly creating condensates of $2 \times 10^5$ atoms with no visible thermal component. Recent additions and improvements to the apparatus which facilitated the experiments on optical lattices are highlighted.

- Chapter 3 introduces the theory of cold atoms in optical lattices, with an emphasis on the realization of two-dimensional Josephson-junction arrays.
1.1. Thesis overview

A description of the local wave function at a lattice site in the JJA regime is investigated analytically.

- Chapter 4 describes a novel optical arrangement based on acousto-optic deflection which can create smoothly rotating optical lattice potentials and a two-dimensional lattice with a variable periodicity in the focal plane of a high numerical aperture objective lens. This work was published in [25]. The alignment of the optical system is detailed, together with a method of stabilizing the depth of a rotating lattice.

- Chapter 5 details the implementation and calibration of a two-dimensional optical lattice with variable periodicity. Calibration of the lattice constant is achieved by direct imaging of the optical lattice intensity and by in situ absorption imaging of atoms in the optical lattice. Matter-wave diffraction from optical lattices of a range of different periodicities is investigated, finding results showing diffraction from larger period optical lattices can be described classically by the emergence of caustics. These results are in agreement with a recent experiment at NIST [26].

- Chapter 6 reviews the physics of rotating condensates relevant for the experiments described in Chapter 7 before introducing the theory of cold atoms in an optical lattice under rotation. The relation between a JJA under rotation and the uniformly-frustrated XY model is discussed along with the expected ground state vortex configurations of the system.

- In chapter 7 we report on the observation of vortex nucleation in a rotating optical lattice. The dependence of vortex nucleation on optical lattice depth and rotation frequency is investigated, as well as heating in the rotating
lattice. This work is described in [27]. The pinning of vortices by a weak rotating lattice potential is also investigated.
An in-depth account of the experimental apparatus and pathway to BEC is described in [28]. This chapter provides a terse overview of our $^{87}$Rb BEC machine, focussing on recent additions and improvements to the apparatus which have been introduced to carry out the experiments described in this thesis, namely axial absorption imaging, described in section 2.4.2, and the optical ‘sheet’ trap, detailed in section 2.5.

2.1 Vacuum system and double MOT arrangement

The apparatus used to cool $^{87}$Rb to quantum degeneracy for the experiments described in this thesis is around four years old. Like many BEC experiments of a
2.1. Vacuum system and double MOT arrangement

Figure 2.1: A scale diagram of the vacuum system. The pyramidal MOT and the second MOT are separated by a distance of 30cm.

similar age it is based on a double MOT arrangement [29]. A double MOT system is a standard method to facilitate the diametrically opposing ideals of having a high pressure vapour region for fast initial loading of a MOT and also requiring a low pressure region so magnetically trapped atoms have a long lifetime against background collisions. While many experimental cold atoms groups with double MOT systems use a 2D MOT [30, 31] as an initial source of cold atoms our apparatus employs the less widely used pyramidal MOT [32]. A scale diagram of the vacuum system is shown in Fig. 2.1. Differential pumping is achieved by a 5 mm diameter, 17 mm long canal in the back of the pyramidal chamber. A background pressure of $\sim 10^{-9}$ mbar is maintained in the pyramidal chamber by a Varian Triode 401/s ion pump, while a 551/s Varian Starcell ion pump is used on the lower pressure side. A non-evaporable getter (NEG) pump is also present on the Science chamber side, removing incident gas molecules (mostly hydrogen) by chemical reaction.

The whole vacuum system is mounted on a pair of rails along which it can be rolled. This is an extremely useful feature, enabling the cell to be moved out of
the region of the various magnetic coils used in the experiment. This allows the cell to be cleaned or for lattice beams at the position of the atoms to be directly imaged (see Section 4.3).

Rubidium is introduced into the pyramidal chamber by five dispensers (SAES getters) connected in series and positioned above the pyramid mirrors. These getters are designed to be used in pulsed operation, however we run the getters continuously at a current of 3.35 A. The current getters have been used for around four years, running ten to twelve hours a day, and are beginning to show signs of running out of rubidium. Over the last year of operation we have noticed a fall in the equilibrium level of the second MOT, from around $5 \times 10^8$ atoms to $3 \times 10^8$, and a decrease in the loading rate of the MOT, both indicative of a gradual decline of the rubidium vapour pressure in the pyramidal chamber. While this ultimately limits the final number of atoms in a condensate, the performance of the apparatus is still acceptable and has some way to fall before the getters have to be replaced and the system baked out again.

The pyramidal MOT is based around four mirrors forming an inverted pyramid with a hole in its vertex. The base of each mirror is 6 cm. The great advantage of the pyramidal MOT is that it only requires a single incident beam; the retro-reflection of this beam from the pyramid mirrors forms the six beams needed for a MOT, and the imbalance in one pair of MOT beams created by the hole in the vertex results in a flux of cold atoms into the Science chamber. This greatly simplifies the day-to-day running of the experiment, as once optimised the pyramid beam requires virtually no further alignment. The pyramid mirrors were coated in the Oxford Physics in-house thin film facility and were designed to have equal reflectivities for s- and p-polarizations to preserve the circular polarization of the
pyramid beam upon reflection\textsuperscript{1}. The beam incident on the pyramidal chamber has a $1/e^2$ width of 17 mm and is composed of 38 mW of cooling light and 7 mW of repumping light. The second MOT in the Science chamber is a standard six beam MOT, with 29 mW of cooling light and 0.5 mW of repumping light split between the six beams.

\textbf{2.2 Laser system}

Laser cooling and probing dilute alkali gases requires low line width (<1 MHz) laser light at a range of specific frequencies. Figure 2.2 shows the energy diagram of the D2 line of $^{87}$Rb with the relevant laser cooling transitions highlighted. The cooling light is red detuned by 15 MHz ($2.5\Gamma$, where $\Gamma$ is the D2 transition line width) to the $5^2S_{1/2} F = 2 - 5^2P_{3/2} F = 3$ transition. The cooling light off-resonantly excites around 1 in 250 atoms to the $5^2P_{3/2} F = 2$ level from which relaxation to the $5^2S_{1/2} F = 1$ level is electric dipole allowed. To prevent this unwanted optical pumping of atoms out of the cooling cycle repumping light resonant with the $5^2S_{1/2} F = 1 - 5^2P_{3/2} F = 2$ transition is applied.

Two external cavity diode lasers (ECDL) and two diode lasers in a slave configuration were used to generate the light for laser cooling and probing. The ECDLs were Toptica DL100 lasers; one was used for the cooling light, known as the master laser, and the other for the repumping light. We previously used Sanyo DL7140 diodes in the ECDLs however we have recently switched to using Eagleyard diodes (EYP-RWE-0790-04000-0750-SOT01) which have an anti-reflection coating on the front diode facet. This allows for much more convenient laser operation. There is no competition between external cavity modes and the internal diode modes.

\textsuperscript{1}An interesting question is whether pyramidal MOT performance degrades over time due to rubidium coating the surface of the mirrors, but we have not seen any evidence of this.
2.2. Laser system

![Diagram of hyperfine structure of the $^{87}\text{Rb}$ $^5\text{S}_{1/2} - ^5\text{P}_{3/2}$ transition (not to scale). The transitions relevant for laser cooling are highlighted and discussed in the text. The optical pumping transition is used to pump atoms into the $^5\text{S}_{1/2} F = 1 m_F = -1$ state for magnetic trapping.]

and the diodes require little or no coercion to hit the $^{87}\text{Rb}$ D2 lines (unlike with uncoated diodes such as the Sanyos which can be a little obstreperous). This made replacement of a dead diode a quick and relatively pain-free process. Unfortunately the anti-reflection coated diodes are a factor of 10 times more expensive.

The repumper laser was directly locked to the repumping transition using Doppler-free saturated absorption spectroscopy. The master laser was locked to the $^5\text{S}_{1/2} F = 2 - ^5\text{P}_{3/2} F = 2$ to $F = 3$ crossover and then shifted to the desired frequency detuning using an acousto-optic modulator (Crystal Technologies 3110-140). The Eagleyard diodes gave around 60 mW of light out of ECDLs. This was sufficient to provide repumping light, but more cooling light was required.
To achieve this two diode lasers in a slave configuration were used. These diodes were injected with $100 - 200 \mu W$ of light from the master laser, and their spectral characteristics followed this injection radiation. One slave laser provided light for the pyramidal MOT (Sharp GH0781JA2C\textsuperscript{2} giving 80mW) and the other for the second MOT (Sanyo DL7140 giving 60mW). The slave diodes were situated in homemade mounts and temperature stabilized by a Peltier element.

A portion of light from the master laser was also used for optical pumping and probing beams. The optical pumping light is resonant with the $5 \, ^2S_{1/2} \, F = 2 - 5 \, ^2P_{3/2} \, F = 2$ transition and was used at the end of a molasses stage to transfer atoms to the $5 \, ^2S_{1/2} \, F = 1 \, m_F = -1$ state used for magnetic trapping (see Section 2.3). The probe light was tuned to resonance with the $5 \, ^2S_{1/2} \, F = 2 - 5 \, ^2P_{3/2} \, F = 3$ transition for absorption imaging (see Section 2.4).

A scale diagram of the layout of the lasers and optics used in the experiment is shown in Fig. 2.3. Polarization-maintaining fibres were used to transfer light from the laser table to the vacuum table, decoupling the alignment of the pyramid and second MOTs from the alignment of the lasers and AOMs. Provided the temperature in the lab was stable (maintained at $20 \pm 1^\circ C$ when the air conditioning was working properly) alignment into the fibres only needed to be adjusted about once a week. The fibres also ensured the light on the vacuum table had a clean Gaussian spatial profile. Both optical tables were enclosed in boxed cages; this prevented background light in the lab from affecting the atoms and air currents from disturbing the laser alignment.

\textsuperscript{2}Now discontinued.
Figure 2.3: Scale diagram of the vacuum and laser tables. This diagram focuses on the arrangement to laser cool and probe the atoms. Optics associated with forming the optical sheet trap and optical lattice have been omitted for clarity and are introduced in Sections 2.5 and 4 respectively. Optics for the vertical MOT beams have also been omitted for clarity.
2.3 Magnetic trapping

The limitations to the lowest temperature and highest density achievable by conventional laser cooling restrict cold atomic clouds to phase-space densities a factor of $10^5 - 10^6$ below that needed to achieve Bose-Einstein condensation. In a MOT the densities are typically limited to $\sim 10^{11}$ cm$^{-3}$ by radiation pressure; at these densities reabsorption of an emitted photon by a neighbouring atom results in repulsion between atoms$^3$. Atoms in a MOT are usually cooled below the Doppler temperature, $k_B T_D = \hbar \Gamma/2$ ($146 \mu$K in $^{87}$Rb) $^3[34]$, by sub-Doppler processes $[35, 36]^4$, however the temperature is ultimately limited by an atom’s momentum recoil upon spontaneous emission. To reach quantum degeneracy trapping in a conservative potential and runaway evaporation must follow laser cooling. Various different methods now exist to achieve this such as hybrid magnetic/optical trapping $[37]$ or an all-optical path $[38]$ to BEC. We used magnetic trapping and forced radiofrequency evaporation.

2.3.1 Compressed MOT, molasses and optical pumping.

In order to transfer the coldest, densest cloud possible into the magnetic trap a compressed MOT (CMOT) stage and optical molasses stage were employed. The CMOT stage achieved two things: (i) the phase space density of the cloud of atoms was increased by suddenly jumping the quadrupole field gradient (from 10 G/cm to 12.5 G/cm) and changing the detuning of the cooling light from -15 MHz to -30 MHz (reducing radiation pressure limiting the MOT density); (ii) during the

$^3$Techniques exist to improve MOT density to $\sim 10^{12}$ cm$^{-3}$, such as using a dark MOT $[33]$ which was employed in the first experiment to reach Bose-Einstein condensation at JILA $[1]$.

$^4$Or at least they occur in alkali metals due to the hyperfine multilevel structure of a ground state. The alkaline earth elements with $^1S_0$ ground states (hence no multilevel structure) cannot exploit these sub-Doppler cooling mechanisms. However laser cooling on an intercombination line can produce sub-microkelvin temperatures as these transitions have very narrow line widths.
CMOT stage shim coils were used to shift the centre of the MOT to the centre of the magnetic trap (these do not generally coincide for a baseball trap, see Section 2.3.2). It is important to load the atomic cloud into the centre of the magnetic trap to avoid generating a dipole centre-of-mass motion (‘sloshing’) of the cloud which will lead to heating (indeed monitoring the sloshing of the cloud is how the matching of the trap centres was optimised).

After the CMOT stage an optical molasses stage was employed to further decrease the cloud temperature. The shim coils were used to cancel out background magnetic fields (the sub-Doppler cooling mechanisms such as Sisyphus cooling and polarization gradient cooling require the hyperfine Zeeman states to be degenerate) and the laser frequency was detuned to -60 MHz. The theoretical limit to the temperature after molasses is set by an atom’s recoil momentum upon spontaneous emission, and is known as the recoil temperature

\[ T_r = \frac{\hbar}{2m\lambda^2}, \]

where \( \lambda \) is the wavelength of the spontaneously emitted photon. \( T_r = 362 \text{ nK} \) for \(^{87}\text{Rb} \), however experimentally temperatures achieved are generally an order of magnitude above \( T_r \), though can still be well below \( T_D \). The equilibrium temperature at the end of the molasses stage depends on the laser detuning, \( \delta \), and intensity, \( I \), as \( k_B T \propto I/\delta \) [36], (hence increasing the detuning to 60 MHz) and we typically achieve temperatures of 40 – 50 \( \mu \text{K} \).

At the end of the molasses stage the atoms are distributed across the hyperfine Zeeman states of the \( 5{^2}\text{S}_{1/2} F = 2 \) level. We used the \( 5{^2}\text{S}_{1/2} F = 1, m_F = -1 \) state for magnetic trapping as discussed in the next section. The atoms were driven into this state by applying a small bias field and illuminating the atoms with circularly polarized light resonant with the \( {^2}\text{S}_{1/2} F = 2 \rightarrow {^2}\text{P}_{3/2} F = 2 \) transition to drive \( \sigma^- \) transitions. Around 60% of the atoms were optically pumped into the \( {^2}\text{S}_{1/2} F = 1, m_F = -1 \) state by a 800 \( \mu \text{s} \) pulse of optical pumping light.
2.3. Magnetic trapping

2.3.2 The baseball trap

Alkali atoms have an unpaired valence electron so their ground states possess a magnetic dipole moment. Cold atoms loaded into a magnetic trap after optical molasses generally move with sufficiently slow velocities that their magnetic dipole moment adiabatically follows the direction of the local magnetic field. Hence an atom in hyperfine state $|F, m_F\rangle$ with Landé g-factor $g_F$ experiences a potential

$$U(r) = -\mu \cdot B(r)$$  \hspace{1cm} (2.1)

$$= g_F m_F \mu_B B(r)$$  \hspace{1cm} (2.2)

where $\mu_B$ is the Bohr magneton. Thus depending on the orientation of the magnetic dipole moment its interaction with an inhomogeneous magnetic field can result in an atom experiencing a force towards the minimum of a magnetic field (low-field seeking states with $g_F m_F > 0$) or the maximum of a magnetic field (high-field seeking states with $g_F m_F < 0$). A maximum in the magnitude of a magnetic field is forbidden in a region devoid of current sources [39], so a high-field seeking state cannot be trapped. Maxwell’s equations do allow for a minimum in the magnitude of a magnetic field, enabling a low-field seeking dipole to be trapped.

The most prosaic type of magnetic trap for neutral atoms is a quadrupole field, formed for example by a pair of anti-Helmholtz coils. However in such a trap the field minimum occurs at $B = 0$ which leads to the following problem. A magnetic dipole precesses about the local field direction at the Larmour frequency, $\omega_L \sim \mu_B B/\hbar$. In order for the dipole to follow the field adiabatically $\omega_L$ must be much larger than the rate of change in magnetic field experienced due an atom’s motion. In a small volume of the trapping region around $B = 0$ this condition is no
longer obeyed and transitions occur to untrapped states (Majorana flops). Atoms initially loaded at microkelvin temperatures have orbits which do not significantly encompass this small volume around \( B = 0 \), and orbit in regions of the trap where the adiabaticity conditions are satisfied. However as a cloud is cooled towards condensation an atom’s orbit is more likely to be in the region where spin flips occur. These spin flips become a major source of loss from the trap as the cloud becomes colder and prevent quantum degeneracy from being reached.

Two generic types of traps have been employed in the field of ultracold atoms to circumvent this problem: (i) the TOP (time-averaged orbiting potential) trap, invented at JILA [40]; (ii) the Ioffe-Pritchard trap, originally described by Ioffe for plasmas before Pritchard introduced similar ideas to the neutral atom community [41]. The TOP trap employs a rotating bias field to shift the position of the magnetic field zero faster than the atoms can respond. A Ioffe-Pritchard trap uses a configuration of current-carrying wires which give a field with minimum \( B \neq 0 \). A range of variations of the Ioffe-Pritchard trap exist including QUIC traps and cloverleaf traps, with each specific design having its particular pros and cons regarding complexity, optical access, flexibility etc. Our magnetic trap is a baseball trap, a Ioffe-Pritchard variant where the current-carrying coils are wrapped in a shape somewhat like the seam on a baseball, as shown in Fig. 2.4.

The idea behind a Ioffe-Pritchard trap is to combine a quadrupole field in the radial plane with a bottle field in the axial direction. Consider a radial quadrupole field of the form

\[
B_q = b(xe_x - ye_y) 
= br(\cos 2\theta e_r - \sin 2\theta e_\theta),
\]
where $b$ is the radial field gradient. The magnitude of the magnetic field increases with $r$ so a low-field seeking state experiences a minimum potential along the line $x = y = 0$. Trapping in the $z$ direction can be introduced using a bottle field of the form (and here a uniform field, $B_0$ in the $z$ direction is also introduced)

$$B_b = B_0[1 + \frac{k^2}{2}(z^2 - \frac{r^2}{2})]e_z - \frac{B_0}{2}k^2ze_re_r,$$  \hspace{1cm} (2.5)

where $k^2$ is the fractional strength of the bottle field. This field produces a minimum in $|B|$ in the axial direction at $z = 0$ but also introduces an antitrapping field radially. However for a sufficiently strong radial quadrupole field radial trapping can be maintained.

The beauty of the baseball trap is that fields of the forms expressed in Eqns. 2.4 and 2.5 are produced by the single set of coils, a simplification on the original Ioffe-Pritchard design where four parallel wires were needed to produce $B_q$ and a pair of
pinch coils produced \( B_b \). By expanding a magnetic field of the form \( B = B_q + B_b \) about the origin and neglecting higher order terms one can show

\[
U = g_F m_F \mu_B |B(r)| \quad \text{(2.6)}
\]

\[
= g_F m_F \mu_B B_0 + \frac{1}{2} m \omega_r^2 r^2 + \frac{1}{2} m \omega_z^2 z^2, \quad \text{(2.7)}
\]

where

\[
\omega_r \propto \frac{b}{\sqrt{mB_0}}. \quad \text{(2.8)}
\]

Good analytical approximations can be derived for \( \omega_r \) and \( \omega_z \) for a standard Ioffe-Pritchard trap (relating \( \omega_r \) and \( \omega_z \) to the coil geometry, currents etc.) though we do not consider them here as they do not apply to the baseball trap (the trapping frequencies must be calculated numerically for a baseball trap and even then are not usually accurate to more than 10\% because of difficulties modelling the finite thickness of the wires etc. [42]). The important point to note is that \( \omega_r \) is inversely proportional to the square root of \( B_0 \), the uniform field along \( e_z \). Bias coils (a pair of Helmholtz coils) were used to produce a uniform field along \( e_z \) which opposed \( B_0 \) produced by the baseball trap, thus giving control over \( B_0 \) and enabling the trap to be tightened, or weakened, in the radial direction.

The cold atomic cloud after the molasses stage was initially loaded into a weak magnetic trap with 200 A through the baseball coils and 0 A through the bias coils. The currents were then ramped up over 1 s to 255 A in the baseball and bias coils, increasing \( b \) to 80 G/cm and reducing \( B_0 \) to 0.4 G. This resulted in a tight magnetic trap with frequencies \( \{\omega_r, \omega_z\}/2\pi = \{167, 5.4\} \), suitable for radiofrequency forced evaporation to take place.

The bias and baseball coils were made from hollow copper wires and so could
be directly water cooled. The baseball coils typically reached a stable temperature of 33°C (having increased from 17°C) during an experimental run.

**Gravitational sag and relaxing to the weak trap**

The action of gravity in the baseball trap is to offset the potential energy minimum from the minimum of the magnetic field in the vertical direction. For a harmonic trap the displacement is

\[ \Delta x = \frac{g}{\omega^2}, \]  

(2.9)

where \( g \) is the acceleration due to gravity. In the tight trap the displacement \( \Delta x \) was only a few microns, thus the cloud resided at the zero of the magnetic field and Eq. 2.7 is valid (as it considered an expansion about the field minimum).

As will be discussed in Section 2.5 it was desirable to relax the magnetic trap to a weaker configuration (referred to as the ‘weak’ trap), allowing (in conjunction with a light sheet potential) a more convenient condensate geometry for loading into a two-dimensional optical lattice. Ramping to the weak trap was achieved adiabatically by decreasing the current in the bias coils over 1 s, until the final trapping frequencies were \( \{ \omega_r, \omega_z \}/2\pi = \{20.1, 4.6\} \) Hz.

**Evaporative cooling: tight trap vs weak trap**

The tight trap was convenient for evaporative cooling for two reasons:

- The elastic collision rate in a thermal cloud (an important parameter for successful evaporative cooling, determining how fast thermalisation occurs and hence how quickly the rf knife can be lowered) is proportional to cloud density, \( n \), and \( n \propto \omega^2_r \omega_z \) for a thermal cloud.
• The small sag of the potential minimum from the magnetic field minimum in the tight trap means the rf knife will cut atoms from a uniform surface centred on the trap minimum. This allows 3D evaporative cooling. In the weak trap the large sag means the atoms lie far from the magnetic field minimum. As a result of the field gradient across the cloud the bottom surface of the condensate comes into resonance with the rf knife before any other surfaces do. This 1D evaporation is significantly less efficient than 3D cutting.

Thus quantum degeneracy was first achieved in the tight trap before moving to the weak trap.

2.4 Absorption imaging

The attenuation of a laser beam of frequency $\omega$ and intensity $I_0$ travelling in the $y$ direction through a cloud of atoms of density $n(r)$ is given by Beer’s Law

$$I(x, z) = I_0(x, z)e^{-D},$$

(2.10)

where $D$ is the optical density,

$$D(x, z) = \int n(r)\sigma_0\,dy \times \frac{1}{1 + I/I_{sat} + 4(\omega_0 - \omega)^2/\Gamma^2}.$$  

(2.11)

$\sigma_0$ is the absorption cross-section on resonance, $I_{sat}$ is the saturation intensity and $\Gamma$ the natural line width of the transition at $\omega_0$. Absorption imaging of cold atoms works by imaging the ‘shadow’ created in the probe beam by a cloud of atoms onto a CCD camera. It is convenient to work with weak probe beams such that
2.4. Absorption imaging

$I \leq I_{\text{sat}}$. For higher intensities the optical density decreases as a smaller fraction of photons are absorbed. (In fact Eqn 2.10 is only valid in this weak beam regime). In this limit the optical density experienced by a weak, resonant probe beam is

$$D(x, z) = \int n(r)\sigma_0 dy.$$  \hspace{1cm} (2.12)

The typical optical density of a condensate in our magnetic trap is around 100, which is too large to gain any information through absorption imaging as complete absorption of the probe beam occurs over the profile of the condensate. Out of necessity clouds are usually imaged in time-of-flight expansion after release from the magnetic trap. At turn off the interaction energy of the condensate is converted to kinetic energy, driving an initially rapid hydrodynamic expansion. The size of an interacting condensate after release can be calculated via a scaling transformation introduced by Castin and Dum [43]. Optical densities $\sim 1$ are reached after a time-of-flight of 20-30 ms, which is ideal for absorption imaging. Off-resonant dispersive imaging methods have been used to non-destructively image trapped condensates \textit{in situ}, including phase-contrast imaging, polarization-contrast imaging and dark-ground imaging [44, 45, 46]. However for the work described in this thesis absorption imaging was used exclusively.

Imaging could be implemented in two orthogonal directions: (i) in the $xz$ plane (transverse imaging); (ii) in the $xy$ plane (axial imaging), as discussed below. The same CCD camera, an Andor Ixon was used for both ($512 \times 512$ pixels with a pixel size of $16 \times 16 \mu m$). The camera has on-chip electron-multiplication gain and a quantum efficiency of 40\% at 780 nm, offering close to single-photon detection although we did not push the camera to these limits for absorption imaging.
2.4. Absorption imaging

2.4.1 Transverse imaging

The transverse imaging probe beam is shown as the yellow beam in Fig. 2.3, and allowed imaging of atoms in the $xz$-plane. Two achromatic lenses ($f = 12.5$ cm and $f = 25$ cm) focussed the probe beam on the Andor camera giving a magnification of $2^5$. The field of view was around 4 mm meaning the cloud could be tracked in time-of-flight without moving the probe beam (an object takes around 29 ms to fall 4 mm under gravity). Transverse imaging was used to monitor the condensate and extract parameters such as atom number, phase-space density and temperature. It also allowed observation of the diffraction from a one-dimensional accordion lattice, see Chapter 5. Just before imaging with the probe beam the atoms were transferred from the $5^2S_{1/2}|1, -1\rangle$ state to the $5^2S_{1/2}F = 2$ manifold by applying a 200 $\mu$s pulse of repumping light. The transverse probe pulse length was 200 $\mu$s with $I/I_{\text{sat}} = 1/10$. The probe beam was circularly polarized and propagated along a small bias field, driving the $5^2S_{1/2}|2, 2\rangle - 5^2P_{3/2}|3, 3\rangle$ transition. This transition has the largest Clebsch-Gordan coefficient, maximising the absorption signal.

2.4.2 Axial imaging

The axial axis of symmetry in the baseball trap is in the $z$ direction as shown by Fig. 2.4. The symmetry of the trap meant that for experiments involving two-dimensional optical lattices or rotation the radial ($xy$ plane in Fig. 2.4) plane was of the greatest interest: this was the plane in which an optical lattice was formed and rotation occurred about the axial direction. As such imaging in the axial direction was required, for example, to observe vortices.

\footnote{The magnification was measured by observing the displacement of a cloud, $s$, after release from the magnetic trap as a function of time and fitting to $s = s_0 + 0.5gMt^2$, where $g$ is the acceleration due to gravity and $M$ the magnification of the imaging system.}
2.4. Absorption imaging

Figure 2.5: Path of the axial probe beam (shown in yellow) through the apparatus. Two motorised flip mirrors (one on the laser table not shown in this figure) divert probe light through the axial probe fibre and along the path of the pyramid MOT beam. These motorised mirrors are triggered to rise only after the experimental sequence has started and the pyramid MOT is no longer needed. A manual flip mirror is used to switch between transverse and axial probe beams at the camera.

Figure 2.5 displays the axial imaging path and highlights the key features which made imaging in this direction possible. Unlike the case for transverse imaging the probe beam could not simply pass through two sides of the vacuum cell, rather the beam had to be sent all the way through the axis of the vacuum system, entering at the slit in the vertex of the pyramidal MOT mirrors. As the probe beam had to initially follow the same path as the pyramid beam (but could not come directly through the pyramid fibre due to the high magnification telescope after the fibre) a motorised flip mirror, triggered by computer control, was used to switch from the pyramidal MOT beam to the axial probe path after the experimental sequence had started. A second motorised flip mirror on the laser table (Fig. 2.3)
diverted probe light to the axial probe fibre. The motorised flip mirrors were made by Radiant Dyes (model RD-KLS-1) and were extremely reliable. They were also extremely stable; despite being raised and lowered $\sim 100$ times a day beam pointing deviations were negligible (the manufacturer’s specifications claim pointing deviations $< 10^{-4}$ rads which agrees with our observations).

**Resolution and depth of field**

As will be discussed in Chapter 4, the experiments described in this thesis involved forming a two-dimensional optical lattice in the focal plane of a high numerical aperture lens. The lens used was a custom-made multi-element design consisting of four individual lenses, and is detailed in Section 4.3.4. As well as being used to create the optical lattice it also served as the objective lens for absorption imaging. The four-lens objective was designed to have a numerical aperture of N.A. = 0.27, giving diffraction-limited resolution of $1.22\lambda/2\text{N.A.} = 1.8 \mu\text{m}$. As with transverse imaging, a simple optical system consisting of two components (the four-lens objective and a $f = 75$ cm achromatic doublet) focussed the image of the atoms onto the CCD, giving a measured magnification of 18.7 (consistent with the four-lens objective focal length $f = 40$ mm [47].)

While a high numerical aperture allowed high resolution imaging it also resulted in a small depth of field. The depth of field (or focal tolerance) is a measure of how far an object may move along the axis of an imaging system and remain in focus. A conventional definition of the depth of field, $\Delta u$, is the distance from the focal plane at which the first minimum of the Airy disk appears at the centre of the image. This criterion gives [48]

$$\Delta u = \pm \frac{2\lambda}{\text{N.A.}^2}$$  \hspace{1cm} (2.13)
For the axial imaging system $\Delta u \simeq 20 \mu m$. The depth of field is an important consideration when imaging an object of finite axial extent, for example the axial size of a condensate in time of flight could exceed $40 \mu m$. We discuss how this affected the visibility of vortices in Section 6.4.

**Axial beam parameters**

Having the constraint that the axial beam had to pass through the pyramid slit placed limitations on the size of the probe beam. As a result the axial probe beam was collimated at an $1/e^2$ waist of 1 mm (the pyramid slit was $1 \times 2$ mm). Without any flexibility in choosing the beam waist at the atoms the beam power and pulse length were chosen to best satisfy the requirements:

1. $I \leq I_{\text{sat}}$ is desired to ensure a significant fraction of probe photons are absorbed as discussed above;

2. The intensity and pulse length must be large enough to ensure there are enough counts at the CCD camera to achieve a good signal to noise ratio;

3. The intensity and/or the pulse length cannot be too large otherwise blurring of the image will occur due to atom recoil upon spontaneous emission in a direction perpendicular to the probe beam.

Consider the number of photons scattered by an atom during time $t$, $N_{\text{ph}} = R_{\text{scatt}} t$, where $R_{\text{scatt}}$ is the scattering rate$^6$. To estimate to what extent blurring due to the motion of atoms after spontaneous recoil may be a problem recall the random walk in velocity space in one dimension perpendicular to the probe beam will give an average velocity $\bar{v} = \sqrt{N_{\text{ph}} v_r/3}$, where $v_r = h/m\lambda$ is the recoil velocity (5.88 mm/s

\[^6\]R_{\text{scatt}} = \Gamma \frac{1/I_{\text{sat}}}{\Gamma/1/I_{\text{sat}} + 4(\omega_{0} - \omega)^2 \Gamma^2}$
for $^{87}\text{Rb}$). The average displacement during a probe pulse of length $\tau$ is then

\[
\bar{x} = \int_0^{\tau} \bar{v} \, dt 
\]

\[
= \frac{2}{9} \sqrt{R_{\text{scatt}} v_r \tau^{3/2}}. 
\]

(2.14) \hspace{1cm} (2.15)

For $I = 0.75I_{\text{sat}}$, $R_{\text{scatt}} = 0.4 \Gamma$ we find $\bar{x} \simeq 1 \mu m$ for a pulse length of $\tau = 50 \mu s$. This was acceptable given the resolution of the imaging system was $1.8 \mu m$, so blurring was not a problem for these parameters. While the counts at the camera where lower for these parameters than for transverse imaging we were able to compensate by increasing the on-chip electron multiplication gain of the CCD for axial imaging.

### 2.5 Dipole sheet trap

Bose-Einstein condensates have been made in a wonderful array of shapes and sizes, highlighting the impressive degree of control available to the ultracold physicist. Ioffe-Pritchard traps (with $\omega_r/\omega_z > 1$) produce cigar-shaped condensates while TOP traps ($\omega_r/\omega_z < 1$) allow oblate BECs. Extreme examples include 1D condensates on atom chips [49] and quasi-2D condensates in light sheet traps [50].

When loading a BEC into an optical lattice it is important to match the condensate dimensions to the optical potentials being considered. For example when working with a 3D optical lattice the trap is usually relaxed to $\omega_r = \omega_z$ to give a spherical condensate. This enables a more homogeneous filling of the optical lattice.

The experiments in this thesis focus on a 2D optical lattice in the radial plane of the condensate. In order to fill as many lattice sites as possible an oblate,
2.5. Dipole sheet trap

pancake-like condensate was desired. However the condensate was formed in the baseball trap with $\omega_r/\omega_z = 31$. Even upon relaxing to the weak trap as described in Section 2.3.2 the dimensions were still unfavourable: $R_z = 45 \, \mu m$, $R_r = 10 \, \mu m$ for a condensate of 200,000 atoms.

In order to achieve the desirable geometry we introduced a single light sheet propagating in the $y$ direction, with a frequency detuning below that of the atom’s resonance (red-detuned). The theory of optical dipole potentials is introduced in Section 3.1, where it is shown that the a.c. Stark shift experienced by an atom in a laser beam is proportional to the beam intensity. In a red-detuned beam an atom experiences a potential minimum at the point of maximum intensity. For a Gaussian beam with waists $w_i$ (for $i = x, z$) which vary as $w_i = w_{i,0}(1 + (y/b_i)^2)^{1/2}$ where $b_i = \pi w_{i,0}^2/\lambda$ are the corresponding Rayleigh lengths, the optical potential has the form

\[
V(x, y, z) = -V_0 \frac{e^{-2x^2/w_x^2}}{(1 + (y/b_x)^2)^{1/2}} \frac{e^{-2z^2/w_z^2}}{(1 + (y/b_z)^2)^{1/2}}. \tag{2.16}
\]

$V_0$ is the peak trap depth at $x = y = z = 0$, proportional to the peak intensity, $I_0$.

2.5.1 Implementation

To produce strong confinement in the axial direction requires $w_z \ll w_x$. The arrangement to achieve this was constrained by the fact the light sheet had to propagate either parallel or antiparallel to the transverse imaging beam and therefore the objective lens used for transverse imaging had to form part of the system\textsuperscript{7}. The solution is shown in Fig. 2.6. A cylindrical lens ($f_{\text{cyl}} = 15 \, \text{cm}$), orientated to

\textsuperscript{7}We experimented with light sheets coming in from angles out of the plane of the table but the solution here, keeping everything horizontal, was much more convenient.
2.5. Dipole sheet trap

Figure 2.6: Optical arrangement to create a light sheet at the centre of the magnetic trap with beam waists $w_z \ll w_x$. The objective lens and the cylindrical lens are separated by the sum of their focal lengths, acting as a telescope in the $x$ direction. The cylindrical lens does not affect $w_z$. The photodiode signal was fed into a PID circuit to servo the laser power. MOT optics are not shown on this diagram.

Focus light in the vertical ($x$) direction, and an achromatic doublet ($f_{\text{obj}} = 12.5\text{ cm}$ used as the objective lens for transverse imaging) were used in a telescope configuration (separated by the sum of their two focal lengths). The waist at the atoms in the vertical direction was thus given by

$$w_{x,0} = \frac{f_{\text{obj}}}{f_{\text{cyl}}} w_0,$$  \hspace{1cm} (2.17)

where $w_0$ was the waist of the collimated beam straight out of the light sheet fibre. The waist in the $z$ direction, unaffected by the cylindrical lens, was focussed by the second lens to a size

$$w_{z,0} = \frac{f_{\text{obj}} \lambda}{\pi w_0}.$$  \hspace{1cm} (2.18)
The beam after the fibre was imaged on a CCD camera and its $1/e^2$ waist measured to be $w_0 = 1.55\text{ mm}$. We also confirmed the beam profile was clean and not distorted which would affect the trapping potential experienced by the atoms. The waists of the sheet at the atoms were determined by measuring the beam size at several points after the cell and fitting the results to $w_i = w_{i,o}(1 + (y/b_i)^2)^{1/2}$. We found $w_z = 22 \pm 2\text{ µm}$ and $w_x = 1.30 \pm 0.01\text{ mm}$, in agreement with predicted values calculated from $w_0 = 1.55\text{ mm}$ and Eqs. 2.17 and 2.18. Given that $b_x \approx 6\text{ m}$ Eq. 2.16 can be approximated by

$$V(x, y, z) = -V_0 e^{-2x^2/w_x^2} e^{-2z^2/w_z^2} (1 + (y/b_z)^2)^{1/2}. \tag{2.19}$$

Cold atoms in the ground state of the combined magnetic plus light sheet potential are localised in the region $y \ll b_z$, $z < w_z$ and $x \ll w_x$, such that it is a good approximation to only keep terms up to order $y^2/b_x^2$, $z^2/w_z^2$ and $x^2/w_x^2$ in a power series expansion of Eq. 2.19. Thus close to the focus of the light sheet the potential can be written

$$V(x, y, z) = -V_0 + \frac{1}{2} m \tilde{\omega}_x^2 x^2 + \frac{1}{2} m \tilde{\omega}_y^2 y^2 + \frac{1}{2} m \tilde{\omega}_z^2 z^2, \tag{2.20}$$

where the oscillation frequencies of the atoms are

$$\tilde{\omega}_x = \left( \frac{4V_0}{mw_x^2} \right)^{1/2}, \quad \tilde{\omega}_z = \left( \frac{4V_0}{mw_z^2} \right)^{1/2}, \quad \tilde{\omega}_y = \left( \frac{V_0}{mb_z^2} \right)^{1/2}. \tag{2.21}$$

The tilde has been used to prevent confusion with the frequencies in the magnetic trap. The dependence of $V_0$ on the intensity and frequency detuning of the light sheet is given in Eq. 3.13. To ensure the light sheet was focussed at the centre of the condensate the axial frequency was measured as a function of focal position. The
2.5. Dipole sheet trap

A 12.5 cm objective lens was mounted on a translation stage allowing the focus to be scanned, while the trapping frequency was measured by observing the dipole oscillation of a condensate’s centre of mass (initiated by displacing the centre of the magnetic trap). The total axial trapping frequency ($\tilde{\omega}_z$ added in quadrature with the magnetic trapping frequency) will vary with $y$ according

$$\omega_z = (\omega_{\text{mag},z}^2 + \tilde{\omega}_z^2)^{1/2} = \left[\omega_{\text{mag},z}^2 + \left(\frac{4V_0}{m w_{z,0}^2 (1 + (y/b_z)^2)^{3/2}}\right)\right]^{1/2}. \quad (2.22)$$

The measured axial trapping frequencies are shown in Fig. 2.7, and fit well to the predictions of Eq. 2.22, shown by the solid blue line. We were thus able to ensure the focus of the light sheet was located at the centre of the condensate, which is important for achieving the maximum axial trapping frequency for a given laser power.

2.5.2 Narrowband light vs broadband light

Optical dipole traps are widely used in cold atom experiments, typically as an alternative to magnetic traps, offering advantages including the ability to trap any hyperfine state of an atom or atoms without a magnetic dipole moment. However, whereas magnetic traps offer intrinsically smooth trapping potentials, making a correspondingly smooth optical potential requires more effort in the laboratory. Diffraction from bits of dust on optical elements degrades beam quality and multiple reflections at a vacuum cell wall or optical components cause interference fringes to appear in the beam profile.

The light sheet used to increase the axial trapping frequency must be particularly smooth for two reasons: (i) noise on the trapping potential would give rise to irregularities in the filling of an optical lattice; (ii) for rotation experiments a
smooth potential is crucial as static perturbations in the lab frame will act to spin down the rotating cloud. Unfortunately upon first realising the light sheet it was clear that it was not a smooth potential.

Figure 2.8 shows images of condensates taken in axial imaging after 20 ms time-of-flight; those released from the light sheet display significant distortions. Figure 2.8 (a) displays an undistorted condensate (released from the magnetic trap and not held in the light sheet), while (b)-(d) show the distortions evident when the cloud sloshed in the combined magnetic and optical potential. This level of perturbation implied an unacceptable amount of noise in the optical trapping potential. The beam profile after the light sheet fibre was clean and high quality optical components were used to create the slice. This suggested the problem arose from interference fringes caused by multiple reflections at the vacuum cell wall or
2.5. Dipole sheet trap

Figure 2.8: False colour images of condensates taken using axial imaging after 20 ms time-of-flight: (a) a condensate released from the magnetic trap; (b)-(d) examples of the distorted condensate shapes seen with the combined magnetic and light sheet potential.
dust on the vacuum cell causing diffraction.

The light for the optical trap had been from a single-mode Ti:sapphire laser, with line width < 1 MHz giving a coherence length > 100 m. In order to get rid of any stray interference fringes in the light sheet we switched to a multimode diode laser with a short coherence length. The use of broadband light to create conservative trapping potentials is a relatively new method in the cold atom community, despite initially being considered ten years ago at Strathclyde University by Clarke et al. [51]. The Strathclyde group compared narrow-bandwidth and mode-locked lasers for dipole traps with the idea that pulsed lasers can be more efficiently frequency doubled, allowing a greater range of wavelengths to be reached in order to trap different atomic species (species with transitions in the blue or uv such as chromium and indium). They noted that cw and pulsed lasers produced essentially the same trapping effect providing light was not sufficiently broadband that photoassociation became a problem.

We were inspired to try a broadband light source after hearing anecdotal evidence of multimode fibre lasers producing smoother trapping potentials than single-mode fibre lasers. Recently however the idea has been impressively extended by Markus Greiner’s group in Harvard, in their spectacular paper showing in situ single atom resolution in a 620 nm period optical lattice [52]. The Harvard group formed the optical lattice with broadband light, using a holographic projection technique rather than standard interference methods. This allowed them to form an optical lattice free from any stray interference effects.

We used a multimode diode laser from Roithner Lasertechnik (Model QL86T4HD) with bandwidth $\Delta \lambda \sim 3$ nm centred on 865 nm, giving a coherence length $\sim 250 \mu m$. This was small enough that multiple reflections from the vacuum cell wall (thickness 2 mm) were incoherent with each other. The advantages of using a
multimode diode rather than a solid state pulsed laser were that the diode solution was cheap and we have a large amount of experience dealing with diode lasers. Although the Roithner diode produced 1 W of optical power the light was spatially multimode as well as longitudinally multimode, severely restricting the amount of light that could be coupled through the single-mode fibre (10% coupling efficiency was achieved). This restricted the maximum axial trapping frequency to 53 Hz. After switching to the broadband light source and cleaning the vacuum cell wall distortions of the condensate were no longer observed, and condensates released from the combined magnetic and optical potential looked essentially the same as in Fig. 2.8(a).

### 2.5.3 Adiabaticity requirements

When increasing the axial frequency of the trap the rate of change of $\omega_z$ must be sufficiently slow to avoid heating the condensate. The adiabaticity condition

$$\frac{d\omega_z}{dt} \ll \omega_z^2$$

was satisfied by exponentially ramping the amplitude of the light sheet trap to its final value over 500 ms. The ramp for the light sheet power had the form $P = P_{\text{final}}(\exp(3t/\tau) - 1)/(\exp(3) - 1)$, with $\tau = 500$ ms, meaning the total axial trapping frequency varied as

$$\omega_z = \left(\omega_{\text{mag},z}^2 + \omega_{\text{final},z}^2 \frac{\exp(3t/\tau) - 1}{\exp(3) - 1}\right)^{1/2}.$$  \hspace{1cm} (2.24)

For the experiments detailed in this thesis the final axial frequency was $\omega_z/2\pi = 53$ Hz.
2.5.4 Contribution to the transverse trapping frequencies

Equation 2.21 shows the contribution of the light sheet to the trapping frequencies in the $x$ and $y$ directions. For the measured axial frequency of $\omega_z/2\pi = 53$ Hz the corresponding transverse trapping frequencies are calculated to be $\tilde{\omega}_x/2\pi = 0.9$ Hz, $\tilde{\omega}_y/2\pi = 0.3$ Hz. When added in quadrature with the magnetic trap frequency of 20 Hz the change in the total trapping frequency is only around 0.1%.

It is important that the light sheet does not significantly contribute to $\omega_x$ and $\omega_y$, since generally $\tilde{\omega}_x \neq \tilde{\omega}_y$, and it would act to increase the asymmetry of the magnetic trap. The degree of asymmetry in the radial trapping is a very important consideration for rotation experiments. A non-zero trap ellipticity means angular momentum will not be conserved in the trap and a rotating cloud will gradually spin down. A standard definition for the trap ellipticity is

$$\epsilon_{\text{trap}} = \frac{\omega_x^2 - \omega_y^2}{\omega_x^2 + \omega_y^2}. \quad (2.25)$$

The limiting factors determining the asymmetry in the trap were imperfections in the geometry of the baseball trap and the asymmetry introduced by the sag in the weak trap. A careful measurement of the dipole motion oscillation frequencies in the combined baseball and light sheet trap yielded

$$\{\omega_x, \omega_y\}/2\pi = \{20.07(3), 20.23(3)\} \text{ Hz}, \quad (2.26)$$

giving $\epsilon_{\text{trap}} = 0.008$. For a Ioffe-Pritchard type magnetic trap this value is slightly better than average. Eric Cornell’s group in JILA carefully distort the rotating bias field of their TOP trap to achieve trap ellipticities of less than 0.001, allowing angular rotation to persist for minutes [53, 54].
2.6 Experimental sequence overview

To summarise the path to quantum degeneracy:

- $^{87}\text{Rb}$ atoms are loaded into the second MOT from the pyramidal MOT. At the point which the second MOT fluorescence (monitored on a photodiode) reaches the level corresponding to $3 \times 10^8$ atoms (this takes around 60 s) the experimental sequence begins;

- the CMOT stage increases the phase space density and shifts the position of the atomic cloud to the centre of the baseball trap. An optical molasses stage further cools the cloud before optical pumping is employed to transfer the atoms in the $|F = 1, m_F = -1\rangle$ state.

- The baseball trap is switched on in $\sim 1\text{ms}$ and the current in the bias coils are ramped up over 1 s to lower the bottom of the magnetic trap and increase the trapping frequencies to $\{\omega_r, \omega_z\}/2\pi = \{167, 5.4\}$ Hz.

- Radiofrequency evaporation takes around 80 s to increase the phase-space density of the $^{87}\text{Rb}$ cloud to $\sim 1$ at which point quantum degeneracy is achieved.

At this point we typically have a condensate with $2 \times 10^5$ atoms with no visible thermal component which is held in the tight magnetic trap. Figure 2.9 shows a series of snapshots of the cloud density profile as the cloud was cooled through the transition to Bose-Einstein condensation. The condensate was cigar shaped with an axial length of around 200 $\mu$m and radial diameter 6 $\mu$m while held in the magnetic trap. Upon relaxing the weak trap and ramping up the light sheet the final trap frequencies were $\{\omega_x, \omega_y, \omega_z\}/2\pi = \{20.07, 20.23, 53.0\}$ Hz. The dimensions of a condensate of 200,000 atoms were $R_r = 17 \mu$m, $R_z = 6 \mu$m in the final trap.
Figure 2.9: A series of experimental images of atomic optical density after 31ms time-of-flight highlighting the transition of a thermal cloud to BEC. A dense parabolic profile, characteristic of a condensate, rises out of the Gaussian background of thermal atoms as the final radiofrequency evaporation value (noted beside each image) is lowered.
Chapter 3

Theory of ultracold atoms in optical lattices

Optical lattices, formed by standing wave intensity patterns of laser light, provide a periodic array of microtraps for neutral atoms. With the advent of Bose-Einstein condensation atoms could be loaded into the lowest band of an optical lattice for the first time, realising analogues of condensed matter systems with unparalleled experimental control and purity. In this chapter we discuss the theory of cold atoms in an optical lattice, including the realisation of a bosonic Josephson-junction array in a two-dimensional optical lattice.

That a standing light wave could be used to trap atoms was first proposed in 1968 by Vladilen Letokhov [55], and subsequently realised in the late eighties. Initial experiments focussed on trapping atoms in near-resonant optical lattices, where cooling due to sub-Doppler processes could occur [56, 57]. It was understood early on that cold atoms in optical lattices could be used to create a system analogous to electrons in a crystal. However experiments in the eighties and nineties
could not achieve a sufficiently high filling fraction to realise an optical crystal. This was because optical lattices were loaded from a MOT or optical molasses with atom densities of $10^{11} - 10^{12}$ cm$^{-3}$, giving filling factors of only a few percent. However the realisation of Bose-Einstein condensate in dilute alkali gases changed this dramatically. A 1998 paper by Jaksch et al. [3] showed that ultracold atoms in an optical lattice could be described by a Bose-Hubbard model which predicts a phase transition from a superfluid phase to a Mott insulator. Subsequent experimental work in Munich [4] observed the superfluid to Mott insulator transition, launching ultracold atomic physics into the regime of strongly-correlated systems.

Since then ultracold atoms in optical lattices have been responsible for a series of striking experimental results including the realisation of Tonks-Girardeau gases [58], Anderson localisation [6, 7], and insulating phases of Fermi gases [59, 60].

3.1 Theory of the dipole force

While laser cooling relies on the dissipative force neutral atoms experience due to absorption and subsequent spontaneous emission of photons, this is not the only important force that arises in light-atom interactions. An oscillating electric field induces an electric dipole moment in a neutral atom; the interaction between the in-phase component of this dipole moment and the oscillating field produces a shift in an atom’s internal energy (the a.c. Stark effect). In a spatially inhomogeneous light field we will show the potential energy of an atom follows the intensity profile of the light field, allowing a conservative trapping potential to be realised.

Consider an atom with complex polarizibility $\alpha$ in an oscillating electric field
3.1. Theory of the dipole force

$E = E_0 \cos \omega t$. The dipole moment of the atom has the form

$$d = \alpha(\omega)E.$$  \hfill (3.1)

The time-averaged energy of the induced dipole in the electric field is

$$V = -\frac{1}{2} \langle d \cdot E \rangle$$  \hfill (3.2)

$$= -\frac{1}{2\epsilon_0 c} \text{Re}(\alpha)I$$  \hfill (3.3)

that is, linearly dependent on the real part of the atomic polarizability and the intensity of the light field ($I = (1/2)\epsilon_0 c E_0^2$). The power absorbed by the oscillating dipole and subsequently reradiated via spontaneous emission is $\langle \dot{d} \cdot E \rangle$. Thus the rate of spontaneous scattering of photons, $\Gamma_{sc}$, is related to the imaginary part of the polarizability by

$$\Gamma_{sc} = \frac{\langle \dot{d} \cdot E \rangle}{\hbar \omega}$$  \hfill (3.4)

$$= \frac{1}{\hbar \epsilon_0 c} \text{Im}(\alpha)I.$$  \hfill (3.5)

Equations 3.3 and 3.5 apply to any polarizable neutral particle, whether the light is near-resonance or far-detuned from a transition.

A rigorous calculation of $\alpha$ is intensive work: see [61] for a calculation of polarizabilities in Rb using relativistic many-body calculations. Here we consider the polarizability due to the valence electron exclusively (a good approximation as the inner core of electrons has zero net orbital angular momentum and a small polarizability). In Appendix A we derive the light shift experienced by an atom in a monochromatic field by treating the light-atom interaction Hamiltonian
3.1. Theory of the dipole force

\[ H_{\text{int}} = -\mu \cdot \mathbf{E} \] using time-dependent perturbation theory (where \( \mu = -e \mathbf{r} \) is the electric dipole operator). For a two-level atom with ground and excited states \( |g\rangle \), \( |e\rangle \) separated in energy by \( \hbar \omega_0 \) we find,

\[ V = -\frac{E_0^2}{4\hbar} \langle \langle e | \mu | g \rangle \rangle^2 \left( \frac{1}{\omega_0 + \omega} + \frac{1}{\omega_0 - \omega} \right). \tag{3.6} \]

The line width of the transition can be written (see for example [62] pg. 197) as

\[ \Gamma = \frac{\omega_0^3}{3\pi \epsilon_0 \hbar c^3} \langle \langle e | \mu | g \rangle \rangle^2. \tag{3.7} \]

Hence Eq. 3.6 is related to the line width by

\[ V(r) = -\frac{3\pi c^2}{2\omega_0^3} \Gamma \left( \frac{1}{\omega_0 - \omega} + \frac{1}{\omega_0 + \omega} \right) I(r). \tag{3.8} \]

This result agrees with that derived in the review by Grimm et al. [63] by considering the classical oscillator model of a dipole. Similarly the scattering rate can be shown to be [63]

\[ \Gamma_{\text{sc}}(r) = -\frac{3\pi c^2}{2\hbar \omega_0^3} \left( \frac{\omega}{\omega_0} \right)^3 \Gamma^2 \left( \frac{1}{\omega_0 - \omega} + \frac{1}{\omega_0 + \omega} \right)^2 I(r). \tag{3.9} \]

Validity of the rotating wave approximation

When considering the interaction of a two-level atom with a monochromatic field it is common to make the rotating wave approximation (RWA). For Eqs. 3.8 and 3.9 this corresponds to neglecting the \( 1/(\omega_0 + \omega) \) term which is a good approximation when \( |\omega - \omega_0| \ll \omega_0 \), as then the \( 1/(\omega_0 - \omega) \) term dominates. Under the rotating wave approximation the Hamiltonian describing the interaction of an oscillating electric field with a two-level atom can be solved exactly, without recourse to
perturbation theory, as is shown in Appendix B. There the RWA allows rapidly rotating terms of the form \( \exp(i(\omega_0 + \omega)t) \) to be neglected. In Appendix B we derive the light shift to be

\[
V(r) = \frac{3\pi e^2}{2\omega_0^2} \frac{\Gamma}{\delta} I(r),
\]

(3.10)

where \( \delta = \omega - \omega_0 \). Clearly Eq. 3.8 agrees with this result in the limit \( \delta \ll \omega_0 \). Similarly under the rotating wave-approximation Eq. 3.9 becomes

\[
\Gamma_{sc}(r) = \frac{3\pi e^2}{2\hbar \omega_0^2} \left( \frac{\Gamma}{\delta} \right)^2 I(r),
\]

(3.11)

However is this always a good approximation for dipole traps? The answer is no for dipole traps with very large detunings, for example traps formed with a 10.6 \( \mu \)m CO\(_2\) laser for rubidium have \( \omega_0 \simeq 14\omega \) so \( \delta \approx \omega_0 \) [64, 65]. For such traps the polarizability tends to its static value, hence the nomenclature of Quasi-electrostatic (QUEST) traps (useful as atoms in different internal states will experience the same trapping potential).

The experiments described in this thesis used optical potentials created with light at \( \lambda = 830 \) nm (the optical lattice) and \( \lambda = 865 \) nm (the dipole sheet trap, described in Section 2.5). For these detunings Eq. 3.10 disagrees with Eq. 3.8 by 3\% for \( \lambda = 830 \) nm and 5\% for \( \lambda = 865 \) nm. The error in \( \Gamma_{sc} \) is larger than that for \( V \) under the rotating wave approximation: 6\% for \( \lambda = 830 \) nm and 10\% for \( \lambda = 865 \) nm. Thus even for our relatively modest (by dipole trap standards) detunings the RWA introduces a non-negligible level of error. So far the discussion has been limited to a two-level atom, however in Section 3.1 it is shown neglecting the multi-level structure of an atom leads to a much larger error in the light shift than just neglecting the counter-rotating terms.
The advantage of using large detunings for dipole traps is clear from Eqns. 3.10 and 3.11: the scattering rate falls off with $\delta$ as $\Gamma_{sc} \propto \Gamma^2 / \delta^2$ while $V \propto \Gamma / \delta$, allowing effectively conservative potentials to be realised, e.g. the peak scattering rate for the $\lambda = 865$ nm light sheet was $\Gamma_{sc} = 0.01$ s$^{-1}$.

Multi-level atoms

$^{87}$Rb is not a simple two-level atom: the $5P$ level is split by spin-orbit coupling, while the interaction of the nuclear magnetic moment with the electronic magnetic moment gives rise to the hyperfine structure. In Appendix A we derived the light shift for a multi-level atom:

$$V = -\frac{E_0^2}{4\hbar} \sum_{n \neq g} |\langle n|\mu|g \rangle|^2 \left( \frac{1}{\omega_{ng} + \omega} + \frac{1}{\omega_{ng} - \omega} \right). \quad (3.12)$$

Following [63] it is convenient to write matrix element $\langle n|\mu|g \rangle$ in the form $c_{ng}||\mu||$, where $||\mu||$ is the reduced matrix element and $c_{ng}$ a real coefficient representing the relative line strength for transitions between given sub-levels, the value of which is determined by the laser polarization and angular momentum considerations. The reduced dipole matrix element only depends on the electronic orbital wave functions and is related to the decay rate $\Gamma$. In $^{87}$Rb the $^2S_{1/2} - ^2P_{1/2}$, $^2P_{3/2}$ doublet is split by nearly 15 nm, while the ground state hyperfine splitting is 6.8 GHz. As the optical lattice was detuned by 35 nm to the $D1$ line it is a very good approximation to consider the hyperfine structure unresolved. The same is not true for the fine structure and taking into account the $D1$ and $D2$ lines the
3.2. Optical potentials

light shift becomes \[63\]

\[
V(\mathbf{r}) = \frac{\pi c^2 I(\mathbf{r})}{2} \left[ \left( \frac{\Gamma_{D_2}}{\delta_{D_2}} - \frac{\Gamma_{D_2}}{\omega_{D_2} + \omega} \right) \frac{2 + \mathcal{P}_{gFmF}}{\omega_{D_2}^3} + \left( \frac{\Gamma_{D_1}}{\delta_{D_1}} - \frac{\Gamma_{D_1}}{\omega_{D_1} + \omega} \right) \frac{1 - \mathcal{P}_{gFmF}}{\omega_{D_1}^3} \right],
\]

(3.13)

where \(\delta_{D_i} = \omega - \omega_{D_i}\) and \(\mathcal{P} = 0, \pm 1\) for linear and \(\sigma^\pm\) polarized light respectively.

Similarly the scattering rate in \(^{87}\text{Rb}\) is

\[
\Gamma_{sc}(\mathbf{r}) = \frac{\pi c^2 I(\mathbf{r})}{2\hbar} \left[ \left( \frac{\Gamma_{D_2}}{\delta_{D_2}} - \frac{\Gamma_{D_2}}{\omega_{D_2} + \omega} \right) \frac{2 + \mathcal{P}_{gFmF}}{\omega_{D_2}^3} + \left( \frac{\Gamma_{D_1}}{\delta_{D_1}} - \frac{\Gamma_{D_1}}{\omega_{D_1} + \omega} \right) \frac{1 - \mathcal{P}_{gFmF}}{\omega_{D_1}^3} \right].
\]

(3.14)

Equations 3.13 and 3.14 were the equations used to calculate lattice depth and the scattering rate in this thesis.

3.2 Optical potentials

The result that the optically created potential for an atom is \(\propto I(\mathbf{r})\) allows the experimentalist to create a diverse range of potential landscapes. Generating arbitrary optical potentials is possible using spatial light modulators \([66]\) or sufficiently fast acousto-optic deflectors \([67]\). In Chapter 4 we describe how to create dynamic optical lattice potentials.

The trapping frequencies for a single red-detuned dipole trap where discussed for the case of a light sheet in Section 2.5 and here we note the equivalent results for a symmetric Gaussian beam of the form

\[
I = \frac{I_0}{1 + z^2/\sigma^2} e^{-2r^2/\omega(z)^2},
\]

(3.15)
where the Rayleigh length \( b = \pi w_0^2/\lambda \) and \( w(z) = w_0(1 + z^2/b^2)^{1/2} \). The peak intensity is given by \( I_0 = 2P/\pi w_0^2 \). In the harmonic region about \( r = z = 0 \) the trapping frequencies are

\[
\omega_r = \left( \frac{4V_0}{mw_0^2} \right)^{1/2} \quad \text{and} \quad \omega_z = \left( \frac{2V_0}{mb^2} \right)^{1/2}.
\] (3.16)

The trap depth \( V_0 \) is given by inserting \( I_0 \) into Eq. 3.13.

Optical lattice potentials are traditionally created by interfering laser beams, giving a potential of the form

\[
V = V_0 \sin^2 \left( \frac{\pi x}{d} \right),
\] (3.17)

in one-dimension, where \( d \) is the periodicity of the optical lattice and the overall confinement due to the Gaussian profile of the lattice beams has been neglected. We delay a treatment of the form of \( d \) (which depends on the angle of intersection of the beams) until Chapter 4. It is important to note that constructive interference makes the lattice depth, \( V_0 \), four times larger than the equivalent depth for a single beam, considered above. That is, if \( P \) is the power in one beam of waist \( w_0 \) the peak intensity is \( I_0 = 4 \times 2P/\pi w_0^2 \).

The potential at a single lattice site can be approximated as harmonic if the lattice is sufficiently deep. In this limit the harmonic trapping frequency at a lattice site is

\[
\omega_{\text{lat}} = \left( \frac{2V_0\pi^2}{md^2} \right)^{1/2}.
\] (3.18)

It is convenient to define the lattice recoil energy as

\[
E_L = \frac{\pi^2 \hbar^2}{2md^2}.
\] (3.19)
3.3 Bloch band theory

In this section we give a brief introduction to single-particle physics in a periodic potential. The Bloch function analysis here is important for considering the diabatic loading and diffraction from an accordion lattice discussed in Chapter 5.

The Hamiltonian describing a single particle in a periodic potential $V(x) = V(x + d)$ is

$$H\phi_{nq} = \left(\frac{1}{2m}p^2 + V(x)\right)\phi_{nq} = E_{nq}\phi_{nq}. \quad (3.20)$$

As is well known (see any condensed matter textbook, for example [68]) the solutions to this equation are Bloch functions:

$$\phi_{nq}(x) = e^{iqx/\hbar}u_{nq}(x), \quad (3.21)$$

where the functions $u_{nq}$ display the same periodicity as the lattice, $u_{nq}(x) = u_{nq}(x + d)$, $q$ denotes the quasi-momentum and the subscript $n$ is the band index. Substituting Eq. 3.21 into Eq. 3.20 gives the Schrödinger equation for $u_{nq}$,

$$Hu_{nq} = \left(\frac{1}{2m}(p + q)^2 + V(x)\right)u_{nq} = E_{nq}u_{nq}. \quad (3.22)$$

Given $V(x)$ and $u_{nq}$ are both periodic with period $d$ they can be expanded in the plane wave basis spanned by the reciprocal lattice vector $2\pi d$, $\{\exp 2\pi lx/d\}$, where $l$ is an integer, i.e. written as a discrete Fourier series. We consider the case $V(x) = -V_0 \sin^2 \pi x/d$. In the plane wave basis the Hamiltonian becomes

$$H = \frac{1}{2m} \left(\frac{2\pi \hbar}{d}l + q\right)^2 + \frac{V_0}{4}(2 - e^{i2\pi x/d} - e^{-i2\pi x/d}). \quad (3.23)$$

If we are only concerned with the lowest energy bands it is convenient to consider
Figure 3.1: The band structure for a single particle in an optical lattice of varying depths, shown in the reduced zone scheme (i.e. the first Brillouin zone). The black line shows the depth of the optical lattice, measured from the zero energy origin of the graph. As the lattice becomes deeper the bands become flatter (most noticeable for the lowest band) and the band energies tend toward those in a harmonic oscillator with frequency $\omega_{\text{lat}}$.

the Hamiltonian in the truncated basis $|l| \leq l_{\text{max}}$. The Hamiltonian can then be written as a tridiagonal matrix of dimensions $(2l_{\text{max}} + 1) \times (2l_{\text{max}} + 1)$:

$$H = \begin{pmatrix}
\vdots & -\frac{V_0}{4} & 0 & 0 & 0 \\
-\frac{V_0}{4} & \frac{1}{2m} \left( -\frac{2\hbar}{d} + q \right)^2 & -\frac{V_0}{4} & 0 & 0 \\
0 & -\frac{V_0}{4} & \frac{q^2}{2m} & -\frac{V_0}{4} & 0 \\
0 & 0 & -\frac{V_0}{4} & \frac{1}{2m} \left( \frac{2\hbar}{d} + q \right)^2 & -\frac{V_0}{4} \\
0 & 0 & 0 & -\frac{V_0}{4} & \vdots
\end{pmatrix} + \frac{V_0}{2} I, \quad (3.24)$$

where $I$ is the identity matrix. The eigenvectors and eigenvalues of the truncated Hamiltonian can be easily calculated and by diagonalising $H$ for $-\pi\hbar/d \leq q \leq \pi\hbar/d$ the bandstructure can be plotted as shown in Fig. 3.1. For energies above
3.4. Energy scales for large periodicity lattices

V$_0$ an atom has a close to free-particle dispersion relation. However the curvature of the lowest bands becomes much flatter as the lattice depth increases. For deep lattices the lower band energies tend to the energy levels of a harmonic oscillator, $E_{nq} \rightarrow E_n = (n + 1/2)\hbar\omega_{lat}$, where the lattice frequency is given by Eq. 3.18. By considering the rate of change of quasimomentum of a particle upon the application of an external force it is possible to define the effective mass of a particle in a single band as [68]

$$\frac{1}{m^*} = \frac{1}{\hbar^2} \frac{\partial^2 E_q}{\partial q^2}. \quad (3.25)$$

The effective mass is inversely proportional to the curvature of the band which fits with the plots of Fig. 3.1; for a deeper lattice the curvature decreases and the effective mass increases as it is harder for a particle responding to a force to move through the lattice. The effective mass is closely related to the tunnelling matrix element $J$. It can be shown that in the tight-binding limit (where tunnelling beyond nearest-neighbours is neglected, a good approximation for $V_0 > 5E_L$) $J$ is related to the width of the lowest band by [69]

$$J = \frac{\max(E_{0q}) - \min(E_{0q})}{4}. \quad (3.26)$$

3.4 Energy scales for large periodicity lattices

In Eq. 3.19 we defined the lattice recoil energy as $E_L = \pi^2\hbar^2/2md^2$, which is a convenient energy scale for the band structure of an optical lattice, shown in Fig. 3.1. Optical lattices are often formed by counterpropagating laser beams, giving $d = \lambda/2$, and lattice depths are quoted in units of the recoil energy $E_R = 2\pi^2\hbar^2/m\lambda^2$ (here $\lambda$ is the lattice beam wavelength, not the wavelength of the
3.4. Energy scales for large periodicity lattices

$D2$ transition). Given the work in this thesis is concerned with larger periodicity ($d > \lambda/2$) lattices it is important to consider how the relevant energies in an optical lattice scale with $d$.

Table 3.1 considers these energy scales in a lattice of depth 33.3 kHz for different lattice constants $d$. For $\lambda = 830$ nm, giving a minimum periodicity of $d = 415$ nm, this depth corresponds to $V_0 = 10E_R$. The band structure for the same parameters considered in Table 3.1 is plotted in Fig. 3.2. As would be expected the tunnelling energy, $J$, falls very quickly with $d$, as can be seen from the curvature of the lowest band becoming flat. The energy separation between bands falls approximately inversely proportional to $d$, given $\omega_{\text{lat}} \propto 1/d$. This accounts for the proliferation of bound states, noticeable in Fig. 3.2 as $d$ increases. As we shall see in Chapter 5, this leads to classical-like behaviour of atoms in a deep, large period optical lattice.

<table>
<thead>
<tr>
<th>$d$ ($\mu$m)</th>
<th>$E_L$ (Hz)</th>
<th>$V_0/E_L$</th>
<th>$J$ (Hz)</th>
<th>Number of bound states</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.415</td>
<td>3329</td>
<td>10</td>
<td>64</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>143</td>
<td>232</td>
<td>$10^{-9}$</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>23</td>
<td>1452</td>
<td>$10^{-16}$</td>
<td>24</td>
</tr>
</tbody>
</table>

Table 3.1: Relevant energy scales in an optical lattice of depth $V_0 = 33.3$ kHz for various lattice constants. $J$ is calculated from the width of the lowest band according to Eq. 3.26.

In the limit of a very deep lattice $J$ can be calculated by considering the width of the lowest band in the 1D Mathieu equation

$$J = \frac{4}{\sqrt{\pi}}E_L \left(\frac{V_0}{E_L}\right)^{3/4} \exp\left[-2 \left(\frac{V_0}{E_L}\right)^{1/2}\right].$$ (3.27)

For $V_0 > 15E_L$ this expression is accurate to better than 10\% [70]. We can rewrite

\footnote{Note the 1D Mathieu equation is valid for a sinusoidal potential of any depth. It is the validity of the analytical expression of Eq. 3.27 which relies on the limit of a deep lattice.}
3.4. Energy scales for large periodicity lattices

Figure 3.2: Single-particle band structure in optical lattices of various periodicities for the same lattice depth, $V_0 = 10E_R = 33.3$ kHz. The horizontal and vertical scales are the same for each plot, with the extent of the first Brillouin zone shown by the dashed lines. The lowest band becomes flat very quickly as $d$ increases, and the number of bound states also proliferates rapidly. $\lambda = 0.83 \ \mu m$.

Eq. 3.27 as

$$J = \frac{4}{\sqrt{\pi}} E_R \left( \frac{\lambda}{2d} \right)^{1/2} \left( \frac{V_0}{E_R} \right)^{3/4} \exp \left[ -2 \left( \frac{2d}{\lambda} \right) \left( \frac{V_0}{E_R} \right)^{1/2} \right], \quad (3.28)$$

showing explicitly the dependence on $d$ for deep lattices, $J \propto (1/\sqrt{d}) \exp (-4d/\lambda)$.

The smaller energy scales in large periodicity lattices make life hard for the experimentalist: coherent dynamics occur over longer timescales but heating rates may remain the same in the lab. Adiabaticity timescales also increase accordingly and it is harder to cool to temperatures lower than the inter-band separation.
3.5 Wannier functions and particle interactions

The Bloch functions discussed in Section 3.3 are delocalised across the optical lattice. When considering particle interactions in an optical lattice it is convenient to use wave functions which represent a localised particle. Such a basis is provided by the Wannier functions, \( w_n \), which are an orthonormal set of wave functions maximally localised at a lattice site. The Wannier functions are related to the Bloch functions via a unitary transformation, though finding the transformation to give the Wannier functions with minimum spread (i.e. maximally localised) is not always straightforward. For a single isolated\(^2\) band the unique transformation which gives the Wannier functions with the minimum spread can be found exactly using a recursive method [71].

The two-body interaction potential, \( V_{\text{int}}(r_1 - r_2) \), between neutral atoms is dominated by van der Waals attraction at large distances and takes a more complicated form at short distances. Fortunately, for sufficiently cold atoms\(^3\) only s-wave scattering is possible. This can be completely described by a single parameter: the scattering length \( a_s \) [70]. The interaction potential can then be approximated by a contact pseudo-potential of the form

\[
V_{\text{int}}(r_1 - r_2) = \frac{4\pi\hbar^2 a_s}{m} \delta(r_1 - r_2).
\]  

(3.29)

In a shallow optical lattice where particles move easily through the lattice and

\(^2\)An isolated band is one which does not become degenerate with another band anywhere in the Brillouin zone.

\(^3\)Some authors take the onset of the s-wave scattering regime to define the term ultracold atoms [70]. The centrifugal barrier to d-wave scattering is typically \( \sim 1 \) mK, so the pseudo-potential approximation is a very good one for BEC experiments. For the scattering of identical bosons in the same internal state the spatial wave function must be symmetric with respect to exchange, ruling out wave functions with odd orbital angular momentum, i.e. p-wave scattering is forbidden.
3.5. Wannier functions and particle interactions

Particle interactions do not dominate, the condensate can still be accurately described by mean-field theory. As the depth of the lattice increases the kinetic energy of the system is restricted to nearest-neighbouring tunnelling and interactions become comparably large such that a mean-field description is no longer valid. This regime was first considered by Jaksch et al. [3] who showed bosonic atoms with repulsive interactions in an optical lattice were well described by the Bose-Hubbard model. The Bose-Hubbard model was introduced by Fisher et al. [72] in 1989 when possible experimental representations were limited to systems of $^4$He absorbed in porous media.

The second-quantised Hamiltonian describing interacting bosons in an optical lattice $V_{\text{int}}(\mathbf{r})$ and an additional confining potential $V_{\text{ext}}(\mathbf{r})$ is

$$\hat{H} = \int d^3 \mathbf{r} \hat{\psi}^{\dagger}(\mathbf{r}) \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{int}}(\mathbf{r}) + V_{\text{ext}}(\mathbf{r}) \right) \hat{\psi}(\mathbf{r}) + \frac{g}{2} \int d^3 \mathbf{r} \hat{\psi}^{\dagger}(\mathbf{r}) \hat{\psi}(\mathbf{r}) \hat{\psi}(\mathbf{r}) \hat{\psi}(\mathbf{r}),$$

(3.30)

where $g = 4\pi \hbar^2 a_s/m$ is the interaction strength and we have taken advantage of the delta-function pseudo-potential defined in Eq. 3.29. $\hat{\psi}$ is the boson field operator. When the temperature and interaction energy of the atoms is small compared to the gap between the lowest and first excited bands then the bosonic field operators may be expanded in terms of the Wannier functions of the lowest band, $\hat{\psi} = \sum_i \hat{a}_i w_0(\mathbf{r} - \mathbf{r}_i)$, where $a_i$ is the destruction operator for a particle in site $\mathbf{r}_i$. Substituting this into Eq. 3.30 and neglecting off-site interactions and beyond nearest-neighbour tunnelling (a good approximation for $V_0 > 5E_L$ [3]) one arrives at the Bose-Hubbard Hamiltonian

$$H_{\text{BH}} = -J \sum_{\langle i,j \rangle} \hat{a}_i^{\dagger} \hat{a}_j + \frac{U}{2} \sum_i \hat{N}_i (\hat{N}_i - 1) + \sum_i (\epsilon_i - \mu) \hat{N}_i,$$

(3.31)
3.5. Wannier functions and particle interactions

where

\[ J = -\int \! d^3r w_0(r - r_i) \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{lat}}(r) + V_{\text{ext}}(r) \right) w_0(r - r_j), \quad (3.32) \]

and

\[ U = g \int \! d^3r |w_0(r - r_j)|^4. \quad (3.33) \]

\( \hat{N}_i = \hat{a}^\dagger_i \hat{a}_i \) is the atom number operator at site \( i \) and the notation \( \langle i, j \rangle \) in the first summation in Eq. 3.31 indicates the summation is over nearest neighbours only. The last term in the Bose-Hubbard Hamiltonian describes the change in energy at a lattice site arising from the external confinement, \( \epsilon_i = \int \! d^3r V_{\text{ext}}(r - r_j)|^2 \approx V_{\text{ext}}(r) \). The chemical potential, \( \mu \), has also been introduced to fix the mean number of particles, i.e. Eq. 3.31 is the Hamiltonian in the grand canonical ensemble. For a deep three-dimensional cubic optical lattice where the chemical potential is much smaller than the onsite trapping frequencies the wave function in the lowest band can be described as the product of the Wannier functions in the three orthogonal directions,

\[ w_0(r - r_i) = w_{0x}(x - x_i)w_{0y}(y - y_i)w_{0z}(z - z_i). \quad (3.34) \]

As discussed above in reference to the band structure the wave function in the lowest band tends to that of a harmonic oscillator in a deep lattice, that is,

\[ w_{0x}(x - x_i) = \frac{1}{(\sqrt{\pi}a_x)^{1/2}}e^{-x^2/(2a_x^2)}, \quad (3.35) \]

where \( a_x = \sqrt{\hbar/m\omega_{\text{lat}}} \) is the harmonic oscillator length. The experiments detailed in this thesis do not involve three-dimensional lattices with fillings of \( \sim 1 \) particle per lattice site, where the three-dimensional extension of Eq. 3.35 would be a good
description of the Wannier function. In the next section we consider the onsite wave function in a two-dimensional lattice with weak confinement in the third direction and a large (> 100) number of atoms per lattice site. In this case broadening of the atomic wave function by atom-atom interactions must be taken into account.

3.6 Two-dimensional bosonic Josephson junction arrays

In the limit of a large number of particles per lattice site (≫ 1) the destruction operator at a site may be approximated by a c-number $\hat{a}_j = \sqrt{N_j} \exp{i\theta_j}$, where $N_j$ is the number of atoms at the $j$th site and $\theta_j$ the phase of the condensate. That is, we consider the tight binding regime where a well-localised macroscopic wave function is centred on each lattice site. This array of condensates is a realisation of a bosonic Josephson-junction array (JJA) (bosonic rather than involving pairs of fermions as was the case in the first theoretical and experimental work on such junctions). A Josephson junction is formed by two weakly-linked (by which we mean having small spatial overlap) macroscopic, phase-coherent wave functions. The existence of supercurrents between two such weakly-linked wave functions and associated effects were predicted by Josephson in 1962, who considered the tunnelling of Cooper pairs between superconductors, earning him the 1973 Nobel Prize in physics.

Josephson-junctions arrays have been studied in condensed matter physics using arrays of superconducting islands (for a review see [73]). They were first realised in a cold atom experiment in 2001 [74], and later a single Josephson junction was also created [75]. JJA’s are extremely interesting as they are an ideal model
system in which to investigate a range of phenomena including vortex dynamics, frustration and phase transitions. A two-dimensional JJA represents a physical realisation of the two-dimensional XY model (this is discussed further in Section 6.3 where the frustrated XY model is considered). The proliferation of vortices on a 2D JJA near the BKT regime has been investigated in a BEC experiment [76], after the BKT crossover in a trapped two-dimensional gas was first observed by Hadzibabic et al. [5]. The creation of a bosonic JJA under an effective magnetic field is one of the major results of this thesis, and is described in Chapter 7.

\subsection{Description of the condensate wave function at a lattice site}

In this section we analytically investigate the form of the wave function localised at a node of a two-dimensional optical lattice potential with weak confinement in the axial direction. We consider the tight-binding regime discussed above where the lattice depth is larger than the chemical potential of the condensate, forming a spatially localised macroscopic wave function at each site. The wave function at a lattice site can be described by

$$\psi_j(r - r_j) = \sqrt{N_j} e^{i\theta_j} w(r - r_j).$$

When the on-site transverse trapping frequencies ($\omega_x = \omega_y = \omega_{\text{lat}}$) are larger than the chemical potential at a lattice site a Gaussian wave function is a good approximation for the transverse condensate wave function. This ansatz is suitable in this limit of weak interactions $\hbar\omega_{\text{lat}} > \mu$ as the exact ground state of a non-interacting condensate is given by the harmonic oscillator wave function of Eq. 3.35.
Two-dimensional bosonic Josephson junction arrays

(or more exactly by the appropriate Wannier function). Thus a wave function of the form

$$w(r) = u(x, y)v(z), \quad (3.37)$$

where

$$u(x, y) = \frac{1}{\sqrt{\pi\sigma}}e^{-(x^2+y^2)/2\sigma^2}, \quad (3.38)$$

is an appropriate starting point. $\sigma$ is the transverse width of the wave function, which can be determined by a variational analysis. In the limit of very tight transverse confinement, $\hbar\omega_{\text{lat}} \gg \mu$, $\sigma = a_{\text{lat}} = \sqrt{\hbar/m\omega_{\text{lat}}}$ and the condensate is quasi-one-dimensional.

The Gross-Pitaevskii equation (GPE) describes a three-dimensional macroscopic condensate at zero temperature (here we consider the time-independent GPE)

$$Hw(r) = \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m(\omega_{\text{lat}}^2(x^2 + y^2) + \omega_z^2 z^2) + g\text{N}_j|\psi|^2\right) w(r) = \mu w(r). \quad (3.39)$$

Substituting the ansatz of Eq. 3.38 into the GPE, multiplying by $u^*(x, y)$ and integrating over the $x, y$ directions gives

$$H'v(z) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{1}{2}m\omega_z^2 z^2 + g_{1D}\text{N}_j|v(z)|^2\right) v(z) = \mu' v(z), \quad (3.40)$$

with

$$\mu' = \mu - \left(\frac{\hbar^2}{2m\sigma^2} + \frac{1}{2}m\omega_{\text{lat}}^2\sigma^2\right) \quad \text{and} \quad g_{1D} = \frac{g}{2\pi\sigma^2}. \quad (3.41)$$

In the case of relatively weak axial confinement it is a good approximation to neglect the axial kinetic energy term. This can be explicitly checked using a
variational approach using the ansatz
\[ v(z) = \frac{1}{(\sqrt{\pi}\sigma_z)^{1/2}} e^{-z^2/2\sigma_z^2}. \] (3.42)

The energy as a function of \( \sigma_z \) is then
\[ E'(\sigma_z) = \langle v(z)|H'|v(z)\rangle = \frac{\hbar \omega_z}{4} \left( \frac{a_z^2}{\sigma_z^2} + \frac{\sigma_z^2}{a_z^2} \right) + \frac{g_{1D}N_j}{\sqrt{2\pi\sigma_z}}. \] (3.43)

Minimising the energy with respect to \( \sigma_z \) gives the condition
\[ x^4 - \beta x - 1 = 0 \quad \text{with} \quad x = \frac{\sigma_z}{a_z} \quad \text{and} \quad \beta = \frac{2}{\hbar \omega_z} \frac{g_{1D}N_j}{\sqrt{2\pi a_z}}. \] (3.44)

For our typical experimental parameters (considering central lattice sites where \( N_j \sim 1500 \) for \( d = 2 \mu m \) for example, and \( \omega_z/2\pi = 53 \) Hz, giving \( a_z = 1.5 \mu m \)) 55 \( \leq \beta \leq 90 \) giving 3.8 \( \leq x \leq 4.5 \). The ratio of axial kinetic energy to potential energy is \( a_z^4/\sigma_z^4 = x^{-4} \sim 10^{-3} \) (consider the first two terms in Eq. 3.43), while the ratio of the kinetic energy to the interaction energy is of the same order. Thus we work in the regime where the axial kinetic energy is negligible. Neglecting the kinetic energy in Eq. 3.43 (the Thomas-Fermi approximation) gives
\[ \frac{1}{2}m\omega_z^2z^2 + g_{1D}N_j|v(z)|^2 = \mu', \] (3.45)
and thus the axial wave function has the form
\[ v(z) = \left( \frac{\mu'}{g_{1D}N_j} \right)^{1/2} \left( 1 - \frac{z^2}{r_z^2} \right)^{1/2} \quad \text{with} \quad r_z = \left( \frac{2\mu'}{m\omega_z^2} \right)^{1/2}. \] (3.46)
3.6. Two-dimensional bosonic Josephson junction arrays

The total wave function at a lattice site is thus

\[ w(\mathbf{r}) = \frac{1}{\sqrt{\pi \sigma}} e^{-(x^2+y^2)/2\sigma^2} \left( \frac{\mu'}{g_{1D} N_j} \right)^{1/2} \left( 1 - \frac{z^2}{r_z^2} \right)^{1/2}. \]  

(3.47)

Using the normalisation condition \( \int d^3r |w|^2 = 1 \) (recall we have split the wave function into a dynamical part \( \sqrt{N_j} \exp i \theta_j \) and a real spatial wave function \( w(\mathbf{r}) \)) the following expressions for \( \mu' \) and \( r_z \) can be derived

\[ \mu' = \left( \frac{3\hbar^2 \omega_z N_j a_s}{2\sqrt{2m} \sigma^2} \right)^{2/3}, \quad r_z = \left( \frac{3N_j a_s a_z^4}{\sigma^2} \right)^{1/3}. \]  

(3.48)

A variational approach can be used to calculate \( \sigma \). Recall the total energy at a lattice site, \( E \), is related to the chemical potential by \( \mu = \partial E/\partial N_j \), and thus the energy per particle can be written

\[ \frac{E}{N_j} = \frac{\hbar^2}{2m\sigma^2} + \frac{1}{2} m\omega_{\text{lat}}^2 \sigma^2 + \frac{3}{5} \mu', \]  

(3.49)

where the factor of 3/5 is a consequence of the fact \( \mu' \propto N_j^{2/3} \). For a given lattice constant, depth and site filling the transverse width of the wave function can be calculated by minimising Eq. 3.49 with respect to \( \sigma \). For example for a lattice depth of \( V_0 = 1200 \) Hz, \( d = 2 \) \( \mu \)m and 1500 atoms at a lattice site we find \( \sigma = 1.14 a_{\text{lat}} = 0.43 \) \( \mu \)m.

3.6.2 Lattice filling in an inhomogeneous system

As the wave function at a lattice site depends on the number of atoms at the site (through the axial part \( v(z) \), while \( u(x, y) \) is presumed to be site-independent\(^4\)) it

\(^4\)While the number of atoms at a lattice site affects \( \sigma \) and hence \( g_{1D} \) it has been shown it is a good approximation to consider \( \sigma \) site-independent over the central lattices sites [20].
is important to consider how the lattice sites are filled in an inhomogeneous system created in an external trapping potential. The results here were originally shown by Krämer et al. [77] and restated by Kasamatsu [20]. Here the ground state of the inhomogeneous system is considered, where the chemical potential at a site $\mu_j$ is constant across the lattice. If the chemical potential was not uniform over the system the phases of the wave functions at different lattice sites would evolve at different rates, and the system would not be in the ground state (for which the quasi-momentum is zero). Recalling an external potential adds an energy $\epsilon_j \simeq V_{\text{ext}}$ at the $j^{\text{th}}$ site and using the result Eq. 3.48 for the local interaction energy at a site, the chemical potential in the Thomas-Fermi approximation is given by

$$\mu_j = \frac{1}{2} m \omega_r^2 (j_x^2 + j_y^2) a^2 + \left( \frac{3 \hbar^2 \omega_z N_j a_s}{2 \sqrt{2 m} \sigma^2} \right)^{2/3} = \text{constant}, \quad (3.50)$$

where $\{j_x, j_y\}$ index the $j^{\text{th}}$ site. The atom distribution in the ground state is then

$$N_j = \frac{5N}{2 \pi j_{\text{max}}^2} \left( 1 - \frac{j_x^2 + j_y^2}{j_{\text{max}}^2} \right)^{3/2}, \quad (3.51)$$

where the normalisation condition $\sum_j N_j = N$ has been used (approximated as an integral) and allows us to define

$$j_{\text{max}} = \frac{a_r}{d} \left( \frac{15 N \omega_z a_d d^2}{2 \pi \omega_r a_r \sigma^2} \right)^{1/5}. \quad (3.52)$$

Introducing the lattice potential enhances the mean-field interaction energy due to the tighter confinement of the atoms. This results in a flow of atoms radially outwards, such that the inverted parabola envelope of ground state in the presence of the lattice will have a larger radius than the profile of the condensate before it is loaded into the lattice. This is shown in Fig. 3.3.
3.6. **Two-dimensional bosonic Josephson junction arrays**

Figure 3.3: Atomic distribution in the $xy$ plane in a two-dimensional optical lattice, calculated using the analytical results of Sections 3.6.1 and 3.6.2. The lattice parameters are $V_0 = 1200$ Hz, $d = 2$ µm with $N = 2 \times 10^5$ atoms. The external confinement is $\{\omega_r, \omega_z\}/2\pi = \{20.1, 53.0\}$. (a) 2D atomic density integrated along the axial direction. The red line marks the Thomas-Fermi radius of the condensate before the optical lattice is introduced. (b) Cross-section of the two-dimensional density along $y = 0$. The blue line shows the atomic distribution in the optical lattice while the red line again corresponds to the Thomas-Fermi profile of the condensate in the absence of the optical lattice. The larger envelope of the atomic distribution in the lattice is evident in both (a) and (b).

### 3.6.3 Energy scales and phase fluctuations

It is instructive to consider the Bose-Hubbard Hamiltonian of Eq. 3.31 for the simplified case of only two adjacent lattice sites, i.e. for a single Josephson junction (and here we neglect the effect of the external confinement over a distance $d$),

$$H = -J(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_2) + \frac{U}{2}(\hat{N}_1(\hat{N}_1 - 1) + \hat{N}_2(\hat{N}_2 - 1)).$$  \hspace{1cm} (3.53)

When the number of atoms per site is sufficiently large that the number and phase at a site is well-defined the destruction operator may be approximated by
\[ \hat{a}_j = \sqrt{N_j} \exp(i\theta_j), \] and the (classical) Hamiltonian becomes

\[ H = -2J\sqrt{N_1N_2}\cos(\theta_1 - \theta_2) + \frac{U}{4}(N^2 - (N_1 - N_2)^2) \tag{3.54} \]

with \( N = N_1 + N_2 \). The relevant tunnelling energy for a Josephson junction is \( \tilde{J}_{jj'} = 2\sqrt{N_jN_{j'}}J \) (sometimes referred to as the Josephson energy and denoted by \( E_J \)). Analytical forms for \( J \) and \( U \) can be found from Eqs. 3.32 and 3.33 upon substitution of wave function Eq. 3.47. One finds

\[ U_j = \frac{4}{5} \frac{\mu_j'}{\sqrt{N_j}}, \tag{3.55} \]

\[ J \simeq \left( \frac{\hbar^2}{2m\sigma^2} \left( \frac{d^2}{4\sigma^2} - 1 \right) - V_0 \right) e^{-d^2/4\sigma^2}, \tag{3.56} \]

where in the calculation for \( J \) the overlap integral for the axial part of the wave function has been approximated by \( \int_{-r_s}^{r_s} dz v_j v'_j \simeq 1 \) and the area of integration in the transverse plane is \(-\infty \leq x \leq \infty, 0 \leq y \leq d\). Factors of \( \text{Erf}\{d/2\sigma\} \) arising in the calculation are equal to unity to a good approximation as \( d/2\sigma \geq 2 \) for typical experimental parameters.

Inspection of Eq. 3.54 shows the ground state occurs for \( N_1 = N_2 \) and the absence of a relative phase difference across the junction, \( \theta_1 - \theta_2 = 0 \). It is important to consider the conditions under which this phase coherence, mediated by tunnelling, survives. This requires examining the role of phase fluctuations in a Josephson junction.

**Quantum fluctuations**

The number difference \( \hat{N}_1 - \hat{N}_2 \) and relative phase \( \hat{\theta}_1 - \hat{\theta}_2 \) are conjugate operators and obey a Heisenberg uncertainty relation of the form \( \Delta(\hat{N}_1 - \hat{N}_2)\Delta(\hat{\theta}_1 - \hat{\theta}_2) \geq 1 \).
The quantum phase and number fluctuations are [78]

\[
(\Delta(\hat{\theta}_1 - \hat{\theta}_2))^2 = \left(\frac{U + \tilde{J}/N^2}{\tilde{J}}\right)^{1/2},
\]

(3.57)

\[
(\Delta(\hat{N}_1 - \hat{N}_2))^2 = \left(\frac{\tilde{J}}{U + \tilde{J}/N^2}\right)^{1/2}.
\]

(3.58)

There are two regimes of interest:

- **Fock regime:** \( U \gg \tilde{J} \). Phase fluctuations become close to unity for \( U \sim \tilde{J} \) and hence in this regime phase coherence is lost between lattice sites.

- **Josephson regime:** \( \tilde{J} \gg U \gg \tilde{J}/N^2 \). Phase fluctuations are small and phase coherence is maintained via atom tunnelling. The relative number fluctuations \( (\Delta(\hat{N}_1 - \hat{N}_2))/N = ((\tilde{J}/N^2)/(U + \tilde{J}/N^2))^{1/2} \) are also small (by making the approximation \( \hat{a}_j = \sqrt{N_j} \exp i\theta_j \) we have already presumed to operate in this regime).

Typical orders of magnitude for our experimental parameters are \( \tilde{J}_j \sim 1 \text{kHz}, N_j \sim 1000, U_j \sim 1 \text{Hz} \) for central lattice sites, lying firmly in the Josephson regime.

**Thermal fluctuations**

While quantum fluctuations are not expected to play an important role in the JJA dynamics the same cannot be said for thermal fluctuations, which are of the order \( \Delta\theta_{\text{th}} \sim \sqrt{k_B T/\tilde{J}} \). Typical cloud temperatures were \( \approx 40 \text{nK} \) for our system while tunnelling energies lay in the range \( 40 \text{nK} \leq \tilde{J} \leq 200 \text{nK} \) for \( 2500 \text{Hz} \geq V_0 \geq 900 \text{Hz} \), \( d = 2 \mu\text{m} \). Hence it was critical to consider the effect of thermal fluctuations in our system.
The role of thermal fluctuations in a two-dimensional lattice of Bose-Einstein condensates has been explored theoretically by Trombettoni et al. \cite{trombettoni79} and in a beautiful experiment at JILA \cite{trombettoni76}. Both papers focus on the dynamics of two-dimensional JJA’s as the ratio $\tilde{J}/T$ is varied. For a free vortex to be thermodynamically stable in a JJA its free energy must be $F = E - TS < 0$, where $E$ is the vortex energy, $E \approx J \log (R/d)$, and $S$ the entropy gain $S \approx \log (R/d)$. $R$ is the size of the array, and the gain in entropy comes from there being approximately $R^2/d^2$ possible sites for the vortex. Thus for temperatures $T \geq \tilde{J}$ free vortices are expected to be present in the Josephson-junction array. Thermal fluctuations in two-dimensional systems support bound vortex-antivortex pairs even below this temperature as famously shown by Berezinskii \cite{berezinskii80, berezinskii81}, and Kosterlitz and Thouless \cite{kosterlitz82}.

### 3.6.4 Adiabaticity considerations

To load cold atoms into the ground state of an optical lattice potential it is important that the turn on of the lattice is adiabatic with respect to the various energy scales in the system. Adiabaticity with respect to inter-band transitions is one of the easier considerations to satisfy. The adiabaticity criterion is

$$|\langle \phi_{nq}| \dot{H}|\phi_{0q} \rangle| \ll \frac{|E_{nq} - E_{0q}|^2}{\hbar}.$$  

(3.59)

As the lattice depth increases $E_{nq} - E_{0q} \rightarrow n\hbar\omega_{\text{lat}}$. However even for very weak lattices there is a finite energy difference of around $4E_L$ between the lowest and first excited bands at $q = 0$ (the quasimomentum is close to zero when considering loading a stationary BEC into a static optical lattice). Thus the rate at which the
lattice is ramped on is constrained by [83]

\[ \frac{d}{dt} \left( \frac{V_0}{E_L} \right) \ll 32\sqrt{2}\frac{E_L}{\hbar}, \]  

For example \( \dot{V}_0/E_L \ll 4 \times 10^4 \text{ s}^{-1} \) for a lattice constant \( d = 2 \mu \text{m} \).

Remaining adiabatic with respect to the effect of atom-atom interactions generally places more stringent demands on the lattice turn on. As discussed in section 3.6.2 the enhanced mean-field interaction in a lattice results in a radially outward flow of atoms. The lattice must be turned on sufficiently slowly that the atoms have time to redistribute themselves and remain in the instantaneous ground state. For deep lattices the timescale for atomic redistribution is set by the tunnelling time \( \sim \hbar/J \). A time-dependent 2D GPE simulation for our experimental parameters showed that an exponential ramp of lattice depth up to \( V_0 = 1.9 \text{ kHz} \) over 500 ms for a lattice with \( d = 2 \mu \text{m} \) gave a final state with a 94% overlap with the ground state [84].
Experimental implementation of dynamic optical lattices

While it is common in ultracold atom experiments to use interfering counterpropagating laser beams to produce optical lattices with spacings of $\lambda/2$ another often-used technique involves two beams intersecting at an angle of $2\theta$, producing a lattice with a spacing $d = \lambda/(2 \sin \theta)$. In particular, two parallel beams incident on a lens will form a 1D optical lattice in the focal plane of the lens where they intersect. We took advantage of this simple geometry to generate both two-dimensional rotating and accordion lattices. A concise description of the experimental method to generate these dynamic lattices can be found in [25]. This chapter goes beyond this paper in describing the specific details of our experimental arrangement and alignment.

Only one group has previously made a rotating lattice, where a mechanically rotated mask was rotated at $\sim 8$ Hz [17]. We have developed a novel experimental
arrangement which allows a 2D lattice to be rotated smoothly at any frequency from 0 to \( \sim 1 \) kHz. The rotation is generated by a dual-axis acousto-optic deflector (AOD) and thus does not suffer from mechanical vibrations and instabilities which would lead to significant heating of atoms in the lattice. The limit of 1 kHz in the current apparatus was set by the update rate of the radio frequency synthesis system controlling the AOD, however the optical arrangement can accommodate even higher frequencies.

In addition the arrangement allows for the realization of a 2D accordion lattice, that is, an optical lattice with a dynamically variable periodicity. 1D accordion lattices have previously been demonstrated by a number of groups [85, 86, 87]. Experiments probing the superfluid-Mott insulator transition typically use optical lattices with periodicities \( \sim 400 - 500 \) nm where tunnelling between lattice sites is significant [4]. Direct observation of the in-lattice density distribution of a number-squeezed state has not been achieved due to difficulties in achieving an optical system with sufficient resolution (though Markus Greiner’s group are close to achieving this [52]). However an accordion lattice could allow for the expansion of the lattice spacing to above the resolution of an optical imaging system (\( > 1 \) \( \mu \)m) after the Mott insulator had been achieved. Detecting and manipulating atoms at single lattice sites are important criteria for quantum information processing schemes with neutral atoms [23].

4.1 Creating dynamic lattices

Figure 4.1 highlights a prosaic but key advantage in forming an optical lattice in the focal plane of a lens. The intrinsic cylindrical symmetry of such an arrangement allowed us to achieve a rotating lattice upon rotation of the parallel beams incident
4.1. Creating dynamic lattices

Figure 4.1: Two orthogonal standing wave intensity patterns in the focal plane of a lens combine to form a 2D lattice. The axial symmetry of the system allows rotation of the lattice to be realised.

on a lens. Two orthogonal 1D lattices were combined to form a 2D rotating or accordion lattice. A similar arrangement with a single pair of incident beams has been used to realize a 1D accordion lattice [87]. Figure 4.2 shows a simplified version of the arrangement used to generate two parallel beams of light which rotated $180^\circ$ out of phase with each other about the optical axis of the system, highlighting the novel aspects of the system. Clearly an important criterion for a pair of beams is that they must have the same frequency and polarization in order to interfere with one another. The Michelson interferometer-like arrangement shown in Fig. 4.2 generates two output beams from a single input beam. The long arm of the device contained a lens, L2, which introduced a parallel displacement about the optical axis between the two beams. By generating this second, displaced beam from the first we ensure both beams have the same polarization and frequency and thus will interfere with each other. This would not be the case if two individual AODs were used to rotate two independent beams. The entire setup had cylindrical symmetry about the optical axis of the system and hence rotation could be realized.

The distances between the non-polarizing beam splitter cube (BS), L2 and
4.1. Creating dynamic lattices

M2 in the long arm were chosen such that the output beams from the short and long arms had the same waist and convergence despite the unequal path lengths. This can be seen most easily by realizing that the long arm of the arrangement in Fig. 4.2 contains a conventional 4f optical imaging system, that is, a one-to-one telescope, mapping the waist and wavefront curvature of the incoming beam onto the beam leaving the long arm [88]. The beam incident on the BS did not therefore need to be collimated to ensure that the two output beams had identical Gaussian characteristics. The unequal path lengths of the different arms mean that the laser must have a coherence length greater than the path difference, however this requirement is easily satisfied by lasers typically used for creating optical lattices and by a reasonable choice for L2. For L2 we used a 10 cm focal length achromatic doublet, while a Titanium-doped Sapphire laser operating at 830 nm (coherence

Figure 4.2: Experimental arrangement for the generation of rotating interference fringes. BS is a non-polarizing beam splitter cube, L1-L3 are converging lenses, M1 and M2 are mirrors.
4.1. Creating dynamic lattices

length $\gg 1$ m) was used for the lattice beams.

We relate the electric field distributions in the front and back focal of L3 using the geometrical optics argument shown in Fig. 4.3, considering both the ideal case of a thin lens and the more realistic case of a high numerical aperture objective lens. Neglecting the Gaussian envelope of the beams the electric field in the back focal plane of L3 is

$$E(r_F, F) = E_0 \left[ \cos(k_1 \cdot r_F - \omega t) + \cos(k_2 \cdot r_F - \omega t) \right]$$ (4.1)

$$= 2E_0 \cos \left( \frac{(k_1 + k_2) \cdot r_F}{2} - \omega t \right) \cos \left( \frac{(k_1 - k_2) \cdot r_F}{2} \right)$$ (4.2)

where $r_F$ designates coordinates in the back focal plane of L3, $F$ is the focal length of lens L3, $k_1$ and $k_2$ are the wavevectors of the intersecting beams as shown in Fig. 4.3 and $\omega$ is the angular frequency of the laser beams. The resultant intensity thus has the form

$$I(r_F, F) = 2I_0 [1 + \cos(k_1 - k_2) \cdot r_F].$$ (4.3)

Equation 4.2 describes two beams with identical linear polarization parallel to the resultant intensity fringes. While it is always possible to arrange the beam polarizations like this for a two-dimensional accordion lattice this is not the case for a rotating lattice. If using linear polarization with a rotating beam the electric field component in the plane of the lattice will change with rotation, leading to a modulation in lattice depth. This problem can be eliminated by using the same circular polarization for a pair of interfering beams. Given the symmetry of the arrangement the electric field component in the plane of the interference fringes remains constant as the lattice rotates, and there is no modulation of the lattice depth.
Figure 4.3: It is convenient to describe the resultant intensity in the back focal plane of L3 ($z = F$) in terms of the coordinates of the incident beams in the front focal plane of the lens ($z = -F$). The arrangement of Fig. 4.2 produced the second beam such that $x_2(t) = -x_1(t)$, $y_2(t) = -y_1(t)$, while $x_1(t)$, $y_1(t)$ were controlled by steering the laser beam with the AOD. $D$ is the separation of the two beams incident on L3.

From the geometry shown in Fig. 4.3 (assuming a thin lens) wavevectors $k_1$, $k_2$ can be expressed as $k_i = (2\pi/\lambda)(-x_i\hat{e}_x - y_i\hat{e}_y + F\hat{e}_z)/(x_i^2 + y_i^2 + F^2)^{1/2}$, where $x_i(t), y_i(t)$ are the coordinates of the beams in the front focal plane of L3. However high numerical aperture objective lenses have a finite thickness and are designed to obey the Sine Condition [48] to ensure good imaging of off-axis objects. A consequence of obeying the Sine Condition is that the wavevectors $k_1$, $k_2$ make a different angle to the optical axis, with $k_i = (2\pi/\lambda)(-x_i\hat{e}_x - y_i\hat{e}_y + F\hat{e}_z)/F$. The optical system of Fig. 4.2 created two beams in the front focal plane of L3 such that $x_2(t) = -x_1(t)$, $y_2(t) = -y_1(t)$. Equation 4.3 then becomes

$$I(x_F, y_F, F) = 2I_0[1 + \cos\frac{2\pi}{\lambda}\left(\frac{2x_1x_F + 2y_1y_F}{F}\right)].$$

(4.4)

We controlled $x_1(t)$ and $y_1(t)$ by steering the beam with the AOD, e.g. for a lattice rotating at an angular frequency of $\Omega$, $x_1(t) = (D/2)\cos\Omega t$, $y_1(t) = (D/2)\sin\Omega t$, where $D$ is the separation of the two beams in the front focal plane of...
L3. Substituting this into Equation 4.4 one finds

\[ I(x_F, y_F, F) = 2I_0[1 + \cos \frac{2\pi}{d}(x_F \cos \Omega t + y_F \sin \Omega t)] \] (4.5)

where

\[ d = \frac{\lambda F}{D} \] (4.6)

is the periodicity of the lattice\(^1\). Li et al. [87] measured the periodicity of lattices formed in the focal plane of an achromatic doublet and found this expression for the lattice spacing to be accurate for intersection angles up to 38°. Equation 4.5 describes a one-dimensional lattice rotating with angular frequency \(\Omega\).

An accordion lattice can also be implemented in this arrangement by using the AOD to change \(D\), the separation of the beams in the front focal plane of L3. \(D\) is determined by the angle of deflection of the beam from the optical axis (\(\Delta \phi\) in Fig. 4.2) and the focal length of L1, \(D = 2f_1 \tan \Delta \phi\). The smallest lattice spacing achievable, \(d_{\text{min}}\), is set by the maximum intersection angle of the beams and hence the numerical aperture (N.A.) of objective lens L3, \(d_{\text{min}} = \lambda/(2\text{N.A.})\). The minimum lattice periodicity possible is of considerable importance in experiments with cold atoms as quantum tunnelling and the mutual interaction energy of two atoms on the same site become negligible at larger lattice spacings. High N.A. objective lenses have been successfully interfaced with cold atoms, for example in [89] an N.A. of 0.7 was used. For 830 nm light this would give \(d_{\text{min}} \approx 600\) nm. The largest possible lattice period was limited by the Gaussian profile of the laser beams, that is, to achieve a reasonable number of fringes in the central, uniform

\(^1\)This result seems counterintuitive as it is what one would expect in the limit of small intersection angles but one must go beyond this small angle regime in order to achieve lattice periods suitable for a cold atom experiment. Essentially the Sine Condition employed in the manufacture of a high numerical aperture lens of non-negligible thickness means that the paraxial approximation is obeyed over much larger angles than what one would usually expect.
intensity region of the beam profile $d$ is required to be smaller than the beam waist.

### 4.2 Acousto-optic deflection and Direct Digital Synthesis of radiofrequency signals

#### 4.2.1 Acousto-optic deflection

Acousto-optic devices rely on the interaction of light and sound in a medium to provide an important means of controlling optical radiation. Acousto-optic modulators are commonplace in atomic physics experiments, providing a way to vary the frequency of a laser beam and control its amplitude on short timescales ($\sim 100\,\text{ns}$). An AOD works on the same principle, that is incident light is diffracted by a periodic modulation of the refractive index of a crystal, but has several features which optimise the device for deflection. We used a dual axis acousto-optic deflector (Isomet LS110A-830XY) to generate the rotation of a laser beam. Each device consisted of two orthogonally mounted crystals of tellurium dioxide\(^2\) (TeO\(_2\)) allowing two-dimensional steering of a beam. One feature enhancing deflection is that the Isomet LS110A-830XY employs a particularly slow shear sound wave in the TeO\(_2\) crystal, with $v_s = 616\,\text{m/s}$, to achieve a relatively large (for an acousto-optic device) deflection. The LS110A-830XY has a central frequency of $\nu_c = 50\,\text{MHz}$ and a 25 MHz bandwidth, corresponding to a maximum deflection of 1.9°.

In order to achieve the Bragg condition for a diffracted maximum for a range of sound wave frequencies steering of the acoustic wave in the crystal is commonly employed in AODs [90]. The acoustic wave steering is controlled passively by splitting the rf signal to a crystal into two and sending one signal down a delay

\(^2\)Also known as paratellurite.
4.2. Acousto-optic deflection and Direct Digital Synthesis of radiofrequency signals

Figure 4.4: Photograph of an Isomet LS110A-830XY. Note each crystal has two piezoelectric transducers to launch an acoustic wave into the crystal. Changing the phase between the rf signals at each transducer allows the sound wave to be steered, as discussed in the text.

Figure 4.4 displays one of our AODs. Two piezoelectric transducers can be seen on each crystal. The relative phase between the rf signals at each transducer introduced by the delay line is responsible for the beam steering. The length of the delay line is chosen such that as the radio frequency \( \nu \) is changed the acoustic wave is appropriately steered to meet the new Bragg angle for diffraction. This allows a reasonable uniform efficiency (\( \sim \pm 5\% \)) over the AOD bandwidth. However power fluctuations of this magnitude are undesirable when implementing an optical lattice, and we were able to compensate for the deterministic change in diffraction efficiency with deflection angle by calibrating the rf amplitude appropriately at each frequency, see Section 4.3.6.
4.2.2 Direct Digital Synthesis

We control the beam steering of a beam diffracted by an AOD by using a custom Direct Digital Synthesis (DDS) based system to generate the desired rf signals. The system was designed and built by the Central Electronic Workshop at Oxford and Ben Fletcher [47]. The DDS system was in fact originally constructed for an experiment which had envisaged using a large number (~10) of individual radiofrequency waves in an AOD (creating an array of deflected beams). Thus a compact, modular system with multiple DDS chips (the Analog Devices AD9852) had been built rather than buying a large number of commercial signal generators.

For the generation of dynamic optical lattices a maximum of four DDS chips were required (two AODs, each with two crystals). Attractive features of the DDS system included:

- High precision frequency generation: 48-bit frequency resolution (corresponding to the µHz level) was possible with the AD9852 though we generally used 24-bits, allowing the frequencies to be specified to better than 1 part in $10^6$;

- Frequency agility: the output frequency could be changed in a phase-continuous manner at a rate of more than ten million new frequencies per second (fast and accurate frequency changing would have been more of a problem if fractional phase-locked loops had been used rather than DDS for example).

The traditional drawbacks of direct digital synthesis technology, i.e. restriction to frequencies below 400 MHz, and a fixed noise floor due to the final digital-to-analogue conversion, were not problematic for our application. Frequencies were only required in the range $37.5 - 62.5$ MHz and the 12-bit digital-to-analogue converter of the AD9852 chip meant the amplitude noise was suppressed by better
than 52 dB in this frequency range [91]. As shall be discussed in Section 4.3.6 this was not the limiting factor determining the power stability of the deflected laser beams. The DDS system was controlled using Labview software, allowing fast and flexible user-friendly control of a beam steered by an AOD.

4.3 Experimental arrangement and alignment

4.3.1 Laser system

The light employed to create the optical lattice came from a Coherent MBR-110 single-frequency Ti:Sapphire laser. The laser could be tuned over a considerable wavelength range, from $\sim 770 - 850$ nm. We chose to use 830 nm light, combining a desired large detuning from the $^{87}$Rb D2 transition with a wavelength at which commercial optics were widely available. The MBR-110 laser has several features to ensure single frequency operation with a small line width ($< 1$ MHz). The cavity is a bow-tie configuration and contains a Faraday isolator to ensure unidirectional lasing. This ring cavity reduces spatial hole burning to a minimum. The laser also contains an intra-cavity etalon which is used in conjunction with an electronic servo system to prevent the laser mode-hopping. This allowed stable operation on a single longitudinal mode of the cavity. Unlike most off-the-shelf MBR-110s our laser did not contain a reference cavity for active frequency stabilization. This meant the laser line width could not be reduced to the optimal manufacturer’s specifications of 10 kHz, however this was not necessary for our application. The MBR-110 laser was pumped with 10 W of 532 nm light from a Coherent Verdi V-10 laser, and produced a maximum of 1.3 W of light at 830 nm. The Ti:Sapphire laser was situated on its own optical table, with a laminar air flow unit (Bassaire)
4.3. Experimental arrangement and alignment

Figure 4.5: Optical arrangement on Ti:Sapphire laser table. L1 and L2 are singlet lenses with focal lengths 20 cm and 50 cm respectively.

preventing contamination of the laser optics with dust.

Figure 4.5 shows the arrangement on the Ti:Sapphire table. An acousto-optic modulator (AOM) (Crystal Technology 3080-122) was used as an element to control the laser power, both for ramping the power and as an element in a PID feedback loop to stabilize the laser power (see Section 4.3.6). A single-mode polarization maintaining fibre was used to transmit the light to the main optical table to the optics responsible for creating the dynamic lattice. The AOM and fibre transmitted 75% and 60% of incident light respectively, to give around 0.5 W of 830 nm light on the main optical table.

4.3.2 Two AOD set-up for two-dimensional dynamic lattices

While Fig. 4.2 shows the arrangement for generating a single set of rotating fringes it is possible to combine two orthogonal sets of fringes to create a two-dimensional optical lattice. The most straightforward and effective way to do this is to introduce
4.3. Experimental arrangement and alignment

a second AOD to rotate a beam $\pi/2$ out of phase with the first. These two rotating beams can then be combined at a beam splitter cube before both entering the Michelson interferometer-like optics. We used such a two-AOD arrangement to create our two-dimensional optical lattices. A plan of the optical arrangement used is shown in Fig. 4.6.

Although Fig. 4.2 highlights the crux of the scheme there are several additional important features apparent in Fig. 4.6. The first of note is the use of a polarizing beam splitter (PBS) cube in the interferometer and others throughout the set-up. Use of a non-polarizing beam splitter cube results in a loss of half the incident power in the interferometer. By using a PBS cube and having a quarter-wave plate in each arm no power was lost in the interferometer. However a more serious problem has emerged; two beams leaving different arms of the interferometer have orthogonal polarizations and thus will not interfere with one another. To circumvent this problem a half-wave plate after the interferometer rotated the polarization of each beam by $45^\circ$, such that each beam had equal horizontal and vertical components of the electric field. A second PBS then split each beam into horizontal and vertical components. We retained the reflected output of this PBS cube, where all the beams were vertically polarized. Although ultimately the same power is lost using a polarising or non-polarising beam splitter we found it convenient to work with polarization-dependent cubes and the light transmitted through this second PBS cube was monitored by a video camera to aid alignment (see Section 4.3.5).

The second point of note is the lattice was finally formed in the focal plane of an objective lens a large ($\sim 4 \text{m}$) distance from the interferometer. Between the interferometer and the objective lens several lenses were employed to achieve the desired lattice period and beam waist at the atoms. A simplified diagram of the lens system is shown in Fig. 4.7. All five lenses, L1 and L3-L6, (L2 forms part of
4.3. Experimental arrangement and alignment

Figure 4.6: Layout of optical lattice optics.

Note: this diagram only shows optics and components responsible for generating an optical lattice at the position of the $^8$ Rb condensate. Lasers and optics responsible for laser cooling, imaging etc are not shown for simplicity.
4.3. Experimental arrangement and alignment

Figure 4.7: Schematic lens arrangement between the AOD and objective lens. L1, L5, and L6 are achromatic doublets with focal length 75 cm. L3 and L4 are achromatic doublets with focal lengths of 10 cm and 20 cm respectively. The lenses have the same labels as in Fig. 4.6. The interferometer optics are not explicitly shown in this diagram, rather only the action of the interferometer in generating a second beam is displayed. $D$ is the separation of a pair of beams in the front focal plane of the objective lenses, as defined in Section 4.1.

the interferometer) and the four-lens objective were spaced such that they form a series of telescopes, i.e. two lenses were separated by the sum of their focal lengths. For an initial deflection of angle $\Delta \phi$ from an AOD, the separation in the front focal plane of the objective lens is

$$D = 2 \left( \frac{f_6}{f_5} \right) \left( \frac{f_4}{f_3} \right) f_1 \tan (\Delta \phi)$$  \hspace{1cm} (4.7)

where $f_i$ is the focal length of lens L$_i$. The angle of deflection about the centre frequency of an AOD is given by

$$\Delta \phi = \frac{\lambda \Delta \nu}{v_s}$$  \hspace{1cm} (4.8)

where $\lambda$ is the wavelength of light, $\Delta \nu = \nu - \nu_c$ is the difference in frequency of the sound wave propagating in the AOD crystal from the centre frequency and $v_s$ is the speed of sound in the crystal$^3$. Substituting this expression for $D$ into Equation

---

$^3$More correctly $\sin(\phi/2) = \lambda \nu/2v_s$ which is the Bragg condition for a diffracted maximum, but Equation 4.8 is a good approximation for the small angles of deflection achievable.
4.3. Experimental arrangement and alignment

4.6 and making the approximation \( \tan(\Delta \phi) = \Delta \phi \) (a very good approximation up to the maximum deflection angle \( \Delta \phi = 1.9^\circ \) for our AODs) we find

\[
d = \frac{1}{2} \left( f_5 \right) \left( f_3 \right) \left( f_{\text{obj}} \right) \left( \frac{v_s}{\Delta \nu} \right).
\]

(4.9)

Note this expression is independent of \( \lambda \) as the angular deflection of the beam by the AOD for a given \( \Delta \nu \) is proportional to \( \lambda \). For a fixed deflection \( \phi \) smaller \( \lambda \) would of course give a smaller lattice spacing. For our chosen focal lengths

\[
d \approx \frac{8.2 \mu \text{m MHz}}{\Delta \nu},
\]

(4.10)

is predicted, where we have taken \( v_s = 616 \text{ m s}^{-1} \) from the literature [92, 93] where it is believed to be accurate to \( \pm 3\% \); we did not independently calibrate the angular deflection of an AOD as a function of \( \Delta \nu \). Calibration of the lattice spacing is described in Section 4.3.3.

Relating the laser beam waist in the back focal plane of the four-lens objective (i.e. at the atoms) and the waist at the AOD is also straightforward with this lens system. The waist at the atoms, \( w_f \), is given by

\[
w_f = \left( \frac{f_{\text{obj}}}{f_6} \right) \left( \frac{f_3}{f_4} \right) \left( \frac{f_3}{f_1} \right) w_i
\]

(4.11)

where \( w_i \) is the initial waist at the AOD and \( f_{\text{obj}} \) is the focal length of the objective lens. We used an initial \( 1/e^2 \) waist of 2.8 mm at the AOD, giving a waist of 70 \( \mu \text{m} \) at the atoms.

In order to prevent interference between the two one-dimensional lattices one pair of lattice beams must be detuned by order of MHz relative to the other pair. This can be achieved by operating the two AODs about different centre frequencies
which are separated by MHz. As the rf signal applied to each AOD is modulated
sinusoidally one must ensure the centre frequencies for each AOD are separated
by more than twice the modulation amplitude. Alternatively an acoustic-optic
modulator can be used to shift the frequency of a beam before it enters an AOD.

### 4.3.3 Calibration of the lattice constant

A convenient aspect of our vacuum system was that it was built on rails and the
quartz cell could be rolled out from the region where the BEC was formed. Re-
moving the vacuum cell allowed a direct measurement of the optical lattice spacing
at the position of the BEC to be made, a rare luxury for cold atom experiments.
As the optical lattice was formed in the focal plane of the high numerical aperture
four-lens objective it was possible to image the lattice using this objective. A flat
plate was inserted at the focus of objective lens, reflecting a small amount of the
lattice light back along the axial imaging path which formed an image of the op-
tical lattice on the Andor camera. By monitoring the image of the optical lattice
and iteratively adjusting the plate to ensure the minimum lattice period could be
resolved it was possible to carefully align the plate \(^4\).

The lattice spacing was calculated by Fourier transforming the image. Fig-
ure 4.8 shows the measured values of the lattice constant as a function of the
frequency in the AOD. The error in \(d\), which was \(\leq 1\%\) (slightly less than the size
of the blue circles), was dominated by the error in the magnification of the axial
imaging arrangement. The black line is a fit to the measured data of the form

\(^4\)Considering the Abbé theory of imaging it is clear if an optical lattice is formed in the focal
plane of a lens it is possible to resolve an image of the lattice using the same lens.
4.3. Experimental arrangement and alignment

Figure 4.8: Direct measurement of optical lattice period, $d$, as a function of AOD frequency. The hollow blue circles denote the measured lattice constant; the black line is a fit of the form $d = C/\Delta\nu$ where $C = 8.53(4)$. The optical lattice was directly imaged along the axial absorption imaging path as described in the text and $d$ calculated from the Fourier transform of this image.

$$d = C/\Delta\nu$$

with free parameter $C$. The best fit gave

$$d = \frac{8.53(4) \mu\text{m MHz}}{\Delta\nu}, \quad (4.12)$$

where we note the discrepancy of a few percent with the predicted value of $C = 8.2 \mu\text{m MHz}$ from geometrical optics (see the preceding section) may be due to aforementioned uncertainty in $v_s$.

While being able to directly measure the lattice periodicity at the focal plane of the four-lens objective is a huge advantage it is important to check this calibration with other methods. One important consideration is that this direct measurement
4.3. Experimental arrangement and alignment

Figure 4.9: Design of the four-lens objective, from [94]. The compound objective was arranged such that the atoms would be on the right-hand side of the diagram. It cannot take into account the effect of the vacuum cell wall which will be present between the atoms and the objective lens in an experiment. In theory a plane glass wall should not effect the lattice spacing (the angle of intersection does not change though the axial position of the focus does) however if the wall is not sufficiently flat it could introduce a difference. In Chapter 5 we discuss additional calibration of the lattice constant using in situ imaging of cold atoms in the lattice and matter-wave diffraction from the lattice.

### 4.3.4 Four-lens objective

The four-lens objective was designed around four years ago as a relatively inexpensive high numerical aperture lens with a large (40 mm) working distance for detecting fluorescence from small number of atoms. However these criteria allowed it to be successfully used for axial absorption imaging and forming an optical lattice as described in this thesis. The design is based on that of Alt [95], with slight
4.3. Experimental arrangement and alignment

modifications for our specific experiment (taking into account the 2 mm thickness of the cell wall for example). Four spherical lenses were mounted in a brass tube, separated with precisely machined spacers, as shown in Fig. 4.9. The main idea behind the multi-lens system was the four lenses together can compensate for the spherical aberration of the individual lenses. The effective numerical aperture of the compound system was 0.27, with a resolution of 1.8 µm at 780 nm. The design and testing of the four-lens objective is described in [94].

In recent years custom-made aspheric lenses have become more widely commercially available and it is possible that the next generation of the experiment could even use a single such aspheric lens inside the vacuum chamber. This would allow higher numerical apertures to be reached, achieving higher resolution imaging and dynamic optical lattices with smaller minimum periodicities. The brass tube containing the lens system was mounted on a five-way adjustable stage (giving x, y, z and pitch and yaw control) to aid alignment.

4.3.5 Alignment

Creating a two-dimensional rotating optical lattice of sufficiently high quality for a cold atom experiment is unfortunately not a trivial task. A careful alignment of the optical system of Fig. 4.6 is vital. In this section a recipe for good alignment is detailed, along with the measurements made to check the lattice was stable upon rotation. (All references to lens L1, mirror M1 etc. refer to Fig. 4.6.)

The critical points to address are:

- The periodicity of the lattice must remain constant as the lattice rotates.

This could be affected by a misalignment of the interferometer part of the

\footnote{An accordion lattice is a bit easier, with a performance more robust to misalignment}
4.3. Experimental arrangement and alignment

arrangement which is responsible for generating a second beam rotating $\pi$ out of phase with the first.

- The centre site that the lattice rotates about must remain stationary. Lenses which are not aligned on the centre of the optical axis of the system (introducing significant astigmatism) could introduce a path difference between interfering beams which would cause a translation of the lattice sites as the lattice rotates.

The interferometer was aligned using the following procedure:

- The beam deflected by an AOD was set to lie along the optical axis of the system, i.e. this was the beam that lay in the centre of the circle traced out by a rotating beam.

- L1 and L2 were removed. Considering initially only the beam from AOD 1 the interferometer mirrors M1 and M2 were aligned in turn by adjusting them such that the beam reflected by each mirror went all the way back through the optical system and through the optical fibre. This ensured that each mirror was perpendicular to the optical axis to a very high degree.

- The beam from AOD 2 was roughly aligned by ensuring it overlapped with the beam from AOD 1 at two points separated by a large distance. Fine adjustment of this beam was then achieved by maximising the back reflection from M1 and M2 through the fibre, using the mirrors immediately after AOD 2 to adjust the beam path (i.e. not adjusting M1 or M2 themselves which have been carefully aligned).

- The $f = 10\text{cm}$ lens, L2, was inserted into the long arm of the interferometer. This lens was mounted on a three-way translation stage and it was
4.3. Experimental arrangement and alignment

At this point the alignment of the system up to the exit of the interferometer was checked by monitoring the output beams on the video camera denoted in Fig. 4.6.

- Finally L1 was inserted. Like the other lenses in the system this lens was mounted on a fine-adjustment \( xy \) positioning mount allowing it to be accurately centred along the optical axis of the system.

critical to have it accurately positioned a focal length away from M2. A good way to achieve this was to monitor the waists of beams from both the long arm and the short arm of the interferometer a large distance away (we used \( \sim 6 \text{ m} \)) and adjust the position of L2 until they were the same. We ensured the beam passed through the centre of L2 by checking the beams from each interferometer arm overlapped perfectly at two points after the interferometer.

Figure 4.10: Effect of misalignment on the interferometer arrangement. The red and blue beams represent the incoming beam being deflected by equal and opposite angles on either side of the optical axis, i.e. monitoring the beams after the interferometer one would observe the red spots changing to the blue spots upon flipping the angle of deflection. In this way it is possible to discern different types of misalignment in the system: (a) if M1 is misaligned the beams will move towards each other or away from each other when the deflection is changed; (b) if lens L2 is not positioned correctly in the long arm of the interferometer a different effect will be seen with beams moving in the same direction. If the system was aligned correctly the spots would overlap upon flipping the deflection angle.
After the interferometer the beams should be parallel with identical waists, and should rotate about a common centre at a fixed radius. The alignment was checked by using the AOD to deflect a laser beam to the opposite side of the optical axis of the system as illustrated in Fig. 4.10. This diagram shows how different types of misalignment can be discerned using this procedure. If the deflection of the initial beam is flipped about the optical axis of the system then in the ideal situation no difference should be seen in the beam position after the interferometer. If there is an angular misalignment of M1 or M2 the beams observed after the interferometer moved towards each other or away from each other when the deflection was flipped. This would cause the periodicity of an optical lattice to change as it rotated. If L2 was not the correct distance away from M2 then a different effect would be seen with both beams being displaced in the same direction upon flipping the deflection. This would cause the centre of a rotating lattice to precess.

After the interferometer alignment had been checked and optimised all that remained was careful centring of lenses L3 to L6 on the optical axis (these lenses were also mounted on precision translation mounts). As well as observing the beam positions at points in the optical system where the beams were parallel it was also possible to look at the lattice formed at points in the system where the beams intersected (that is, between L3 and L4, L5 and L6, see Figs. 4.6 and 4.7). Upon rolling the vacuum cell out as discussed earlier in this chapter it was possible to observe the lattice at the focus of the four-lens objective. By taking pictures of a rotating lattice at different angles of rotation we were able to check that the lattice constant did not vary during rotation (within the measured error on $d$) and that the centre of rotation was stationary.

It was also possible to directly observe the optical lattice on a CCD camera while it was rotating. Watching a slowly rotating lattice allowed a qualitative
4.3. Experimental arrangement and alignment

Analysis of how well the system was aligned. A useful trick was to take a long exposure picture of a rapidly rotating lattice. Consider the intensity pattern of rotating fringes averaged over the period of rotation

$$\langle I \rangle = \frac{\Omega}{2\pi} \int_{-\pi/\Omega}^{\pi/\Omega} 2I_0 [1 + \cos \frac{2\pi}{d} (x \cos \Omega t + y \sin \Omega t)] dt$$  \hspace{1cm} (4.13)

$$= 2I_0 \left[ 1 + \frac{\Omega}{2\pi} \int_{-\pi/\Omega}^{\pi/\Omega} \cos \frac{2\pi r}{d} \cos (\theta - \Omega t) dt \right], \hspace{1cm} (4.14)$$

where we have switched to polar coordinates in the second line. Along a line of constant $\theta$, i.e. radially outwards this gives

$$\langle I(r) \rangle = 2I_0 \left[ 1 + J_0 \left( \frac{2\pi r}{d} \right) \right], \hspace{1cm} (4.15)$$

where $J_0(z)$ is the zero-order Bessel function of the first kind. A long exposure ($t \gg 2\pi/\Omega$) photograph of a rapidly rotating lattice is shown in Fig. 4.11 along with a cross-cut through the centre of the time-averaged intensity pattern. The contrast and radial variation of the intensity fits the form of a zero-order Bessel function extremely well, confirming the lattice is not distorting during rotation. If the centre lattice site about which the lattice rotates was moving significantly we would see a broadening of the central peak.

4.3.6 Lattice depth stability

In optical lattice experiments it is standard practice to actively stabilize the power of the optical lattice beams, thus reducing parametric heating of atoms in the lattice to a minimum. A moving lattice beam introduces new challenges for power stabilization, and fluctuations in lattice depth arise from several different sources:
4.3. Experimental arrangement and alignment

Figure 4.11: (a) Long exposure photograph of a rapidly rotating set of fringes, showing the time-averaged intensity pattern. (b) The blue line is the horizontal cross-cut through the centre of image (a), the red line is theoretically predicted dependence along \( y = 0 \), a zero-order Bessel function as discussed in the text (with a Gaussian envelope). The excellent agreement shows the lattice is well-aligned and not distorting or precessing doing rotation.

1. Fluctuations in polarization (the fibre coupling light from the Ti:Sapphire table to the rotating lattice optics was a polarization-maintaining fibre, but residual fluctuations still occurred);

2. Noise associated with a stationary laser beam, e.g. coming from intrinsic amplitude noise in the laser and power fluctuations in the optical fibre;

3. The deterministic variation in power of a rotating beam due to the non-uniform diffraction efficiency of an AOD across its bandwidth;

4. Amplitude fluctuations introduced by the movement of a beam itself. For example we observed fluctuations in power as a beam moved over the surface of an optical element and hit a bit of dust or a scratch.

Issues 1 and 2 are common to all optical lattice experiments and were dealt with in a standard fashion. Figure 4.6 shows light was picked off for the power servo (a PID servo feeding back to the AOM on the Ti:Sapphire table) after an initial polarizing
beam splitter cube. The first PBS cube converted polarization fluctuations from
the fibre into power fluctuations which were reduced by servo control. If this was
not used polarization fluctuations would lead to an anti-correlation between the
beam power picked off for the servo control photodiode and the power of the main
beam, resulting in the servo amplifying the effect of the fluctuations.

The variation in diffraction efficiency across the bandwidth of an AOD is deter-
ministic and reproducible for a given beam alignment into the AOD. As discussed
in Section 4.2.1 despite steering of the sound wave in the AOD crystals this vari-
ation was quite large in our system, around \( \pm 5\% \). This change in power with
rotation was accounted for by adjusting the amplitude of the rf wave in an AOD
crystal at each frequency appropriately.

Hence it was possible to stabilize the power of a single rotating beam from
each AOD. Difficulties arose after the interferometer where two rotating beams
were created from each single one. As a pair of beams is derived from a single
beam the two beams in an interfering pair could not be individually controlled.
As the two beams rotated they sampled the same surfaces but after a time lag
of \( \pi/\Omega \). If a beam moved over an imperfection on a mirror surface for example
it was possible to monitor the resulting fluctuation in power and correct the rf
power in the AOD at that point to compensate, but this correction would cause an
unnecessary power fluctuation in the other beam in the pair. In order to minimize
any such variations in power between a pair of beams after the interferometer we
ensured all the mirrors, lenses and waveplates in the optical system were free from
scratches and kept free of contamination by dust by a box enclosure surrounding
the optical table. However there are a large number of optical elements in the
arrangement for creating the lattice and some power variation still remained.
The solution to this problem lies in noting that the depth of the lattice formed by two beams with intensities $I_1$ and $I_2$ is proportional to $\sqrt{I_1 I_2}$. As the individual intensities could not be stabilized, $I_1$ and $I_2$ were measured in turn allowing $\sqrt{I_1 I_2}$ as a function of position during rotation to be calculated. The correction which needed to be applied to $I_1$ and $I_2$ in order to smooth out $\sqrt{I_1 I_2}$ was then calculated and implemented by once again adjusting the RF power (and hence the diffraction efficiency) in an AOD crystal by the desired amount.

After all these various steps to ensure stability of the lattice depth we found the depth of a rotating lattice was stable to $\pm 1\%$, which was a factor of ten worse than the depth fluctuations in a static lattice $\sim 0.1\%$. The extra instability associated with rotation was naturally at frequencies of $\Omega$ and its harmonics.

### 4.3.7 Pros and cons

The pros and cons of our approach to making rotating optical lattices are summarized below.

**Pros**

- The use of an acousto-optic deflector to generate the rotation of the lattice allows intrinsically smooth rotation free from mechanical instabilities;

- Acousto-optic deflection also enables a excellent degree of control over the rotation frequency, for example the condensate can be loaded into a static lattice before the rotation rate is smoothly increased from zero, an important feature;

- A large range of rotation frequencies are possible, $0 \leq \Omega \leq 1$ (kHz);

- The lattice constant can be changed.
4.4 Alternative single AOD arrangement

This represents a substantial improvement over the only previous rotating optical lattice experiment where rotation was generated mechanically at $\sim 8$ Hz using a revolving mask [17].

Cons

- A reasonably large number of optical elements were used in the generation of the optical lattice, with fourteen mirrors, seven lenses, two PBS cubes, and three waveplates used after the acousto-optic deflector (see Fig. 4.6). No optical element is perfect and having a large number of them leads to a slight degradation of the Gaussian profile of the lattice beams. Having multiple optical elements also leads to the issues limiting the stabilization of the depth of a rotating lattice discussed in section 4.3.6.

A ‘standard’ optical lattice experiment typically uses only two lenses and one mirror to create a retro-reflected standing wave. It should of course be noted that the chain of four lenses and twelve mirrors mapping the output of the interferometer to the front plane of the four-lens objective in our arrangement are not intrinsically necessary for generating a rotating lattice, but are required for our particular set-up due to limitations on optical table space. Arranging the interferometer and AODs to be closer to the four-lens objective would substantially reduce these extra optical elements and improve the Gaussian profile of the lattice as well as the depth stability.

4.4 Alternative single AOD arrangement

It is worth noting in passing that it is possible to make a two-dimensional rotating lattice using a single AOD. An arrangement for achieving this is shown in Fig. 4.12.
4.4. Alternative single AOD arrangement

The pair of beams from the interferometer arrangement of Fig. 4.2 is split at a 50:50 polarizing beam splitter cube. An out-of-plane reflection can then be used to rotate one pair of beams by 90° before recombining all four beams at the second polarising beam splitter. The out-of-plane reflection can be achieved using a periscope (P1 in Fig. 4.12) which is orientated to deflect the beam direction by 90°.

The use of a periscope to rotate an image by 90° is one of the simpler examples of using out-of-plane reflections to rotate an image. In general any arbitrary rotation can be achieved. A convenient way to calculate the rotation introduced by an optical system is by considering the geometric phase accrued by an image bearing beam. This reduces the analysis of image rotation to simple geometric constructions and the calculations of areas on a sphere [96]. For example one can envisage using double Porro prisms at arbitrary relative angles to achieve an arbitrary rotation. A rotating triangular optical lattice could be created in this way.

There is, however, a catch with the single AOD arrangement. The two or-
Figure 4.13: Local polarization across a two-dimensional lattice formed by two orthogonally polarized standing waves with the same frequency. The polarization at a lattice site depends on the relative phase $\phi$ between the two standing waves: (a) For $\phi = 0^\circ$ the polarization is linear everywhere and an atom experiences the same lattice depth at each lattice site; (b) for $\phi = 90^\circ$ the polarization at adjacent lattice sites changes from $\sigma^+$ to $\sigma^-$, resulting in a 13% change in lattice depth for the $|F = 1, m_F = -1\rangle$ hyperfine state atom if the lattice wavelength is $\lambda = 830$ nm.

Orthogonal pairs of beams created using the single AOD scheme have orthogonal polarizations but the same frequency as all four beams are derived from the same original beam. Unless the two out-of-plane paths of Fig. 4.12 are phase-stabilised the relative phase between the orthogonal standing waves at the atoms, $\phi$, may fluctuate in time. The electric field at the atoms has the form $E = 2E_0[e_x \cos (\pi x/d) \cos(\omega t + \phi) + e_y \cos (\pi y/d) \cos \omega t]$. As $\phi$ changes the effect is to change the local polarisation at a lattice site, as considered in Fig. 4.13 for the extreme cases of $\phi = 0^\circ$ and $\phi = 90^\circ$.

The lattice depth depends on the local polarization according to Eq. 3.13.
The polarization dependence of $V_0$ becomes weaker for larger detunings, but is still surprisingly large at $\lambda = 830\,\text{nm}$. The difference in lattice depth for $\sigma^+$ and $\sigma^-$ light is 13% at $\lambda = 830\,\text{nm}$ for atoms in the $|F = 1, m_F = -1\rangle$ hyperfine state, meaning the light shift at adjacent lattice sites could differ by this much for $\phi = 90^\circ$, as shown in Fig. 4.13. If $\phi$ is stabilised then a near-resonant lattice can be made spin dependent in this way, while sub-Doppler cooling is also possible [56]. Note that this issue of polarization also arises in three-dimensional lattice experiments where it is standard practice for the orthogonal standing waves to have frequencies differing by tens of MHz.

While we used the two-AOD arrangement to avoid the issues discussed here the single-AOD scheme is worth noting for: (i) possible applications where the local polarization is not critical, perhaps in biophysics applications or microfluidic sorting [97] rather than in atomic physics; (ii) for a larger detuning from resonance the effect becomes smaller, e.g. for $\lambda = 1064\,\text{nm}$ the difference in the light shift for $\sigma^+$ and $\sigma^-$ polarizations is only 2%.
A two-dimensional optical lattice with variable spacing

As optical lattices have provided a paradigm shift in BEC experiments it is interesting to consider how flexible such lattices can be made. Optical lattices can be moved at a constant velocity or accelerated by changing the relative frequency difference between interfering beams, and experiments have even been carried out with a ‘shaking’ lattice [98]. Having the ability to change the periodicity of an optical lattice quickly between experimental runs is an extremely convenient tool, allowing an experimentalist to explore large parameter spaces. Trey Porto and Bill Phillips’s group at NIST, Maryland, are the only group to have previously used such an accordion lattice (though in one-dimension only) [85]. Accordion lattices may also be useful for expanding or compressing the lattice period during an experimental run. Although the main focus of this thesis is on experiments with a rotating optical lattice this chapter gives details of the first implementation of a two-dimensional accordion lattice.
5.1 Calibration of lattice depth and periodicity

Section 4.3.3 described the calibration of the optical lattice constant by direct imaging of the lattice. Here further measurements of the lattice constant using \textit{in situ} imaging of atoms in the optical lattice and matter-wave diffraction are considered.

5.1.1 \textit{In situ} imaging of ultracold atoms in a large period optical lattice

In this section we consider \textit{absorption} imaging of a Bose-Einstein condensate loaded into optical lattices with spacings $4.2(1) \mu m \leq d \leq 8.4(1) \mu m$. The condensate contained around $2 \times 10^5$ atoms and thus for this range of lattice spacings there were expected to be $N_j \sim 24,000$ atoms in central sites for $d = 8.4 \mu m$ and $N_j \sim 6000$ atoms in central sites for $d = 4.2 \mu m$. The high resolution of the axial imaging arrangement allowed us to directly resolve the atom distribution in an optical lattice.

It is worth noting that a long term goal of the cold atom community has been the detection and resolution of single atoms in optical lattices. Direct imaging and manipulation of an atom at a lattice site will allow a new breed of quantum simulation and quantum computing experiments with unparalleled control. As of Autumn 2009 experimentalists are on the cusp of entering this exciting new era. Single atom resolution in a 690 nm lattice has been achieved by Marcus Greiner’s group at Harvard [52] building on earlier work by David Weiss’ group at Penn State who imaged single atoms in a 5 \mu m lattice which was loaded from a MOT [99]. These experiments used fluorescence imaging of atoms trapped in very deep potentials. In this thesis we used absorption imaging of a large number of atoms.
5.1. Calibration of lattice depth and periodicity

Figure 5.1: In situ absorption imaging of cold atoms in an $d = 8.4(1) \mu m$ optical lattice. A slice of repumping light was used to selectively repump atoms in the lattice as described in the text. The repumping light was incident from the left hand side of the above images, and insufficient power led to the incomplete repumping of the atoms as can be seen in the images as the power is reduced from (a) to (c): (a) $P \sim 1 \mu W$; (b) $P \sim 100 \text{nW}$; (c) $P \sim 10 \text{nW}$.

per lattice site in relatively shallow lattices as a diagnostic for lattice spacings etc. to facilitate experiments focussing on rotation of the optical lattice but single-atom detection experiments could be possible in the future using our accordion lattice arrangement.

Figure 5.1 displays a series of absorption images of condensates loaded into an optical lattice with periodicity $d = 8.4(1) \mu m$. The lattice constant was measured by Fourier transforming the images. In order to achieve the clearest images it was found necessary to selectively repump atoms in a slice in the plane of the lattice. This was achieved by having a small amount of repumping light co-propagating with the 865 nm light for the light sheet trap (for standard absorption imaging the light to repump the atoms into the $F = 2$ level before probing was sent along the six MOT beam paths). Reducing the axial extent of the atoms in the lattice improved the imaging due to considerations such as the small depth of focus of the four-lens objective. Unfortunately a downside of using a slice of repumping light is highlighted in Fig. 5.1: the condensate and lattice parameters were the same.
for subfigures (a)-(c), the difference in the images came from varying amounts of repumping light. The repumping light was incident from the left-hand-side of the images of Fig. 5.1, and it clear that for small powers atoms at sites on the right hand side of the lattice where not sufficiently repumped. This led to the inhomogeneous appearance of image (c).

The high atomic densities at a lattice site ($\sim 10^{14}$ cm$^{-3}$) gave rise to another issue regarding absorption imaging. We observed non-negligible heating of the atoms due to the probe light, leading to the atoms appearing to ‘spill out’ of a lattice site. This was attributed to light-induced collisions between atoms which can lead to an increase in kinetic energy or molecule formation [100], and also to radiative repulsion between the atoms confined at a lattice site.

These heating processes and the finite axial extent of the atomic distribution meant lattice periods $d < 4.2 \mu$m could not be resolved using absorption imaging. Figure 5.2 shows the measured values of $d$ from the absorption images and the direct measurement data detailed in Section 4.3.3. The results agree within error suggesting the presence cell wall did not significantly affect the lattice spacing.

5.1.2 Matter-wave diffraction from large periodicity optical lattices

Observing matter-wave diffraction from an optical lattice is a standard method to measure the depth and periodicity of the lattice. In the absence of atom-atom interactions the atomic distribution after a long$^1$ time-of-flight expansion

$^1$An interesting paper by Gerbier et al. [101] shows the time-of-flight required for the atomic distribution to represent the original momentum distribution in an optical lattice can be much larger than generally thought, as long as $\sim 100$ ms. While diffraction peaks are usually clearly observed after $\sim 20$ ms the width and height of these peaks do not correspond to a Fourier transform of the initial wave function for these typical experimental expansion times.
5.1. Calibration of lattice depth and periodicity

Figure 5.2: Comparison of the calibration of optical lattice periodicity from in situ imaging of atomic distribution (red circles) and direct imaging of the optical lattice (blue circles and black fit, Section 4.3.3). The measurements agree within error (the size of the circles is equal to or less than the ±1% uncertainty on the data). Inset: further examples of in situ images of atomic distribution (a) $d = 7.0(1) \mu$m and (b) $d = 4.6(1) \mu$m.

represents a Fourier transform of the atomic wave function in the lattice. For example if a condensate is adiabatically loaded into the lowest band of a lattice with zero quasimomentum, first order diffraction peaks are observed after time-of-flight $t_{\text{TOF}}$ and are separated from the zeroth order by $\Delta x = \pm h t_{\text{TOF}}/md$. Thus $d$ can be calculated from this observed separation.

However it is important to consider the role interactions play in matter-wave diffraction. This is not usually necessary in optical lattice experiments with $d = \lambda/2 \sim 400$ nm, as the impulse energy imparted by the lattice ($\sim (1/2m)(h/md)^2 = 4E_L$), is much larger than the interaction energy. For lattices with larger spacings $E_L$ decreases and the interaction energy can significantly affect the observed
diffraction from an optical lattice in two ways:

- If $E_L < \mu$ the conversion of interaction energy to kinetic energy as the lattice is turned off means the individual diffracted orders expand faster than the orders separate in time-of-flight. In this case no matter how long $t_{\text{TOF}}$ was the diffraction orders would always overlap. Thus it was not possible to measure large lattice constants using this method.

- Even for $E_L \geq \mu$ interactions can slightly increase the separation of the diffracted orders in time-of-flight, which would lead to an underestimation of $d$. This occurs because of the initial spatial overlap between the different momentum components; at the turn-off of the optical lattice the momentum components start to move away from each other and get an additional impulse due to the repulsive interaction between them (Figure 5.3). This extra velocity due to repulsive interactions, $\delta v$, has been carefully measured in an atom interferometry experiment [102].

This latter consideration had an important effect on the diffraction pattern from the optical lattice as discussed below.

![Figure 5.3: Schematic of the diffraction process. In the absence of interactions the diffracted components would move apart with velocities $\pm \frac{h}{md}$. However the initial overlap between the different orders results in a small (but for our conditions measurable) repulsive contribution to their velocities, $\delta v$, after the lattice is turned off.](image)

As an alternative to diffraction from the lowest band of an optical lattice the condensate can be diabatically loaded into the lattice. The instantaneous turn-on
5.1. Calibration of lattice depth and periodicity

Figure 5.4: Images of diffraction patterns from two-dimensional optical lattices after diabatic loading of the condensate. The atoms were held in a lattice of depth $V_0 = 23$ kHz for $70 \mu$s and imaged after $20$ ms time-of-flight. The lattice spacings predicted by analysing the separation of the $\pm 1^{\text{st}}$ orders and $\pm 3^{\text{rd}}$ orders were different and both underestimated the predicted $d$ from the direct measurements. This was due to the effect of interactions on the diffraction pattern as discussed in the text. The lattices spacings as given by the direct measurement were: (a) $d = 2.24(2) \mu$m; (b) $d = 2.51(3) \mu$m; (c) $d = 2.84(3) \mu$m; (d) $d = 3.05(3) \mu$m; (e) $d = 3.28(3) \mu$m; (f) $d = 3.55(4) \mu$m.

of the lattice potential leads to the population of many excited bands and multiple diffraction orders can be seen in time-of-flight. The evolution of the Bloch wave functions in different bands as a function of hold time in the lattice gives rise to an interesting evolution of the diffraction pattern, as will be considered in the next section. It is interesting to compare the separation of the $\pm 1^{\text{st}}$ orders to higher diffraction orders to see if interactions are having an effect on the spacing of the diffraction peaks. Presuming the contribution to the kinetic energy from the atom-atom interactions is constant it will represent a smaller fraction of the impulse given to the $\pm 3^{\text{rd}}$ orders (for example) than the $\pm 1^{\text{st}}$ orders.

Figure 5.4 displays a series of diffraction patterns obtained by diabatically loading a BEC of around $2 \times 10^5$ atoms into optical lattices with periodicities in
the range $2.24 \mu m < d < 3.55 \mu m$. The optical lattice in each case was turned on instantaneously ($< 1 \mu s$) at a depth of $V_0 = 23 \text{kHz}$ and held on for $70 \mu s$ before sudden turn-off and $20 \text{ms}$ time-of-flight expansion. The lattice spacing was calculated by measuring the spacing of the $\pm 1^{\text{st}}$ and $\pm 3^{\text{rd}}$ diffraction orders (the centres of the diffraction peaks were fit with a Gaussian to determine the centre position). The extracted lattice constants are shown in Fig. 5.5 and Table 5.1. The $4\%$ error on the extracted values of $d$ is equal to or less than the size of the square markers.

The results show that the lattice constant calculated from both the first and third order separations were consistently smaller than the previous directly measured values for $d$. Furthermore the lattice constant inferred from the third order separations was larger than that inferred from the first order separations, strongly suggesting interactions were indeed having a significant effect.

<table>
<thead>
<tr>
<th>$\Delta \nu$ (MHz)</th>
<th>From $\pm 1^{\text{st}}$ order</th>
<th>From $\pm 3^{\text{rd}}$ order</th>
<th>From direct measurement</th>
</tr>
</thead>
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<tr>
<td>2.0</td>
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<td>3.7(1)</td>
<td>4.27(6)</td>
</tr>
<tr>
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<td>3.3(1)</td>
<td>3.5(1)</td>
<td>3.88(6)</td>
</tr>
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<td>2.9(1)</td>
<td>3.2(1)</td>
<td>3.55(5)</td>
</tr>
<tr>
<td>2.6</td>
<td>2.7(1)</td>
<td>2.8(1)</td>
<td>3.28(5)</td>
</tr>
<tr>
<td>2.8</td>
<td>2.6(1)</td>
<td>2.8(1)</td>
<td>3.05(5)</td>
</tr>
<tr>
<td>3.0</td>
<td>2.5(1)</td>
<td>2.6(1)</td>
<td>2.84(4)</td>
</tr>
<tr>
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<td>2.2(1)</td>
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<tr>
<td>3.8</td>
<td>1.9(1)</td>
<td>2.1(1)</td>
<td>2.24(3)</td>
</tr>
</tbody>
</table>

Table 5.1: Comparison of lattice constants calculated from matter-wave diffraction and direct imaging of the optical lattice. The diffraction measurements underestimate $d$ by up to $20\%$.

If the discrepancy in $d$ is presumed to come exclusively from the repulsive interactions $\delta v$ is found to be $\delta v \sim 0.3 \text{mms}^{-1}$, corresponding to a repulsion energy
5.1. *Calibration of lattice depth and periodicity*

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**Figure 5.5:** Comparison of all measurements of the lattice constant as a function of AOD frequency. The lattice spacings determined from the diffraction images are not consistent with the other data due to the effect of interactions increasing the separation of the diffraction orders. The difference is worse for larger $d$ as expected as the interaction energy provides a larger fraction of the total impulse to a diffraction order.

$U_{\text{repul}} \sim 10$ Hz (it is surprising such small energies can be relatively clearly measured). This is of comparable size to the mean-field repulsion induced velocity found in [102] by matter-wave interferometry. The data has a slight trend suggesting the interaction-induced discrepancy becomes worse for larger lattice spacings. While this is expected as the impulse contributed by the lattice becomes smaller the dependence with $d$ is not entirely straightforward. For larger period lattices the momentum components separate more slowly and thus may significantly expand while still overlapping. Under these conditions one might expect $\delta v$ to also decrease as the density of the momentum components is smaller leading to less mean-field repulsion.
5.2 Quantum vs classical in a large periodicity lattice.

The aforementioned NIST group who also have access to an (1D) accordion lattice recently considered diffraction from a one-dimensional optical lattice for a range of $d$ and hold times in the lattice. They related their results to the crossover from quantum to classical behaviour as the lattice period became larger and showed the role caustics played shaping the diffraction pattern [26] (see also the thesis of John Huckans [85]). We obtained similar results when calibrating our accordion lattice which we discuss here.

As discussed above after diabatic loading of a BEC into an optical lattice many excited bands can become populated. How the Bloch functions evolve in time leads to a more interesting diffraction pattern than the case for adiabatic loading. Monitoring this Kapitsa-Dirac diffraction as a function of hold time is another well known method enabling determination of $d$ and also of $V_0$. If the condensate dimensions are large compared to the lattice spacing the BEC can be treated as a plane wave (and here we consider the condensate to be stationary), $|0\rangle$, and the initial wave function can be expanded in terms of Bloch functions, $|\phi_n^{(q=0)}\rangle$ as

$$|\psi(t = 0)\rangle = \sum_n \langle \phi_n^{(q=0)} | 0 \rangle |\phi_n^{(q=0)}\rangle. \quad (5.1)$$

As Bloch functions are the single-particle eigenstates in the lattice potential the time-evolution of the wave function in the lattice is easily accounted for:

$$|\psi(t)\rangle = \sum_n \langle \phi_n^{(q=0)} | 0 \rangle e^{-i E_n^{(q=0)} t / \hbar} |\phi_n^{(q=0)}\rangle. \quad (5.2)$$
5.2. Quantum vs classical in a large periodicity lattice.

Upon sudden turn-off of the lattice after time $\tau$ the population of each diffraction order can be found by projecting back onto the plane wave basis,

\begin{align}
|b_l|^2 &= |\langle l|\psi(\tau)\rangle|^2 \\
&= \left| \sum_n \langle \phi_n^{(q=0)} | 0 \rangle e^{-iE_n^{(q=0)} \tau/\hbar} \langle l|\phi_n^{(q=0)} \rangle \right|^2.
\end{align}

Monitoring the population of a specific diffraction order as a function of lattice pulse time for Kapitsa-Dirac diffraction is another standard way of measuring the lattice depth and lattice constant. Recall from Section 3.4 that for $d = \lambda/2$ a lattice typically only supports a few bands and as a result it is usual to observe up to second or third diffracted orders depending on the lattice depth. For larger values of $d$ a large number of bands can be populated leading to the dramatic evolution of the diffraction pattern displayed in Fig. 5.6. For deep, large periodicity, lattices the number of diffraction orders can be estimated by equating the depth of the optical lattice to the kinetic energy of the highest diffraction order,

\begin{align}
V_0 &\approx \frac{1}{2m} \left( \frac{2\pi \hbar l_{\text{max}}}{d} \right)^2 = l_{\text{max}}^2 4E_L \\
\rightarrow l_{\text{max}} &\approx \frac{1}{2} \sqrt{\frac{V_0}{E_L}}.
\end{align}

For example a 2.5 $\mu$m lattice of depth 29 kHz can support the $\pm 9^{\text{th}}$ orders.

Figure 5.6 shows a series of experimental and theoretical images of diffraction from a one-dimensional optical lattice as a function of lattice pulse time, $\tau$. Data for three different lattice constants is shown. One clear qualitative trend independent of $d$ is the dramatic collapse of the momentum distribution, that is after the initial spreading of the envelope of the atomic distribution there is a sudden revival of the zero-order population. Quantum-mechanically this arises because of the
5.2. Quantum vs classical in a large periodicity lattice.

Figure 5.6: Experimental results for diffraction from large periodicity (1.9 µm ≤ d ≤ 2.5 µm) optical lattices as a function of lattice pulse time, along with quantum and classical simulations (the classical case is discussed in Section 5.2.1). A 1D optical lattice was switched on suddenly and held on for a variable length, \( \tau \), before being suddenly turned off. The images were taken after 28 ms time-of-flight. The quantum simulation used Eq. 5.4 to find the population of the \( i^{th} \) diffraction order as a function of \( \tau \).
beating of the different Bloch functions according to Eq. 5.4. The reason that the rephasing into the zero momentum order is not 100% is the differing energy separation between bands. For example in a perfectly harmonic potential, rather than a sinusoidal potential, the entire population would rephase into the zeroth order, at times of

\[ T_{\text{rephase}} = \frac{n\pi}{\omega_{\text{lat}}}, \quad (5.7) \]

where \( n \) is an integer, i.e. the momentum distribution would rephase every half period of the harmonic oscillator. For \( d = 2.5 \mu m, \ V_0 = 29 \text{kHz} \) this predicts the rephasing should first occur at \( \sim \pi/\omega_{\text{lat}} = 153 \mu s \) which agrees well with the experimental data.

The simulation results shown in Fig. 5.6 were calculated using Eq. 5.4 to find the population of the various diffraction orders as a function of pulse time. These calculations did not take into account the effect of particle interactions, probably accounting for slight deviations between the theoretical and experimental results as \( \tau \) increases.

The depth of the lattice can be inferred by the maximum extent of the diffraction pattern in time-of-flight. In Eq. 5.5 the maximum diffracted order was estimated by equating the kinetic energy to the lattice depth. In the same way the lattice depth can be calculated by observing \( l_{\text{max}} \),

\[ V_0 \simeq 4E_L l_{\text{max}}^2 \]

\[ \simeq \frac{1}{2}m \left( \frac{\Delta x_{\text{max}}}{t_{\text{TOF}}} \right)^2, \quad (5.9) \]

where \( \Delta x_{\text{max}} \) is the separation of the zeroth and \( l_{\text{max}} \) orders after time-of-flight \( t_{\text{TOF}} \). The resolution of this method is \( \pm 4E_L \) (the energy separation between
diffracted orders), and as such is more accurate for optical lattices with larger lattice constants. In practice we can measure $V_0$ with an error of $\pm10\%$. The measured values of $V_0$ agreed within error with the predicted values from measuring the optical power in the lattice beams and calculating $V_0$ using Eq. 3.13. The only variable parameters in the simulation of Fig. 5.6 were $d$ and $V_0$ and the very good agreement with the experimental results gives confidence that the calibration of $d$ was accurate and not subject to a significant systematic error and $V_0$ was also calculated accurately.

5.2.1 Classical interpretation

As $d$ increases the number of bound states proliferates as discussed above. The Correspondence principle suggests in this limit the system can be well described classically. In this section we show that treating atoms as classical, non-interacting, point-like particles (whose dynamics in the optical lattices is specified by their position and momentum) reproduces some of the features evident in the diffraction results of Fig. 5.6, namely the dramatic spread and collapse of the time-of-flight atomic distribution. While the crossover from quantum to classical behaviour is interesting in its own right, another appealing aspect of the classical problem is its mapping onto the problem of diffraction of light by a sound wave in a crystal, and the connected theory of caustics.

The classical Hamiltonian for a single-particle in a sinusoidal potential is (in 1D)

$$H(x, p) = \frac{p^2}{2m} + V_0 \sin^2 \left( \frac{\pi x}{d} \right),$$

(5.10)
5.2. Quantum vs classical in a large periodicity lattice.

Classical particle trajectory in space in a sinusoidal potential

Classical particle trajectory in momentum space in a sinusoidal potential

![Graphs showing classical trajectories in space and momentum space](image)

**Figure 5.7:** Classical trajectories of a particle in a sinusoidal potential as found by solving Eqs. 5.12 and 5.13: (a) Trajectories in real space; (b) in momentum space. An equally spaced range of initial positions in the range $-d/2 \leq x_0 \leq d/2$ have been considered. As with the quantum-mechanical analysis there is a clear rephasing of the momentum distribution at integer multiples of $\pi/\omega_{lat}$.

or in dimensionless form

$$\tilde{H}(\tilde{x}, \tilde{p}) = \frac{1}{2} (\tilde{p}^2 + \sin^2 \tilde{x}) ,$$  \hspace{1cm} (5.11)

with $\tilde{p} = p/(2mV_0)^{0.5}$, $\tilde{x} = \pi x/d$. This classical motion can be solved analytically in the range $-\pi/2 \leq \tilde{x} \leq \pi/2$, giving [103],

$$\tilde{x}(\tilde{x}_0, t) = \sin^{-1}[\sin \tilde{x}_0 \text{sn}(\tilde{x} + K(\sin^2 \tilde{x}_0) | \sin^2 \tilde{x}_0)],$$  \hspace{1cm} (5.12)

$$\tilde{p}(\tilde{x}_0, t) = \sin \tilde{x}_0 \text{cn}(\tilde{x} + K(\sin^2 \tilde{x}_0) | \sin^2 \tilde{x}_0),$$  \hspace{1cm} (5.13)

where $\tilde{x}_0$ is the initial position of a particle in the sinusoidal potential, $K(q)$ is the complete elliptic integral, and $\text{sn}(g|q)$ and $\text{cn}(g|q)$ are elliptic functions which are defined in the *Handbook of Mathematical Functions* [104].

In Fig. 5.7 $x$ and $p$ are plotted as a function of time for a range of $x_0$. In agreement with the quantum-mechanical analysis and the experimental results the
momentum distribution partially rephases around integer multiples of $\pi/\omega_{\text{lat}}$. This is straightforward to understand in the classical picture: particles starting near the minimum of the sinusoidal potential execute simple harmonic motion at frequency $\omega_{\text{lat}}$. Thus a time $\pi/\omega_{\text{lat}}$ after the lattice has been switched on these atoms have completed half an oscillation and return to having zero velocity. The reason why the rephasing is not 100% complete at $t = \pi/\omega_{\text{lat}}$ is atoms which start further out from the potential minimum have a longer period of oscillation.

Figure 5.8 compares the experimental data for $d = 2.5 \, \mu\text{m}, V_0 = 29 \, \text{kHz}$ with a classical simulation of the momentum distribution calculated from Eq. 5.13. The classical simulation does not reproduce the fine structure of the experimental results the way the quantum simulation did but still provides an excellent description of the spreading of the envelope of the diffraction pattern and the timing of the rephasing. The correspondence between the two would improve for larger lattice constants and larger $V_0$. The agreement between the classical and quantum dynamics can be understood from the properties of a quantum harmonic oscillator. While a wave packet always has expectation values of its position and momentum which follow that of a classical particle it is important to note that classical dynamics is a good description of the motion only if the spreading of the wave packet can be neglected. This is the case for an oscillating wave packet in a harmonic potential, where the wave packet remains localized.

The classical and quantum simulations are compared together with the experimental results in Fig. 5.6. It is evident that for smaller lattice constants the quantum simulation agrees much more closely with the experimental data, with the structure of individual orders clearly visible. This shows coherent effects are important in our system and the qualitative agreement with classical predictions is not due to a lack of coherence.
5.2. Quantum vs classical in a large periodicity lattice.

Figure 5.8: Comparison of experimental images of diffraction as a function of hold time in an optical lattice ($d = 2.5 \mu m$, $V_0 = 29$ kHz) with a classical simulation. The classical simulation does not accurately reproduce the structure of the diffraction pattern (unlike the quantum simulation of Fig. 5.6) but does give a good description of the spreading of the envelope of the diffraction pattern and its collapse.

Caustics

The classical Hamiltonian 5.11 also describes the propagation of light rays through a sinusoidal volume grating (as might occur in an acousto-optic device for example), where the propagation in time is replaced by propagation in space ($t \rightarrow z$) [103]. The importance of caustics in describing such optical systems is well known having been extensively studied by Michael Berry (see [105] and references therein). It is evident here that caustics dominate the spread and collapse of the envelope of the diffraction pattern (the caustics are the smooth envelopes coming out from the cusps in $p$ at $t = n\pi/\omega_{\text{lat}}$ and the cusps in $x$ at $t = (n + 1/2)\pi/\omega_{\text{lat}}$, see Fig. 5.7).
The connection between condensate diffraction from a large periodicity lattice and caustics was first stressed by Huckans et al. [26, 85].

It is interesting to consider whether caustic formation is useful for describing more complicated time-of-flight distributions in cold atom experiments. Chalker and Shapiro recently argued [106] that the time-of-flight images seen by Chen and co-workers upon releasing a \(^7\)Li BEC from a disordered potential [107] are an example of caustic formation in an expanding condensate.

### 5.3 Summary

The periodicity of the accordion lattice as a function of the AOD frequency was calibrated using the direct measurement of Chapter 4 and \textit{in situ} absorption imaging of the atomic distribution in the optical lattice. The lattice constant was found to be

\[
d = \frac{8.53 \mu \text{m MHz}}{\Delta \nu},
\]

with an error of ±1%. The lattice spacing calculated from matter-wave diffraction was not consistent with this calibration due to interactions increasing the separation of the diffraction peaks in time-of-flight. The measured impulse of the diffracted components due to the repulsive interaction between them was found to be consistent with previous work [102].

Monitoring Kapitsa-Dirac diffraction from the optical lattice allowed the lattice depth \(V_0\) to be measured to ±10% accuracy and confirmed there was no measurable systematic error in Eq. 5.14. We found the evolution of the diffraction pattern from a large periodicity lattice was shaped by the emergence of caustics, in agreement with earlier work at NIST [26].
Rotating condensates and rotating optical lattices: theoretical background

In this chapter the physics of rotating condensates will be briefly reviewed, from condensates containing a small number of vortices to rapidly-rotating vortex lattice arrays. The effect of rotation upon cold atoms in an optical lattice is then considered with emphasis placed on the analogy to Josephson-junction arrays under effective magnetic fields and the frustrated XY-model. The fractional quantum Hall effect is also introduced, as although our experiments (see next chapter) did not enter this regime, it is of substantial interest and could be investigated in future experiments with the rotating optical lattice.
6.1 Rotating condensates

Treating the condensate wave function in terms of its magnitude and phase, \( \psi = \sqrt{n} \exp{i\theta} \), allows a local velocity to be defined by considering the particle-current density \( j = (\hbar/2m)i(\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi}) = nv \). One finds

\[
\mathbf{v} = \frac{\hbar}{m} \nabla \theta. \tag{6.1}
\]

The fact the velocity is given by the gradient of a scalar potential is of important consequence, as it follows that the superfluid flow must be irrotational, \( \nabla \times \mathbf{v} = 0 \), wherever \( \theta \) is not singular. The circulation, \( \kappa \), is defined as the line integral of \( \mathbf{v} \) around a closed path,

\[
\kappa = \oint \mathbf{v} \cdot d\mathbf{l} = \frac{\hbar}{m} \oint (\nabla \theta) \cdot d\mathbf{l}. \tag{6.2}
\]

Using Stokes’ theorem this can be written \( \int \mathbf{d} \mathbf{S} \cdot \nabla \times \mathbf{v} \) so again in a region where \( \theta \) is not singular a superfluid has zero circulation. However non-zero circulation can be supported about a line of zero density, i.e. a vortex. As \( \kappa \) is given by the change in phase around a closed loop and the condensate wave function must be single-valued it follows that the circulation must be quantised in a condensate,

\[
\kappa = l \frac{\hbar}{m}, \tag{6.3}
\]

where \( l \) is an integer. Vortices with quantised circulation where first proposed to exist in superfluid liquid helium due to the Bose-condensed nature ground state by Onsager [108] and Feynman [109].
6.1. Rotating condensates

6.1.1 Slow rotation: single/few vortex physics

For a review of single vortex physics see [78] or [110] for example. In this section we will consider two aspects of a vortex in a trapped BEC which will be useful for understanding the experiments described in the following chapter: the vortex size (important for considering imaging of vortices, see section 6.4); and the minimum frequency needed to nucleate a vortex in a condensate.

The length scale over which a condensate can change density significantly can be found by equating the kinetic energy of a perturbation on such a length scale $\xi$ with the chemical potential, $\hbar^2/(2m\xi^2) \sim \mu = gn$. A variation in the wave function on a shorter length scale would result in atoms having energies greater than $\mu$ and leaving the condensate. The ‘healing length’ can then be defined as

$$\xi = \frac{1}{\sqrt{8\pi na}}$$  \hspace{1cm} (6.4)

where $n$ is the local condensate density. This gives the size of a vortex core for low and intermediate rotation frequencies. For our experimental parameters ($2 \times 10^5$ atoms in a BEC, $\{\omega_r, \omega_z\}/2\pi = \{20.1, 53\}$ Hz) we find $\xi = 0.34 \mu$m.

It is interesting to consider at what rotation frequency a single vortex will enter a condensate. Let $E_1(r_0, \Omega)$ be the energy in the rotating frame of a BEC with a vortex at radial position $r_0$ and $E_0$ represent the energy of a vortex-free state. Following Fetter [111] consider the formation energy of a vortex $\Delta E(r_0, \Omega) = E_1(r_0, \Omega) - E_0$. Presuming an oblate condensate geometry ($\omega_z > \omega_r$) such that it is reasonable to assume an off-centre vortex will be straight, an evaluation of the energy function in the rotating frame under the Thomas-Fermi approximation (a
6.1. Rotating condensates

![Figure 6.1](image-url)  

**Figure 6.1:** Energy of a vortex in an axially symmetric condensate as a function of its radial position $r_0$ (calculated under the Thomas-Fermi approximation). Several values of $\Omega$ are considered and labelled in the legend. For $\Omega \leq 0.6\Omega_c$ a vortex is globally and locally unstable and will slowly spiral out of the condensate if a thermal component is present to exchange energy with. Only above the critical frequency $\Omega_c$ is the global minimum in the centre of the condensate.

A good approximation for all experiments on vortices to date) gives [110, 111]

\[
\frac{\Delta E(r_0, \Omega)}{\Delta E(0, 0)} = \left(1 - \frac{r_0^2}{R^2_r}\right)^{3/2} - \frac{\Omega}{\Omega_c} \left(1 - \frac{r_0^2}{R^2_r}\right)^{5/2},
\]

(6.5)

where

\[
\Omega_c = \frac{5}{2} \frac{h}{m R^2_r} \ln \left(\frac{R_r}{\xi}\right),
\]

(6.6)

$R_r$ is the radial Thomas-Fermi size of the condensate as usual. Figure 6.1 shows this energy plotted as a function of $r_0$ for various values of $\Omega$. For $\Omega \leq 0.6\Omega_c$ it is evident that a vortex is locally and globally unstable in the condensate. However at zero temperature there is no mechanism by which the vortex can dissipate energy, and such a vortex would move in a circle at constant $r_0$. At finite temperatures
6.1. Rotating condensates

there is dissipation from the interaction with the normal component of the cloud and a vortex will move outwards and spiral out of the condensate. The timescales of these processes turn out to be very long \[112\], \( \sim 10 \text{s} \), often greater than the lifetime of the condensate.

For \( 0.6 \Omega_c < \Omega < \Omega_c \) there is a local minimum in the vortex energy at the centre of the condensate while for \( \Omega \geq \Omega_c \) a vortex is locally and globally stable. \( \Omega_c \) is the thermodynamic critical rotation frequency for vortex creation (in the Thomas-Fermi limit). For our typical experimental parameters (see above) \( \Omega_c \simeq 4 \text{ Hz} \).

Initial experiments\(^1\) on vortices involved physically stirring a condensate in analogy to the ‘rotating bucket’ experiments on liquid helium. The first of these stirring experiments was at ENS in Paris where a toggled rotating laser beam was used to spin up a cloud \[10\]. They found a minimum rotation frequency for vortex nucleation, \( \Omega_{\text{min}} \), considerably above the predicted \( \Omega_c \), at \( 0.7 \omega_r \). Similar results were observed here at Oxford \[113\]. The nucleation mechanism was attributed to the excitation of the \( m = 2 \) surface mode which has a resonance at \( \omega_r/\sqrt{2} \). A later stirring experiment at MIT \[114\] used laser beams with smaller waists which were predicted to excite higher order surface modes (with corresponding resonance frequencies \( \omega_r/\sqrt{m} \) for the \( m^{\text{th}} \) mode). They found vortices could be nucleated below \( 0.7 \omega_r \) and the critical rotation frequency observed for vortex nucleation agreed with \( \Omega_c \). In chapter 7 we discuss \( \Omega_{\text{min}} \) for cold atoms in a rotating optical lattice, a very different situation to a bulk condensate.

\(^1\)The very first experiment to nucleate a vortex was at JILA and involved phase imprinting in a two-component BEC mixture \[9\], which we do not discuss here.
6.1. Intermediate rotation: vortex arrays

For rotation frequencies above $\Omega_c$ multiple vortices can be nucleated, which in the ground state form a triangular Abrikosov lattice. Small vortex lattices were first formed by the ENS group [115] before the MIT group generated large vortex arrays (containing $\sim 130$ vortices) using a similar stirring method [11]. Later the JILA group also created large vortex lattices using an evaporative spin up technique [53].

It was Feynman who first argued that a rapidly rotating superfluid should contain a uniform array of vortices. He noted solid-body rotation with $\mathbf{v} = \Omega \times \mathbf{r}$ and hence vorticity $\nabla \times \mathbf{v} = 2\Omega$ could be mimicked by a superfluid containing an array of vortices. This array would reproduce solid-body rotation on average despite the velocity field near a vortex being quite different from the classical limit. This insight allows a simple derivation of the density of vortices in such an array. Figure 6.2 shows a vortex lattice in a condensate; as each vortex contributes a unit of circulation $h/m$ the total circulation inside contour $C$, enclosing $N_v$ vortices in an area $A$, is $\kappa = N_v h/m$. The corresponding classical value for solid body rotation is $\kappa = \oint \mathbf{v} \cdot d\mathbf{l} = 2\Omega A$. Equating these expressions gives the mean areal vortex density

$$n_v = \frac{m\Omega}{\pi \hbar}.$$  \hspace{1cm} (6.7)

Having calculated the density of vortices knowledge of the condensate dimensions yields the vortex number. In a rotating bucket experiment with liquid helium the number of vortices will be linearly dependent on $\Omega$ as the radial size of the superfluid is fixed. However for a rotating BEC in a harmonic trap the radial trapping potential in the rotating frame is weakened as $\Omega$ increases, and the condensate will spread out radially. In Appendix C we derive the dependence of the condensate
6.1. Rotating condensates

Figure 6.2: Schematic of a vortex lattice: the array of vortices mimics the solid-body velocity field \( \mathbf{v} = \Omega \times \mathbf{r} \). (b) An image of a real small vortex array from the experiment displaying the classic ground state triangular structure.

As expected \( R_r(\Omega) \) diverges rapidly as \( \Omega \to \omega_r \) due to the centrifugal distortion. The dependence on \( \Omega \) of the number of vortices \( N_v \) is thus

\[
N_v = \frac{m\Omega}{\hbar} R_r(0)^2 \left( 1 - \frac{\Omega^2}{\omega_r^2} \right)^{-6/10}. \tag{6.9}
\]

6.1.3 Fast rotation \( \Omega \to \omega_r \): quantum Hall physics

It is interesting to consider how the single-particle energy levels change as the rotation frequency increases from zero to \( \omega_r \) in the situation of tight axial confinement \( \omega_z \gg \mu, k_B T \), such that the system is effectively two-dimensional\(^3\). We do not

\(^2\)The TF approximation is still appropriate for intermediate rotation speeds where the vortex core size, \( \xi \), is small compared to the separation between vortices, \( \sim 1/\sqrt{n_v} \sim \sqrt{\hbar/m\Omega} \). The kinetic energy contribution due to the density modulation of the vortices is then still small compared to the other energies in the problem.

\(^3\)Note as \( \Omega \to \omega_r \), \( R_r \) diverges while \( R_z \) and \( \mu \) vanish so the system tends to a two-dimensional configuration in this regime anyway.
repeat the argument here (see Figure 25 in review paper by Bloch et al. [70]) but note that the two-dimensional harmonic oscillator levels at $\Omega = 0$ shift to form degenerate Landau levels as $\Omega \rightarrow \omega_r$.

Landau levels are familiar from the physics of a two-dimensional electron gas under a magnetic field, where the way in which electrons fill the Landau levels has dramatic consequences for the electrical transport properties of the system. Von Klitzing observed plateaux in the transverse resistivity of such a two-dimensional system at magnetic field values corresponding to integer filling factors, where the filling factor is the ratio of electron number to the number of available states in a Landau level. This is known as the integer quantum Hall effect (IQHE), the discovery of which led to Von Klitzing being awarded the 1985 Nobel Prize in Physics.

Shortly after the IQHE was discovered the fractional quantum Hall effect (FQHE) was observed. Plateaux were unexpectedly seen at fractional filling factors, suggesting an internal structure to Landau levels. While the IQHE can be understood neglecting electron-electron interactions the FQHE is the result of the collective behaviour of electrons due to their interactions. Tsui and Stormer observed a plateau in the transverse resistivity at a filling factor $1/3$ [116] which Laughlin was able to explain by finding a many-body wave function describing a strongly correlated state with a lower energy than the single particle energy [117]. In a large magnetic field all the electrons lie in the Lowest Landau Level (LLL), and their kinetic energy is constant, hence irrelevant. Coulombic interactions between the electrons thus dominate the system dynamics. This is what makes the fractional quantum Hall effect so interesting and hard to analyse theoretically: it is in some sense the ultimate strongly-correlated system as interactions are the only thing of relevance in the system.
The analogy to the physics of a two-dimensional electron gas under a magnetic field is perhaps not surprising when one considers the Lorentz force on a charged particle $\mathbf{F}_{\text{Lor}} = q(\mathbf{v} \times \mathbf{B})$, has the same form as the Coriolis force on a neutral particle in the rotating frame $\mathbf{F}_{\text{Cor}} = 2m(\mathbf{\Omega} \times \mathbf{v})$, suggesting a mapping of $\mathbf{\Omega} \rightarrow qB/2m$ between the two systems.

For fast rotation, $\mathbf{\Omega} \rightarrow \omega_r$, of a neutral gas there are two limiting regimes, and it is convenient to introduce the filling factor, the number of atoms per vortex $\nu = N/N_v$, to distinguish between them. For large filling factors the system is still a superfluid (the mean-field quantum Hall regime) and a vortex lattice is expected [70, 111]. For small filling factors where the number of vortices is comparable to the number of atoms a series of strongly correlated ground states are expected in analogy to the fractional quantum Hall states of an electron gas [15, 16, 118].

To date experiments have reached the mean field regime with $\nu \sim 500$ and $\mathbf{\Omega} = 0.99\omega_r$ [12], but this is still far from the regime where quantum fluctuations are expected to lead to a melting of the vortex lattice and the emergence of strongly-correlated states. There is a high level of interest in achieving fractional quantum Hall physics in cold atom systems and several groups are focussing on using light-induced gauge potentials to reach the strongly-correlated regime [119, 120, 121]. We discuss the prospects for achieving FQHE physics in a rotating lattice in the following section. Beside the intrinsic interest in studying these extremely exotic many-body states in the clean and flexible environment offered by cold atom experiments an exciting long term aspect of this line of research is the possibility of observing non-Abelian excitations of quantum Hall states. Non-Abelian anyons are predicted to have very interesting properties related to their fractional braiding statistics. There has been no clear experimental evidence of non-Abelian anyons to date and the observation of such particles and their proper-
ties under exchange would be of huge fundamental interest. There is also interest in non-Abelian anyons due to their application in topological quantum computing [122].

6.2 Ultracold atoms in a rotating optical lattice

The analogy between the physics of a charged particle in a magnetic field and a neutral particle in a rotating frame was noted above. As regards the effect of a magnetic field/rotation on a charged/neutral particle in a lattice potential let us first consider the well-studied problem of a charged particle in a lattice under a magnetic field. The momentum operator $p$ allows one to define a translation operator

$$
\hat{T}(d) = e^{ip \cdot d / \hbar},
$$

such that $\psi(r + d) = \hat{T}(d)\psi(r)$. In a magnetic field the momentum operator is given by the Peierls substitution [123] $p \rightarrow p + eA$, where $A$ is the magnetic vector potential. Thus the effect of the magnetic field is to introduce an extra phase factor upon translation

$$
\psi(r + d) = \exp\left(\frac{ie}{\hbar} \int_{r}^{r+d} A \cdot dr'\right) \hat{T}(d)\psi(r).
$$

If a particle is translated along a closed path then the phase gained around the loop is

$$
\exp\left(\frac{ie}{\hbar} \oint A \cdot dr'\right) = \exp\left(i2\pi \frac{\Phi}{\Phi_0}\right),
$$

where $\Phi_0 = h/e$ is the flux quantum and $\Phi/\Phi_0 = (e/h) \int B \cdot dS$ i.e. the number of magnetic flux quanta threading the enclosed path. This ratio is usually denoted by
6.2. Ultracold atoms in a rotating optical lattice

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.3.png}
\caption{(a) Hofstadter’s Butterfly: the energy spectrum for an electron on an infinite two-dimensional lattice in the presence of a uniform magnetic field \cite{124}. \(J\) is the tunnelling matrix elements between adjacent site. (b) As a particle hops round a lattice plaquette it picks up a phase 2\(\pi\alpha\).}
\end{figure}

\(\alpha = \Phi/\Phi_0\). This is the well known Aharonov-Bohm effect. For an electron hopping around a lattice plaquette of side \(d\) under field \(B\), \(\alpha = e Bd^2/h\). The problem of an electron in an infinite two-dimensional lattice with tight-binding dispersion under a uniform magnetic field was first studied by Hofstadter \cite{124}. Hofstadter showed the energy spectrum has a fractal structure and depends very precisely on \(\alpha\). When \(\alpha\) is a rational fraction, \(\alpha = p/q\) where \(p, q\) are integers, the energy spectrum splits into \(q\) sub-bands. Figure 6.3 shows the famous Hofstadter’s Butterfly.

Now let us treat the analogous system of ultracold atoms in a rotating lattice in greater detail. When a time-dependent external potential (the lattice) does work on a system it is convenient to transform to the frame rotating with the external potential in which it is stationary. The Hamiltonian in the rotating frame is then given by \cite{125}

\[ H_{\Omega}(r) = H_0(r) - \Omega \cdot \mathbf{L}, \tag{6.13} \]

where the new frame is rotating at \(\Omega\) with respect to the lab frame, \(H_0\) is the Hamiltonian in the lab frame and \(\mathbf{L} = \mathbf{r} \times \mathbf{p}\) is the angular momentum. Note
these coordinates are the coordinates in the rotating frame. Taking $\Omega = \Omega e_z$ the many-body Hamiltonian in the rotating frame is (recall Eq. 3.30)

$$\hat{H} = \int d^3r \hat{\psi}^\dagger(r) \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{lat}}(r) + V_{\text{ext}}(r) + \frac{g}{2} \hat{\psi}^\dagger(r) \hat{\psi}(r) - \Omega L_z \right) \hat{\psi}(r),$$  

(6.14)

which can be written

$$\hat{H} = \int d^3r \hat{\psi}^\dagger(r) \left( \frac{(-i\hbar \nabla + m \mathbf{A}(r))^2}{2m} + V_{\text{lat}}(r) + \frac{1}{2} m (\omega_r^2 - \Omega^2)(x^2 + y^2) + \frac{1}{2} m \omega_z^2 z^2 + \frac{g}{2} \hat{\psi}^\dagger(r) \hat{\psi}(r) \right) \hat{\psi}(r),$$  

(6.15)

where $\mathbf{A} = \Omega \times \mathbf{r}$ plays the role of a magnetic vector potential except here it comes from rotation. The centrifugal term which arises in the rotating frame reduces the effective radial trapping frequency as noted previously. As discussed in section 3.5 a useful basis in which to expand the bosonic field operator for the $\Omega = 0$ case is that of the Wannier functions, in particular the lowest band Wannier functions are appropriate for a deep lattice. This is still a good basis for low rotation speeds such that $\hbar \Omega \ll E_L$. However the effect of higher rotation speeds is to mix higher bands, the main consequence of which is modifying the phase structure of the wave function within a site. Bhat et al. [126, 22] showed the modified Wannier basis

$$w_{\text{rot}}(\mathbf{r} - \mathbf{r}_j) = \exp \left( -i \frac{m}{\hbar} \int_{\mathbf{r}_j}^\mathbf{r} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' \right) w_0(\mathbf{r} - \mathbf{r}_j),$$  

(6.16)

partially accounts for the mixing of different bands. They quantitatively studied the ground state in the rotating frame by imaginary time propagation and found the modified Wannier basis gives a better description of the phase gradient within a lattice site than the regular Wannier functions. Given $\mathbf{A} = \Omega \times \mathbf{r}$ the idea behind
the ansatz 6.16 is clear: a phase gradient in the azimuthal direction proportional to the velocity \( v_\phi = \Omega r \) has been chosen.

Expanding the bosonic field operator in terms of the modified Wannier basis, 
\[
\hat{\psi} = \sum_j \hat{a}_{\text{rot}}(r - r_j),
\]
and inserting this into Eq. 6.15 gives the Bose-Hubbard Hamiltonian in the rotating frame
\[
H = -J \sum_{\langle i,j \rangle} \left( \hat{a}_i^{\dagger} \hat{a}_j e^{-i\phi_{ij}} + \hat{a}_j^{\dagger} \hat{a}_i e^{i\phi_{ij}} \right) + \frac{U}{2} \sum_i \hat{N}_i(\hat{N}_i - 1) + \sum_i (\epsilon_i - \mu) \hat{N}_i, \tag{6.17}
\]
where the tight-binding regime has been assumed, such that only nearest-neighbour hopping is significant. \( U \) given by the same expression as for the non-rotating case (Eq. 3.33) while \( J \) and \( \epsilon_i \) are modified slightly due to the change in the external trapping potential, with \( \epsilon_i = \int d^3r (V_{\text{ext}} - m\Omega^2(x^2 + y^2)/2)|w_0(r - r_i)|^2 \approx V_{\text{ext}}(r) - m\Omega^2(x^2 + y^2)/2 \) and
\[
J = -\int d^3r w_0(r - r_i) \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{lat}}(r) + \frac{1}{2} m(\omega_r^2 - \Omega^2)(x^2 + y^2) + \frac{1}{2} m\omega_z^2 z^2 \right) w_0(r - r_j). \tag{6.18}
\]
The phase term in Eq. 6.17 associated with tunnelling is given by
\[
\phi_{ij} = -\frac{m}{\hbar} \int_{r_i}^{r_j} A(r') \cdot dr' \tag{6.19}
= \frac{m\Omega}{\hbar} (x_i y_j - x_j y_i). \tag{6.20}
\]
This is the phase change experienced by a particle hopping from one site to another. Thus if we again consider the problem of a particle hopping around a lattice plaquette of side \( d \) as shown in Fig. 6.3(b), we see the particle gains a phase \( 2\pi \alpha \...
where
\[ \alpha = \frac{2m\Omega d^2}{\hbar}. \]  \hspace{1cm} (6.21)

Note this can be written as the ratio of the density of vortices \((n_v = m\Omega/\pi\hbar)\) to the density of lattice sites \((n_{\text{site}} = 1/d^2)\), \(\alpha = n_v/n_{\text{site}}\), i.e. \(\alpha\) is the number of vortices per lattice site in analogy to the case of a charged particle in a magnetic field where \(\alpha\) was the number of magnetic flux quanta threading a lattice site.

**Single-particle physics**

The single-particle energy spectrum is once again given by Hofstadter’s Butterfly. When \(\alpha\) is a rational fraction \(\alpha = p/q\) the ground state on an optical lattice has a modulated density which repeats every \(q\) lattices sites. The number-density for single-particle ground states in a \(68 \times 68\) site optical lattice for various values of \(\alpha\) is shown in Fig. 6.4. Jaksch and Zoller [127] suggested observation of such periodicity in the particle-density would allow one to measure the number of energy bands for a rational value of \(\alpha\). As evident in Fig. 6.4 the periodicity is degraded or can disappear completely when \(\alpha\) is irrational.

The Hofstadter Butterfly has proved hard to observe in solid-state experiments due to the small values of \(\alpha\) which are experimentally accessible. The largest steady-state magnetic fields which can be achieved are \(\sim 20\) T, and as \(d \sim 0.1\) nm, \(\alpha \sim 10^{-5}\). In contrast a \(^{87}\)Rb atom in a \(d = 2\) \(\mu\)m optical lattice rotating at \(\Omega/2\pi = 46\) Hz (achievable with our AOD arrangement) can reach \(\alpha = 1/2\).

**Many-body physics**

Fractional quantum Hall states have been predicted to be accessible to cold atoms under an effective magnetic field in an optical lattice [23, 22, 24, 128, 129]. For
6.2. Ultracold atoms in a rotating optical lattice

Figure 6.4: Calculated density of the single-particle ground state on a $68 \times 68$ site lattice for different $\alpha$: (a) $\alpha = 0$; (b) $\alpha = 1/4$; (c) $\alpha = \sqrt{1.5}/\pi \approx 0.3898$; (d) $\alpha = 3/7$. For rational values of $\alpha$ ($\alpha = p/q$, where $p, q$ are integers) the $q$ site periodicity modulation of the density is evident. For an irrational value of $\alpha$ such as in (c) this periodicity is degraded. I have used box boundary conditions which are responsible for the Gaussian-like envelope of the particle density (an external trapping potential has not been considered in these images). See [22] for similar images for other values of $\alpha$.

Small values of $\alpha \ll 1$ similar strongly correlated ground states to the continuum case have been found (this is expected as $\alpha$ can be written as the ratio of areas $\alpha = d^2/\pi l_v^2$, where $l_v = \sqrt{\hbar/m\Omega}$ is the magnetic length scale/distance between vortices, i.e. for small $\alpha$ the lattice spacing is small compared to the characteristic size of the ground state wave function and lattice effects are not significant). For larger $\alpha \sim 1$ the effect of the lattice becomes important, as seen for the single-particle case where the Landau levels become the fractal energy spectrum of the Hofstadter butterfly. Palmer et al. found that near rational values of $\alpha = p/q$ the system could be described as a bilayer quantum Hall system [23].
of using a rotating optical lattice lies in boosting the energy scales of the correlated states, which can be very small ($< 1 \text{nK}$) in a rapidly rotating condensate [118].

Several methods have also been theoretically proposed for how to generate a phase change for particles hopping round a lattice cell without physically rotating the lattice, which has the significant advantage that the centrifugal force is absent, and large values of $\alpha$ could be reached without the centrifugal expulsion of the cloud. Jaksch and Zoller [127] suggested using Raman transitions between distinct internal sublevels of an atom, with the different sublevels trapped in alternating lattice columns. Sorenson et al. [24] proposed time-dependent tunnelling together with a quadrupole field to imprint a non-vanishing phase around a plaquette. No such schemes have yet been experimentally implemented in a lattice.

6.3 Josepshon-junction arrays under an effective magnetic field

In section 3.6 the Josephson-junction regime of cold atoms in a two-dimensional optical lattice was discussed and deemed an accurate description of the system when there are a large number of atoms per lattice site. The Bose-Hubbard Hamiltonian in the rotating frame was derived in Eq. 6.17. If we again make the approximation $\hat{a}_j = \sqrt{N_j} \exp(i\theta_j)$ suitable when $N_j \gg 1$ Eq. 6.17 becomes

$$H = - \sum_{\langle i,j \rangle} \tilde{J}_{ij} \cos (\theta_i - \theta_j + \phi_{ij}) - \sum_j \frac{U_j}{2} \frac{\partial^2}{\partial \theta_j^2}$$

$$- i \sum_j (\epsilon_j + U_j N_j - \mu) \frac{\partial}{\partial \theta_j} + \sum_j \left( \epsilon_j N_j + \frac{U_j}{2} N_j^2 - \mu N_j \right)$$

(6.22)
6.3. **Josepshon-junction arrays under an effective magnetic field**

where \( \tilde{J}_{ij} = 2\sqrt{N_iN_j}J \) (see section 3.6) and the phase representation \( \tilde{N}_j = N_j - i\partial/\partial\theta_j, \theta_j = \theta_j \) has been used. The rotation-induced phase difference between sites, \( \phi_{ij} \), is given by Eq. 6.19. Minimizing the last term in Eq. 6.22 gives the equilibrium atom distribution, such that \( \epsilon_j + U_j\tilde{N}_j = \mu \). We presume the system has this equilibrium atom distribution (as would be achieved by a sufficiently long ramp of the lattice potential, section 3.6.4), and the Hamiltonian becomes

\[
H = -\sum_{(ij)} \tilde{J}_{ij}\cos(\theta_i - \theta_j + \phi_{ij}) - \sum_j U_j \frac{\partial^2}{\partial\theta_j^2}.
\]

(6.23)

We described the detailed form of \( w_0(r) \) at a lattice site in this regime in section 3.6. Here we note making the mapping \( \omega_r^2 \rightarrow \omega_r^2 - \Omega^2 \) gives the corresponding results in the rotating frame for the atomic distribution

\[
\tilde{N}_j = \frac{5N}{2\pi j_{\text{max}}^2} \left(1 - \frac{j_{x}^2 + j_{y}^2}{j_{\text{max}}^2}\right)^{3/2},
\]

(6.24)

with

\[
j_{\text{max}} = \frac{a_r}{d} \left(\frac{15N\omega_z a_s d^2}{2\pi\omega_r a_r \sigma^2}\right)^{1/5} \left(1 - \frac{\Omega^2}{\omega_r^2}\right)^{-3/10}.
\]

(6.25)

The wave function at the \( j^{\text{th}} \) lattice site is still given by

\[
w(r) = \frac{1}{\sqrt{\pi}\sigma} e^{-(x^2+y^2)/2\sigma^2} \left(\frac{\mu'_j}{g_{1D} N_j}\right)^{1/2} \left(1 - \frac{z^2}{r_z^2}\right)^{1/2},
\]

(6.26)

where an elegantly concise expression for \( \mu'_j \) is given by [20]

\[
\mu'_j = \frac{1}{2} m(\omega_r^2 - \Omega^2)(j_{\text{max}}^2 - j_{x}^2 - j_{y}^2)d^2.
\]

(6.27)
Uniformly frustrated XY model

A bosonic JJA under rotation is of interest as it allows a direct realization of the uniformly frustrated XY model (UFXYM) which is described by the Hamiltonian

\[ H = -\tilde{J} \sum_{\langle i,j \rangle} \cos (\theta_i - \theta_j + \phi_{ij}). \]  

(6.28)

The XY model is characterized by systems which have an order which can be described as a two-dimensional spin, i.e. a unit vector \((\cos \theta, \sin \theta)\) at each position in space (for example Bose-Einstein condensation can be described by the XY model, where the phase of the macroscopic wave function at every point in space corresponds to the angle of a two-dimensional spin, though bear in mind this simplified pictures neglects density fluctuations, i.e. fluctuations in the amplitude of the order parameter). Here the phase of each local wave function at a lattice site can be described as a two-dimensional spin. Frustration refers to the rotation of the system changing the energetically preferred phase differences between lattice sites.

As detailed in section 3.6 for a lattice constant \(d = 2 \mu m\) and our experimental parameters \(\tilde{J}_{ij} \gg U_j\) over a large area of sites. Under these conditions the quantum correction coming from the second term in Eq. 6.23 is negligible, and the Hamiltonian has the form of the UFXYM above, with the difference that in the cold atom system the tunnel coupling is inhomogeneous due to the external trapping potential in a cold atom experiment. Kasamatsu has studied the problem of cold atoms in a rotating optical lattice in the JJA regime for typical experimental conditions and calculated ground state vortex lattice structures in the optical lattice for various simple rational values of \(\alpha\).
6.4. Imaging vortices: tricks of the trade

The healing length and hence vortex core size in our trapped condensate is $\sim 0.3 \mu m$, which is below the resolution of our imaging system, therefore vortices must be viewed after a time-of-flight expansion. The initial decrease in density upon release of a cloud caused by the hydrodynamic pressure gradients drives an increase in the core size, as for short expansion times the core adjusts instantaneously to the local density and scales with the healing length. In this initial expansion period the core size can actually increase faster than the cloud size (for 3D expansion) [131].
When the density falls to the level at which interactions become negligible, the atoms expand as free particles, so the vortex core expands at the same rate as the cloud size.

Given the condensate is required have an oblate geometry (to allow the filling of as many lattice sites as possible) it will tend to a cigar shape after release from the trap. Although the initial rapid expansion in the axial direction drives an increase in vortex size it is preferable to have only radial expansion if possible; keeping the expanding condensate pancake-shaped gives a much better vortex visibility for several reasons such as the depth of focus (section 2.4.2) and the fact that vortices are three-dimensional objects that can bend, reducing core visibility. Tricks to achieve high vortex visibility have been used at MIT and JILA to make very impressive images of large vortex arrays:

- Ketterle’s group at MIT used a slice of repumping light to image a thin central portion of their rotating sodium condensates, allowing good vortex visibility [11]. The use of the repumping slice led to a slight inhomogeneity over the condensate in their images, similar to that observed in Fig. 5.1;

- The beautiful images of vortex arrays made by Eric Cornell’s group in JILA relied on expansion in an anti-trapping potential such that their rotating condensate expanded by a factor of 10 radially but only by a factor of $1 - 2$ axially. This ingenious expansion technique is described in the thesis of Coddington [54], and relies on ‘inverting’ their TOP trap to expand the cloud radially while using a magnetic field gradient to support the cloud against gravity.

In the same spirit as the JILA method we achieved a similar effect by keeping the light sheet on for a short time after releasing the rotating cloud from the magnetic
6.4. Imaging vortices: tricks of the trade

Figure 6.6: Comparison of two small vortex arrays which have undergone different amounts of axial expansion in time-of-flight. (a) The rotating cloud was released from the magnetic trap and the light sheet at the same time, reaching a size $R_r = 50 \mu m$, $R_z = 46 \mu m$ after 20 ms time-of-flight; (b) The light sheet was held on for 3 ms after the magnetic trap was released, reducing the axial extent of the cloud in time-of-flight by around 30\%, $R_r = 55 \mu m$, $R_z = 32 \mu m$. The improvement in vortex visibility is apparent.

trap. Thus the cloud experienced tight axial trapping while expanding in the radial direction. This effect had to be balanced against the fact that the final density of the cloud will be higher when the light sheet inhibits axial expansion, and thus the final vortex size will be smaller. We found the best balance between vortex size and the increased visibility in an axially thinner condensate was given by holding the light sheet on for 3 ms after the magnetic trap was released. Figure 6.6 shows a comparison of two images of small vortex arrays, one which has been released simultaneously from the magnetic trap and the light sheet and hence undergone significant axial expansion, and another for which the light sheet was held on for an additional 3 ms, suppressing the axial expansion.
Observation of vortex nucleation
in a rotating two-dimensional
lattice of Bose-Einstein condensates

In this chapter we report on the first experiments with a rotating optical lattice in the ‘deep’ lattice regime, where lattice depths were reached such that a two-dimensional array of weakly-linked condensates were created, forming a Josephson junction array. Under rotation this system realises the uniformly frustrated JJA described in the preceding chapter. We have investigated the nucleation of vortices in a rotating optical lattice, starting from a non-rotating condensate loaded into a static lattice, and then increasing the rotating rate of the lattice. In the only previous experiment employing a rotating optical lattice at JILA [17], Tung et al.
reported that heating due to mechanical instabilities and aberrations limited the rotating lattice to depths less than 30% of the condensate’s chemical potential. We were not limited to this weak lattice regime and our system is markedly different to the JILA experiment where a vortex lattice was created by an elegant evaporative spin-up technique [12] before a weak co-rotating optical lattice was imposed.

7.1 Vortex nucleation in a rotating optical lattice

The experiments started with a $^{87}$Rb condensate containing $2 \times 10^5$ atoms with no visible thermal component, held in a trap formed by an axial symmetric Ioffe-Pritchard trap and a single red-detuned light sheet in the radial plane increasing the axial trapping frequency to give $\{\omega_r, \omega_z\} = 2\pi \{20.1, 53.0\}$ Hz. The residual radial anisotropy of the combined magnetic and optical trap was $\omega_y/\omega_x = 1.008 \pm 0.003$.

The dipole sheet trap was creating using broadband light with a spectral width $\Delta \lambda \sim 3$ nm centred on 865 nm in order that the light had a short coherence length. As discussed in section 2.5 this ensured there were no interference effects, e.g. from multiple reflections at the vacuum cell wall, adding noise to the trapping potential. With narrow-band light it was observed that such perturbations acted to spin down the cloud and caused irregularities in the filling of the lattice.

The two-dimensional optical lattice was formed in the radial plane of the trapped condensate by four circularly polarized beams at $\lambda = 830$ nm intersecting in the focal plane of a high N.A. objective lens, achieving a lattice constant of 2 $\mu$m. One pair of beams was detuned by 10 MHz with respect to the other pair ensuring interference between the orthogonal optical lattices did not affect the atoms. The optical lattice was initially static as it was ramped to its final depth $V_0$ in the range $100$ Hz $\leq V_0 \leq 4000$ Hz. The lattice depth was calibrated by
Kapitza-Dirac diffraction with an uncertainty of ±10%. The measured $V_0$ agreed with that calculated from measured parameters within this uncertainty. The radial Thomas-Fermi radius of the condensate was 16 µm, resulting in the filling of ~16 lattice sites across the diameter of the condensate and around 200 sites in all. The lattice enhanced interaction energy at the central sites rose above the chemical potential of the unperturbed condensate ($\mu \approx 500$ Hz) meaning an array of individual condensates was not formed until $V_0 \sim 1000$ Hz. The central wells contained around 1500 atoms, calculated from Eqs. 6.24 and 6.25 using measured parameters.

After the condensate had been loaded into the static optical lattice $\Omega$ was increased linearly from zero to its final value in 320 ms, followed by a further 100 ms of rotation at the final value. While the lattice was still rotating the lattice depth was then ramped down to zero in 12 ms, allowing the condensates to merge and converting phase differences around plaquettes to vortices. Previously Scherer et al. [132] have investigated vortex formation by merging three independent condensates, and Schweikhard et al. [76] have observed the thermal activation of vortices on a two-dimensional lattice of Josephson-coupled BECs, however our experiment takes place in the rotating frame. Rotation gives rise to an additional phase $2\pi\alpha$ around a plaquette of the optical lattice, simulating the effect of a magnetic field. After the optical lattice was ramped down the cloud was then immediately released from the magnetic and optical traps and destructively imaged after 20 ms time-of-flight expansion. We also repeated the sequence but holding the cloud after the lattice was ramped down for an additional 500 ms, to allow any vortex-antivortex pairs to annihilate (such pairs could be created in a static lattice [76]). This generally improved visibility of vortices in the harmonic trap and we observed the same number of vortices in the system within experimental error. The uncertainty in
the number of vortices observed for a given $\Omega$ arose from difficulties in determining the presence of vortices near the edge of a condensate.

### 7.1.1 Minimum rotation frequency for vortex nucleation

Figure 7.1 shows the number of vortices, $N_v$, observed as a function of the optical lattice rotation frequency, $\Omega$, for lattice depths of $V_0 = 190$ Hz and $V_0 = 1200$ Hz. The behaviour of $N_v$ has a characteristic shape which depends on the exact vortex nucleation process. The first BEC stirring experiments used stirring beams of a similar size to the condensate and observed that no vortices were nucleated below $\Omega \approx 0.7\omega_r$ (which corresponds to 14 Hz for our system), where there is excitation of the $m = 2$ surface mode [10]. It was later shown that sufficiently small stirring beams nucleate vortices without discrete resonances of surface modes [114], and...
the minimum frequency for vortex nucleation, $\Omega_{\text{min}}$, matched the frequency at which a single vortex at the center of a condensate is energetically stable, $\Omega_c$ [110]. For our experimental conditions $\Omega_c/2\pi \simeq 4$ Hz. For $V_0 = 190$ Hz, far below the $\approx 1000$ Hz needed to split the condensate into weakly linked islands, Fig. 7.1 shows a similar trend to that observed in [114], i.e. $N_v$ did not exhibit sharp resonances and $\Omega_{\text{min}} \approx \Omega_c$. We infer a weak ($V_0 \ll \mu$) rotating lattice stirs up a condensate in a non-resonant fashion. For a rotating lattice of depth $V_0 = 1200$ Hz, however, $N_v$ displayed a distinctly different trend, with the number of vortices observed following a linear dependence on $\Omega$ even below $\Omega_c$. This change in behaviour is indicative of entering the regime where the system forms an array of Josephson-coupled condensates, rather than simply a condensate perturbed by a weak rotating lattice as for $V_0 = 190$ Hz.

The rotation induced phase differences between neighbouring condensates are converted to vortices when the lattice is ramped down in the rotating frame. This nucleation mechanism creates vortices near the centre of the condensate even when $\Omega < \Omega_c$. Vortices do not have to spin in from the edge of the cloud, which is what we observed for stirring with a weak lattice potential. The plot of $\Omega_{\text{min}}$ vs $V_0$ in Fig. 7.2 highlights the transition between the two regimes. $\Omega_{\text{min}}$ falls below $\Omega_c$ as $V_0$ rises above $\mu = 500$ Hz, and falls to 1 Hz for $V_0 \geq 1000$ Hz, corresponding to the lattice depth at which condensates are expected to be well localized on sites and only communicate by tunnelling.
7.1. Vortex nucleation in a rotating optical lattice

Figure 7.2: Minimum rotation frequency, $\Omega_{\text{min}}$, needed to nucleate a single vortex as a function of lattice depth, $V_0$. The chemical potential of the condensate prior to loading into the optical lattice, $\mu \approx 500$ Hz, is shown on the graph, as is $\Omega_c$. The region where $V_0$ becomes equal to the local chemical potential at the central lattice sites, $\mu'_j$, (calculated from Eq. 6.27 and 6.25), is marked by the shaded region between the dashed lines.

7.1.2 Dependence of the number of vortices on rotation frequency

The measurements of the number of vortices $N_v$ was extended to higher rotation frequencies $\Omega$ as shown in Figure 7.3. The ratio $\Omega/\omega_r$ is used where $\omega_r$ is the radial trapping frequency taking into account the slight enhancement by the Gaussian envelope of the lattice beams: for $V_0 = 600$ Hz, $\omega_r/2\pi = 21.5$ Hz and for $V_0 = 1100$ Hz, $\omega_r/2\pi = 22.6$ Hz.

There are two important points to note concerning Fig. 7.3:

1. The density of vortices for a given $\Omega$, whether a lattice is present or not, is
7.1. Vortex nucleation in a rotating optical lattice

$n_v = m\Omega/\pi\hbar$. As detailed in section 6.1.2, a rotating condensate in equilibrium at $\Omega$ has a Thomas-Fermi radius given by Eq. 6.8 and the number of vortices is expected to have the nonlinear dependence on $\Omega$ given by Eq. 6.9, i.e.

$$N_v \propto \Omega \left( 1 - \frac{\Omega^2}{\omega_F^2} \right)^{-6/10}.$$ (7.1)

The predicted number of vortices for a cloud displaying this behaviour is denoted by the dashed line in Fig. 7.3. The solid line presumes the radius stays constant at $R_0 = 16$ $\mu$m. It is evident that the weaker, $V_0 = 600$ Hz, optical lattice has a significantly faster, nonlinear dependence on $\Omega$, while for $V_0 = 1100$ Hz, $N_v$ rises closer to linearly with $\Omega$ ($N_v$ for $V_0 = 600$ Hz initially lies below $N_v$ for $V_0 = 1100$ Hz, but rises above the response of the deeper lattice for higher $\Omega$).
2. Vortices were observed for $\Omega/\omega_r > 1$. A condensate in a harmonic trap should be expelled for $\Omega > \omega_r$, and this is one of the limiting experimental problems in trying to reach the fast rotating regime. Bretin et al. were able to achieve $\Omega = 1.06\omega_r$ using additional quartic confinement [133]. However they found the visibility and density of vortices decreased unexpectedly at high rotation frequencies in their harmonic-plus-quartic trap. For $V_0 = 600$ Hz we detect vortices up to $\Omega = 1.15\omega_r$. Heating of the cloud became significant at these rotation frequencies, and the combination of heating and reduction of vortex visibility meant no vortices could be observed above this rotation frequency. The heating was worse for the deeper $V_0 = 1100$ Hz lattice and no vortices were visible above $\Omega = 1.02\omega_r$.

Both these trends are attributed to the optical lattice inhibiting the spreading out of the cloud as the rotation frequency is increased from zero to its final value. When the lattice depth surpasses the chemical potential atoms can only redistribute themselves by tunnelling. While the tunnelling parameter is several kHz at the central lattice sites this drops to $< 50$ Hz at peripheral lattice sites due to the drop in atom number, giving tunnelling times at the edge of the lattice comparable to the experimental duration. This could explain why the atoms do not reach their equilibrium extent in the lattice. The other more interesting possibility is that there is a Bloch-oscillation-like effect. This is the well-known effect by which a particle in a periodic potential subject to an external force undergoes oscillations in real and momentum space. As $\Omega$ is increased from zero the atoms in the rotating frame feel a force radially outwards and there may be a similar oscillating response in atom distribution rather than the atoms simply being expelled from the lattice.

There is one theoretical paper [134] which proposes that a Bloch oscillation-
Figure 7.4: Images after release from the rotating optical lattice showing the pattern of vortices: (a) Condensate before rotation; (b) $\Omega/2\pi = 4$ Hz; (c) $\Omega/2\pi = 7$ Hz; (d) $\Omega/2\pi = 17$ Hz.

type effect could prevent atoms from being expelled from a rotating optical lattice for $\Omega \geq \omega_r$. However it is not entirely convincing and although it appeared on the preprint archive in 2006 it has not yet, as far as we are aware, been not published in a peer-reviewed journal. Attempts to contact one of the authors regarding the paper have not elicited response. Further theoretical and experimental work is needed to clarify the role, if any, of Bloch oscillation effects in a rotating optical lattice, but this remains a potentially very interesting point.

### 7.2 Vortex array structures

The structure of the vortex patterns formed in a bosonic JJA under rotation are heavily influenced by the presence of the lattice. As discussed in chapter 6 for $\alpha = p/q$, where $p$ and $q$ are integers, the ground state vortex patterns have been calculated [20], and have a $q \times q$ site unit cell structure. Between these rational values of $\alpha$ intermediate patterns are formed. With $d = 2 \mu m$ the range of $\alpha$ which
could be reached as $\Omega$ was scanned was $0 \leq \alpha \leq 1/5$. We do not reproducibly observe the predicted $q \times q$ unit cell structure at simple rational values of $\alpha$. There are several probable reasons for this:

1. **The finite size of system.** As the maximum value of $\alpha$ that can be reached is around $1/5$ the finite size of the system becomes an issue as it will be difficult to realise the periodic structure for $q = 5$ when a maximum of 200 sites (i.e. around $16 \times 16$ lattice sites) are filled by the condensate.

2. **Glassy energy landscape.** The energy landscape of cold atoms in a rotating optical lattice is known to be very glassy, that is closely spaced minima in energy can be separated by large barriers, making it hard to reach the true ground state [135].

The first issue can be addressed by increasing the axial trapping frequency $\omega_z$, creating a flatter condensate with a larger radius allowing more lattice sites to be filled ($\omega_z/2\pi = 53 \mu m$ was the maximum frequency we could achieve, but a change in laser system could allow a larger slice power thus higher $\omega_z$ in the future). Creating larger condensates would also allow the filling of more lattice sites.

Larger values of $\alpha$ could be easily reached by increasing $d$, e.g. $d = 3.5 \mu m$ enables $\alpha = 1/2$ to be reached at $\Omega/2\pi = 14.9 \text{ Hz}$, but correspondingly less sites would be filled (only $\sim 10 \times 10$) so this would be of no benefit. Similarly an increase in $\omega_r$ would allow higher values of $\alpha$ to be reached by the cloud radius would be smaller and less sites would be filled.
7.3. Heating in the rotating optical lattice

The issue of heating in the rotating optical lattice is an important one for the viability of future experiments. Heating comes from fluctuations in the depth of the optical lattice (see section 4.3.6) and residual imperfections in the lattice as it rotates. Unlike the JILA experiment we were not limited to lattice depths well below the chemical potential but heating was still an issue for large lattice depths and long rotation times. Figure 7.5 (a) shows how the condensate fraction\(^1\) falls with increasing lattice depth after ramping up the rotation rate to 15 Hz in 40 ms followed by 35 ms of rotation at 15 Hz. Below the condensate’s chemical potential at \(\mu \simeq 500\) Hz there is negligible heating.

Figure 7.5 (b) displays \(N_0/N\) as a function of rotation time for a lattice in the

\(^1\)The condensate fraction was measured using a bimodal fit to the cloud of atoms in time-of-flight. Unlike for a stationary cloud the temperature cannot be simply calculating from the size of the thermal component in time-of-flight but can be extracted from the condensate fraction (though the dependence of \(N_0/N\) on the temperature is different in the rotating frame compared to the lab frame as rotation suppresses the quantum degeneracy critical temperature, see [136]).
deep lattice regime with $V_0 = 1300$ Hz and a rotation frequency of $\Omega/2\pi = 10$ Hz. It is evident there is significant heating on a timescale of 0.5 s. However it is interesting to note that this is not necessarily a major obstacle for frustrated JJA experiments. At this depth the individual condensates at lattice sites instantaneously acquire the rotation speed of the lattice, i.e. there is no need to stir-up a condensate over a period a time as in traditional stirring experiments with rotating laser beams. As soon as the optical lattice is moving at $\Omega$ the condensates acquire the phase difference $\phi_{ij}$ (see Eq. 6.19) in the rotating frame. The timescale for rotation is set by the tunnelling parameter which enables phase coherence to be established between the condensates, but this can be quite large, several kHz or higher.

The experiments on rotating JJA described in this chapter all involved the rotation frequency being increased smoothly from zero to its final values over $40 - 320$ ms. However in future experiments we intend to try ‘flicking’ the rotating lattice to achieve any $\Omega$ on a very short timescale: this will work by instantaneously displacing the lattice by a small angle after a condensate has been adiabatically loaded into the ground state. Each condensate is displaced to the side of its local potential well and will start to undergo harmonic motion at its lattice site at $\omega_{\text{lat}}$. After a quarter of a period each condensate will have achieved its maximum velocity and will be at the centre of each lattice site. At this point the lattice is suddenly set rotating at the relevant angular frequency and the system is rotating as a solid body.

For example shifting the lattice by an angle $\delta \theta$ results in a condensate in the $j^{th}$ site being displaced by a distance $(j_x^2 + j_y^2)^{1/2} d \delta \theta$ from the local minimum. The condensate reaches a maximum speed after $\pi/2 \omega_{\text{lat}}$ of $v_j = \omega_{\text{lat}} (j_x^2 + j_y^2)^{1/2} d \delta \theta$. As $v_j = \Omega \times r_j$ for solid body rotation this velocity instantaneously corresponds to a
rotation frequency $\omega_{\text{lat}} \delta \theta$. Thus a time $\pi/2\omega_{\text{lat}} < 1 \text{ ms}$ after the lattice is displaced by $\delta \theta$ it can set rotating at $\Omega = \omega_{\text{lat}} \delta \theta$.

### 7.4 Vortex pinning

Reference has been made to the fact there has been only one previous rotating optical lattice experiment, which took place at JILA. Tung et al. were restricted to the weak lattice regime ($V_0 \ll \mu$) and investigated the pinning of vortices by a weak co-rotating lattice. One can show the energy per unit length of a vortex in a condensate of radius $R_v$ is [78] $\epsilon_v \simeq n(\pi \hbar^2/m) \ln(R_v/\xi)$, that is, proportional to the local condensate density. Thus a vortex costs less energy at a position where the condensate density is suppressed, i.e. at the antinodes of an optical lattice potential, and hence it feels an energy landscape mirroring the optical lattice potential.

Vortex pinning in superconductors is a topic of substantial interest, with pinning of vortices at impurities affecting the melting of the vortex lattice (see [137] for a review). The pinning of vortices in a BEC by a weak optical lattice potential was predicted by Reijnders and Duine [138] and Pu et al. [19], the latter showing that a rich range of vortex structures are possible depending (as always) on the commensurability of the optical lattice with the vortex lattice (yet again the parameter of relevance describing this commensurability or lack of it is $\alpha$).

For example for $\alpha = 1$ there is a single vortex for each pinning site and a square vortex lattice structure is expected. This was observed in JILA where a large triangular vortex array was created using an evaporative spin-up technique before a weak co-rotating lattice was applied, leading to a structural transition from a triangular array to a square array of vortices at $\alpha = 1$ [17]. More exotic vortex lattice structures are predicted to occur but have not been seen experimentally.
7.4. Vortex pinning

![Image of vortex pinning](image)

Figure 7.6: Vortex pinning with a weak rotating optical lattice. (a) A condensate rotating at $\Omega/2\pi = 13$ Hz was held in the magnetic trap for 700 ms with forced rf evaporation, resulting in the ground state Abrikosov triangular lattice. (b) A weak rotating optical lattice ($V_0 = 200$ Hz = 0.4$\mu$) was applied for 900 ms, resulting in a transition to a square lattice. The lattice constant was $d = 2 \mu$m, giving $\alpha = 1/7$ around 13 Hz. The structure of the vortex array is evident from the Fourier transforms of these images: (c) the reciprocal lattice vectors are separated by 60°; (d) the square lattice gives peaks at multiples of 90° in reciprocal space.

For example at $\alpha = 2$ there are two vortices for each pinning site, and an array of doubly-quantized vortices is expected [19].

We were also able to observe vortex pinning. Unlike the JILA group we did not have a spin-up method which was independent of the rotating optical lattice, so we spun up a cloud using a relatively deep ($V_0 \sim 1000$ Hz) lattice, before ramping down the lattice. We then held the rotating cloud in the combined magnetic and light sheet trap for 700 ms with forced rf evaporation, producing triangular vortex lattices as seen in Fig. 7.6(a). Upon applying a weak rotating optical lattice ($V_0 = 200$ Hz) for 900 ms a transition to a square vortex lattice, Fig. 7.6(b), was observed. The lattice constant was $d = 2 \mu$m and $\Omega/2\pi = 13$ Hz giving $\alpha = 1/7$.

Investigation of vortex pinning as a function of lattice depth and condensate-
optical lattice rotation frequency mismatch was thoroughly investigated by the JILA group and the interested reader should refer to their work [17]. We attempted to extend their work by pinning two vortices to a single lattice site. This was attempted at JILA but no doubly-quantised vortices could be observed in their system. We were not able to observe doubly-quantized vortices and in general the observed vortex patterns did not have a reproducible structure. However images shown such as that in Fig. 7.7 suggest the concept was partially working. Figure 7.7(a) shows a cloud which was spun-up with a rotating optical lattice at $\Omega/2\pi = 15$ Hz and held in the combined magnetic and light sheet trap for 850 ms. In Fig. 7.7(b) the rotating condensate was held in the trap for 600 ms before a $d = 6.8 \mu m$ co-rotating optical lattice (giving $\alpha = 1.9$) at $V_0 = 100$ Hz was ramped on and held on for 250 ms. There are large dips in the density (including one in the centre of the cloud) which look like two vortices sufficiently close together that they cannot be individually resolved (rather than actual doubly-quantized vortices which should have a higher visibility). Comparison of the number of vortices in (a) and (b) fits the hypothesis that these larger holes in the condensate density are two vortices close together. Increasing the depth of the optical lattice should make
these vortices to overcome their mutual repulsion and form a doubly-quantized vortex, however we did not observe this.

The difficulty in creating these pinned arrays is likely to arise from the three-dimensional nature of the condensate at these rotation speeds and trapping frequencies, which means vortex lines can bend, complicating the pinning dynamics. It should be possible to improve these pinning experiments in the future by increasing $\omega_z$ to make the system more two-dimensional.

7.5 Summary

A new method of vortex nucleation in a quantum degenerate bosonic gas has been realised using acousto-optic deflection to generate smooth rotation of an optical lattice at a well defined frequency. For lattice depths $V_0 < \mu$ we found a weak optical lattice acted as a non-resonant stirring mechanism. For deeper lattices the system was split into well-localised condensates at each site and a linear dependence of vortex number on $\Omega$ was observed for rotation frequencies below the thermodynamic critical frequency for vortex nucleation. This implied vortices were created locally at a lattice plaquette and represents the first ever cold atom realization of a bosonic Josephson-junction array under an effective magnetic field, with values up to $\alpha \sim 1/5$ achieved.

The predicted ground state vortex patterns at rational values of $\alpha$ were not observed, probably due to the relatively small size of the system compared to the scale of the predicted ground states. Heating of atoms in deep $V_0 > 1000$ Hz optical lattices was shown to be significant for timescales $> 500$ ms, with much longer rotation times possible in weaker optical lattices.

The presence of the optical lattice was observed to inhibit the centrifugal expul-
Pinning of vortices by a weak ($V_0 < \mu$) rotating lattice has now been observed in Oxford as well as JILA, although the more exotic pinned states including an array of doubly-quantized vortices has yet to be achieved.
In this concluding chapter we summarize the main results of the thesis and highlight their importance in the wider field. Possible extensions to this work and future experiments are discussed.

\section*{8.1 Summary of results}

Rotating optical lattices for ultracold atoms have generated a large amount of theoretical interest but experimental implementations of these ideas have not been forthcoming due to the pragmatic difficulty in generating a rotating optical lattice which does not cause substantial heating of atoms. This thesis describes a new method of generating a rotating optical lattice based on acousto-optic deflection and forming the optical lattice in the focal plane of a high numerical aperture lens. The use of an acousto-optic deflector to generate the rotation of the lattice allows intrinsically smooth rotation and an excellent degree of control over the rotation frequency (for example the condensate could be loaded into a static lattice before
the rotation rate is smoothly increased from zero). This represents a substantial improvement over the only previous rotating optical lattice experiment where rotation was generated mechanically [17].

Our new technique allowed us to reach the regime $V_0 > \mu$ for a rotating lattice for the first time, realizing a rotating array of weakly-linked condensates. This represents the first realization of the frustrated XY model in an ultracold atom experiment, and of rotation-induced phase differences between neighbouring condensates. This is an important step towards studying the competition between a BKT type phase transition and an Ising-type phase transition at $\alpha = 1/2$, discussed below. It is also an important step towards achieving strongly-correlated quantum Hall states in cold atoms, a goal currently being pursued by a number of well-established groups in the cold atom community.

Although the main focus of this thesis was experiments with a rotating optical lattice we demonstrated that the arrangement for creating a smoothly rotating optical lattice could also be used for generating a two-dimensional accordion lattice, that is an optical lattice with controllable periodicity. Bill Phillips and Trey Porto’s group at NIST has previously demonstrated a one-dimensional accordion lattice which allowed them to explore the quantum and classical dynamics of a BEC in a large period optical lattice [26], and we found similar results when calibrating our accordion lattice. Fast and reproducible control of the lattice constant is an important new experimental tool for the ultracold experimentalist and could prove useful in a variety of experiments, from patterned loading in a superlattice to the direct imaging of single atoms in an optical lattice.
8.2 Extensions to this work

8.2.1 Phase transitions in fully frustrated Josephson-junction arrays

The case of a Josephson-junction array under rotation with $\alpha = 1/2$ is particularly interesting due to the interplay of the continuous $U(1)$ symmetry of the XY model and the discrete symmetry of the underlying commensurate vortex lattice configuration (see Fig. 6.5). Consider the case for $\alpha = 0$: at a critical temperature (the BKT transition) the long-range order of the system disappears as vortex-antivortex pairs unbind. However for $\alpha = 1/2$ there is also an Ising transition associated with the discrete broken translational symmetry (there are two degenerate ground states of opposite chirality).

There has been extensive theoretical and experimental (using superconducting JJA under a magnetic field) work carried out as to whether there is a single transition where the Ising order and long-range order disappear simultaneously or whether there are two separate transitions (see for example [139, 140, 141] and related works cited in these papers). However to date no definitive answer has been given. Polini et al. suggest it is possible to study both the $U(1)$ and $Z_2$ (Ising symmetry) orders in a cold atom realization of a frustrated JJA by studying the momentum distribution of the system in time-of-flight [21]. Upon sudden switch off the matter waves from each lattice site expand and interfere, giving rise to a diffraction pattern in time-of-flight (as seen in Chapter 5). When there is long-range phase coherence the diffraction pattern in time-of-flight has discrete peaks corresponding to the reciprocal lattice vector $2\pi/d$. When Ising broken translation symmetry is also present then additional peaks will appear in the interference
pattern. Given the $2 \times 2$ unit cell of the ground state at $\alpha = 1/2$ these additional peaks are expected at reciprocal lattice vectors $\pi/d$. By monitoring the visibility of the diffraction pattern from the optical lattice it would be possible to observe if the additional peaks due to the Ising broken symmetry disappeared first or if the Ising order disappears simultaneously with the phase coherence.

### 8.2.2 Expansion and compression of optical lattice periodicity

Having demonstrated a high degree of control over the periodicity of an optical lattice a logical continuation of this work is to explore expanding or compressing the optical lattice period in real time during an experimental run. This could aid further work on the detection of single atoms in an optical lattice, for instance a strongly-correlated state could be achieved at small $d$ (where interactions are largest due to the tight confinement and tunnelling timescales are also relatively fast) before the system is expanded to a larger lattice spacing above the resolution of the imaging system. While Bakr et al. in Harvard have recently imaged single atoms in a 690 nm period optical lattice without expanding the lattice [52], their experimental arrangement is technically very demanding. The use of an accordion lattice could relax the considerable engineering requirements needed for combining high resolution imaging with a cold atom experiment.

The key demonstration needed to show that expansion of an optical lattice is feasible is that of motional adiabaticity. That is, expansion should be slow and smooth enough that the atoms remain in the ground state of the optical system. As the angle of intersection between interfering laser beams, is varied an optical lattice expands as shown in Fig. 8.1. The central lattice site, defined by the centre
8.2. Extensions to this work

Figure 8.1: Schematic showing the expansion of the accordion lattice. The lattice expands about the centre of intersection of the lattice beams, leaving the central lattice site stationary. The $n^{th}$ lattice site from the centre moves outwards with a velocity $v_n = n \partial d/\partial t$.

of intersection of the interfering beams\(^1\) remains stationary, while the $n^{th}$ lattice site moves out at a velocity $v_n = n \partial d/\partial t$.

There are two effects which can lead to the heating of the atoms: (i) translational motion of a lattice site; (ii) the change in $\omega_{\text{lat}}$, the trapping frequency at a lattice site. For the central lattice site only the second heating mechanism is present. It is however possible to prevent $\omega_{\text{lat}}$ from varying as the lattice expands by increasing the optical lattice depth appropriately. Recalling that $\omega_{\text{lat}} = (2V_0 \pi / md^2)^{1/2}$ one can see if $d$ had a $t^\gamma$ dependence during expansion, a ramp in the lattice depth of the form $V_0 \propto t^{2\gamma}$ would allow $\omega_{\text{lat}}$ to remain constant.

Translational heating is the more serious issue, especially for the outermost sites which are moving the fastest. This heating rate can be calculated using an adiabatic basis set [142] and standard perturbation theory. Initial calculations were carried out in the thesis of Huckans [85], where it was estimated translational heating in a deep lattice would limit expansion timescales to hundreds of milliseconds.

In recent experiments detecting fluorescence from single atoms at individual lattice sites, very deep, near-resonant optical lattices are used to pin the atoms

\(^1\)Note the central lattice site about which the lattice expands or contracts is only defined by the point of intersection, not the relative phase between the laser beams.
while near-resonant laser beams in a molasses arrangement provide cooling and illuminate the atoms [52, 99]. It is possible a similar sub-Doppler cooling scheme could be used to cool the atoms in a deep accordion lattice as it expands, preventing atoms from being lost from a lattice site and preserving the initial atomic distribution.

8.2.3 Prospects for fractional quantum Hall physics in a rotating optical lattice

The first implementation of a strong rotating lattice has been described in this thesis, an important requirement for achieving Hall effects in a rotating optical lattice. The other requirements are:(i) lattice site occupations of order unity or less; (ii) the filling factor, \( \nu \), (the number of atoms per vortex) is also of order unity [16]. Strongly-correlated states such as the Mott insulator state have already been achieved in optical lattices with order of one particle per lattice site, but this regime has yet to be combined with rotation of the lattice. To achieve this first condition it would helpful to achieve traditional lattice spacings of \( d = \lambda/2 \). The highest numerical aperture lens system to have been integrated into a BEC experiment had N.A. = 0.8 [52]. Optical lattices with \( d \simeq 0.6\lambda \) could be formed in the focal plane of such a lens.

Regarding the second condition the lowest filling factor to have been achieved experimentally is \( \nu \simeq 500 \) [12]. It is apparent that for filling factors of order unity to be reached whether in a rotating optical lattice or not a small number of atoms (\( \sim 10 \)) will be needed. Detection of strongly-correlated states with such small atom numbers is then a very challenging task, though perhaps recent developments in single atom imaging in optical lattices offer hope for the future [52] (the strongly-
correlated state could then be identified by the in-lattice atom distribution).

It is important to consider the temperatures which need to be reached in an experiment to observe fractional quantum Hall states. Following Cooper [118] these temperatures can be estimated for a rotating quantum gas in the absence of a rotating lattice by considering the interaction energy which gives the energy gap scale for these strongly-correlated states. At $\Omega \sim \omega_r$ the vortex density is $n_v = m\omega_r/\pi\hbar$ and hence for $\nu \sim 1$ the areal atom density will also be $n_{2D} = m\omega_r/\pi\hbar$.

The maximum 3D density is then $n_{2D}/a_z$, where $a_z$ is the axial harmonic oscillator length. This gives an interaction energy $\mu \sim gn \sim (a_s/a_z)\hbar\omega_r$. For the best case experimental conditions $a_s/a_z \sim 0.1$ (which would be the case for $\omega_z/2\pi \sim 40\,\text{kHz}$) and $\omega_r/2\pi \sim 100\,\text{Hz}$ this energy is only 0.5 nK. This is around the lowest temperature ever achieved in a BEC experiment, and achieving this in the low atom number regime required for high filling factor would be problematic as evaporative cooling no longer works at these densities due to the long thermalization times.

In contrast optical lattices allow larger energy gaps for the strongly correlated states to be realized, with Sørensen et al. predicting gaps up to $\sim 5\,\text{nK}$ [24], a factor of ten better than the (generously estimated) continuum case above. However whether heating in a rotating optical lattice can ever be reduced enough to allow these temperatures to be realized is another matter. The creation of artificial gauge fields in optical lattices could represent a more promising route, avoiding the technical challenges associated with rotation. Several schemes have been suggested for introducing geometric phases in an optical lattice without rotation but have yet to be realized experimentally [24, 127, 143]. Recently a light-induced vector potential has been demonstrated for a BEC [119]. Although the vector potential created corresponded to zero magnetic field the technique can be extended to give non-zero effective magnetic fields.
Appendix A

Derivation of light shift using time-dependent perturbation theory

It is common to see derivations of the light shift in which the rotating wave approximation (RWA) is made at an early stage (for example we consider such an approach in Appendix B). However as discussed in Section 3.1 the RWA can lead to significant errors in calculating the light shift experienced by an atom in a far-detuned dipole trap. Following [78] we derive the light shift for a multi-level atom using time-dependent perturbation theory without making the RWA. The results match that derived in the review by Grimm et al. [63] by considering the classical oscillator model of a dipole. Yet another derivation reaching the same result can be achieved using a Floquet approach [64].

The Hamiltonian describing the interaction of a monochromatic field $\mathcal{E} =$
\( \mathcal{E}_0 \cos \omega t \) with an atom has the form

\[
H_{\text{int}} = -\mu \cdot \mathcal{E}
\]  

(A.1)

where \( \mu = -e \mathbf{r} \) is the electric dipole operator. The atomic wave function \( |\Psi\rangle \) can be written in the basis of the unperturbed atomic eigenstates \( |n\rangle \) with energies \( E_n \),

\[
|\Psi\rangle = \sum_n a_n e^{-E_n t/\hbar} |n\rangle,
\]  

(A.2)

where \( a_n \) are the expansion coefficients. Substituting Eq. A.2 into the time-dependent Schrödinger equation, a set of coupled differential equations for the expansion coefficients can be obtained

\[
i\hbar \dot{a}_n = \sum_k \langle n|H_{\text{int}}|k\rangle a_k e^{i\omega_{nk} t},
\]  

(A.3)

where \( \hbar \omega_{nm} = E_n - E_k \). Let the atom be in the unperturbed ground state \( |g\rangle \) at \( t = 0 \), and presume that \( a_g \simeq 1 \) for all \( t \), which is a good approximation in an off-resonant dipole trap where the populations of excited states are very small. To the first order the expansion coefficients can then be written

\[
a_n = \frac{1}{i\hbar} \int_0^t dt' \langle n|H_{\text{int}}|g\rangle e^{i\omega_{ng} t'}
\]  

(A.4)

\[
= -\frac{1}{2i\hbar} \int_0^t dt' \langle n|\mu \cdot \mathcal{E}_0|g\rangle (e^{i(\omega_{ng}+\omega) t'} + e^{i(\omega_{ng} - \omega) t'})
\]  

(A.5)

\[
= \frac{\langle n|\mu \cdot \mathcal{E}_0|g\rangle}{2\hbar} \left( \frac{e^{i(\omega_{ng}+\omega) t}}{\omega_{ng} + \omega} - 1 + \frac{e^{i(\omega_{ng} - \omega) t}}{\omega_{ng} - \omega} - 1 \right).
\]  

(A.6)

We have presumed \( \omega \) is not resonant with any of the transition frequencies \( \omega_{ng} \) which will be true in an off-resonance dipole trap. We determine the energy shift
by considering the coefficient for the ground state, writing it as \( a_g = e^{i\phi_g} \) and substituting into A.3 to give

\[
\hbar \dot{\phi}_g = \sum_{n} \langle g | H_{\text{int}} | n \rangle a_n e^{-i\omega_g t} ,
\]

where we have made the approximation \( e^{i\phi_g} = 1 \). Substituting Eq. A.6 into Eq. A.7 gives

\[
\hbar \dot{\phi}_g = \frac{1}{2\hbar} \sum_{n \neq g} |\langle n | \mu \cdot \hat{\epsilon} | g \rangle|^2 e^{-i\omega_g t} \cos \omega t \left( \frac{e^{i(\omega_{ng} + \omega) t} - 1}{\omega_{ng} + \omega} + \frac{e^{i(\omega_{ng} - \omega) t} - 1}{\omega_{ng} - \omega} \right) .
\]

(A.8)

Time-averaging Eq. A.8 gives the light shift

\[
U = -\hbar \langle \dot{\phi}_g \rangle t = -\frac{\mathcal{E}_0^2}{4\hbar} \sum_{n \neq g} |\langle n | \mu \cdot \hat{\epsilon} | g \rangle|^2 \left( \frac{1}{\omega_{ng} + \omega} + \frac{1}{\omega_{ng} - \omega} \right) ,
\]

where \( \hat{\epsilon} \) is a unit vector denoting the polarization of the electric field and we have used the result \( \langle g | \mu \cdot \hat{\epsilon} | g \rangle = 0 \) (due to the parity of the electric dipole operator).
Appendix B

Derivation of light shift for a two-level atom within the rotating wave approximation

Here we consider the interaction of a two-level atom with a monochromatic classical field of the form $\mathcal{E} = \mathcal{E}_0 \cos \omega t$. The eigenstates of the atomic Hamiltonian, $H_0$, are $|e\rangle$ and $|g\rangle$ with corresponding eigenenergies $\hbar \omega_0 / 2$ and $-\hbar \omega_0 / 2$. The interaction Hamiltonian is given by Eq. A.1. The total Hamiltonian of the system can then be written

$$H = H_0 + H_{\text{int}} = \begin{pmatrix} -\hbar \omega_0 / 2 & \hbar \Omega \cos \omega t \\ \hbar \Omega \cos \omega t & \hbar \omega_0 / 2 \end{pmatrix}, \quad (B.1)$$

where $\hbar \Omega = -e \langle e | r | g \rangle \mathcal{E}_0$ (in this Appendix $\Omega$ is the Rabi frequency rather than the rotation frequency of the rotating lattice as considered in the rest of thesis). It is convenient to transform to the frame rotating at $\omega$, which is achieved using
the unitary transformation

$$U = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix},$$  \hspace{1cm} (B.2)

such that the wave function in the rotating frame $|\tilde{\Psi}\rangle = U^\dagger |\Psi\rangle$. The Hamiltonian is transformed according to

$$\tilde{H} = U^\dagger H U + i\hbar \frac{\partial U^\dagger}{\partial t} U.$$  \hspace{1cm} (B.3)

We find

$$\tilde{H} = \begin{pmatrix} \hbar \delta /2 & e^{-i\omega t} \hbar \Omega \cos \omega t \\ e^{i\omega t} \hbar \Omega \cos \omega t & -\hbar \delta /2 \end{pmatrix}.$$  \hspace{1cm} (B.4)

$$= \frac{\hbar}{2} \begin{pmatrix} \delta & \Omega(1 + e^{-i2\omega t}) \\ \Omega(e^{i2\omega t} + 1) & -\delta \end{pmatrix}. $$  \hspace{1cm} (B.5)

Neglecting the rapidly varying terms oscillating at $\pm 2\omega$ corresponds to making the rotating wave approximation. In this approximation the Hamiltonian in the rotating frame becomes time-independent

$$\tilde{H} = \frac{\hbar}{2} \begin{pmatrix} \delta & \Omega \\ \Omega & -\delta \end{pmatrix},$$ \hspace{1cm} (B.6)

with eigenvalues (the ‘dressed’ state energies, i.e. considering the atom plus field energy, see for example [144])

$$U_\pm = \pm \frac{\hbar}{2} \sqrt{\delta^2 + \Omega^2}. $$  \hspace{1cm} (B.7)
For off-resonance dipole traps usually $\delta \ll \Omega$ (an especially good approximation in this thesis where only a relatively shallow optical lattice is used), and the dressed state energies can be written

$$U_{\pm} = \pm \frac{\hbar \delta}{2} + \frac{\hbar \Omega^2}{4 \delta}, \quad \text{(B.8)}$$

and the light shift of the ground state is

$$V = \frac{\hbar \Omega^2}{4 \delta}. \quad \text{(B.9)}$$

Substituting $\Omega^2 = |\langle e | \mu | g \rangle|^2 E_0^2 / \hbar^2$ and using expression 3.7 relating the natural line width $\Gamma$ to the dipole matrix element the light shift for a two-level atom becomes

$$V(r) = \frac{3\pi c^2 \Gamma}{2 \omega_0^2 \delta} I(r). \quad \text{(B.10)}$$
Appendix C

Centrifugal distortion of a rotating condensate in the Thomas-Fermi limit

As discussed in the main text the Thomas-Fermi approximation is valid for vortex lattices at intermediate rotation speeds ($\Omega < 0.99\omega_r$), as the healing length is still small compared to the distance between vortices. Given the area per vortex is $\pi l_v^2 = n_v^{-1}$ the distance between vortices is $2l_v = 2\sqrt{\hbar/m\Omega}$. Clearly as $\Omega$ increases the central density decreases due to the condensate expanding radially, giving a larger healing length. As the size of the vortex core becomes comparable to the inter-vortex spacing the density modulation due to the vortices significantly increases the kinetic energy of the system and the Thomas-Fermi approximation is no longer valid. A theoretical treatment of this regime beyond the Thomas-Fermi approximation by Baym and Pethick [145] showed the ratio of the core size to

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vortex spacing actually saturates as $\Omega \rightarrow \omega_r$. Here we only consider the regime in which the Thomas-Fermi regime is valid, and neglecting the kinetic energy in the rotating frame gives a Hamiltonian

$$
\mu = g|\psi|^2 + \frac{1}{2} m(\omega_r^2 - \Omega^2)(x^2 + y^2) + \frac{1}{2} m\omega_r^2 z^2,
$$

(C.1)

\[\text{giving a TF profile of the usual form}\]

$$
|\psi|^2 = \frac{\mu}{g} \left( 1 - \frac{(x^2 + y^2)}{R_r^2} - \frac{z^2}{R_z^2} \right),
$$

(C.2)

with

$$
R_r^2 = \frac{2\mu}{m(\omega_r^2 - \Omega^2)}, \quad \text{and} \quad R_z^2 = \frac{2\mu}{m\omega_r^2}.
$$

(C.3)

The normalization condition $\int |\psi|^2 d^3r = N$ yields $N = 8\pi \mu R_r^2 R_z^2/15g$ giving the chemical potential for a condensate rotating at $\Omega$,

$$
\mu(\Omega) = \mu(0) \left( 1 - \frac{\Omega^2}{\omega_r^2} \right)^{2/5},
$$

(C.4)

with $\mu(0)$ given by the usual expression for the TF chemical potential for a stationary condensate, $\mu = 0.5\hbar \bar{\omega} (15Na_s/\bar{a})^{2/5}$, where $\bar{\omega} = (\omega_r^2 \omega_z)^{1/3}$ and $\bar{a}$ is the corresponding average harmonic oscillator length, $\bar{a} = \sqrt{\hbar/m\bar{\omega}}$. Substituting this expression for $\mu$ back into C.3 gives

$$
R_r(\Omega) = R_r(0) \left( 1 - \frac{\Omega^2}{\omega_r^2} \right)^{-3/10}, \quad R_z(\Omega) = R_z(0) \left( 1 - \frac{\Omega^2}{\omega_r^2} \right)^{1/5}.
$$

(C.5)
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