Oxford Physics Department

Notes on General Relativity

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These notes are intended for classroom use. There is little here that is original, but a few results are. If you would like to quote from these notes and are unable to find an original literature reference, please contact me by email first.

Thank you in advance for your cooperation.

Steven Balbus

Recommended Texts

Hobson, M. P., Efstathiou, G., and Lasenby, A. N. 2006, *General Relativity: An Introduction for Physicists,* (Cambridge: Cambridge University Press) Referenced as HEL06.

A very clear, very well-blended book, admirably covering the mathematics, physics, and astrophysics of GR. Excellent presentation of black holes and gravitational radiation. The explanation of the geodesic equation and the affine connection is very clear and enlightening. Not so much on cosmology, though a nice introduction to the physics of inflation. Overall, my favourite text on this topic. (The metric has a different sign convention in HEL06 compared with Weinberg 1972 & MTW [see below], as well as these notes. Be careful.)

Weinberg, S. 1972, *Gravitation and Cosmology. Principles and Applications of the General Theory of Relativity,* (New York: John Wiley) Referenced as W72.

What is now the classic reference by the great man, but lacking any discussion whatsoever of black holes, and almost nothing on the geometrical interpretation of the equations. The author is explicit in his aversion to anything geometrical: gravity is a field theory with a mere geometrical "analogy" according to Weinberg. But there is no way to make sense of the equations, in any profound sense, without immersing onself in geometry. More suprisingly, given the author's skill set, I find that many calculations are often performed awkwardly, with far more effort and baggage than is required. The detailed sections on classical physical cosmology are its main strength. Weinberg also has a more recent graduate text on cosmology *per se,* (*Cosmology* 2007, Oxford: Oxford University Press). This is very complete but at an advanced level.

Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1973, *Gravitation,* (New York: Freeman) Referenced as MTW.

At 1280 pages, don't drop this on your toe, not even the paperback version. MTW, as it is known, is often criticised for its sheer bulk, its seemingly endless meanderings, its cuteness, and its laboured strivings at building mathematical and physical intuition at every possible step. But look. I must say, in the end, there really is a lot of very good material in here, much that is difficult to find anywhere else. It is a monumental achievement. It is also the opposite of Weinberg: geometry is front and centre from start to finish, and there is lots and lots of black hole and gravitational radiation physics, 40+ years on more timely than ever. I very much recommend its insightful discussion on gravitational radiation, now part of the course syllabus. There is a "Track 1" and "Track 2" for aid in navigation; Track 1 contains the essentials.

Hartle, J. B. 2003, *Gravity: An Introduction to Einstein's General Theory of Relativity,* (San Francisco: Addison-Wesley)

This is GR Lite, at a very different level from the previous three texts. But for what it is meant to be, it succeeds very well. Coming into the subject cold, this is not a bad place to start to get the lay of the land, to understand the issues in their broadest context, and to be treated to a very accessible presentation. This is a difficult subject. There will be times in your study of GR when it will be difficult to see the forest for the trees, when you will feel overwhelmed with the calculations, drowning in a sea of indices and Riemannian formalism. Everything will be all right: just spend some time with this text.

Ryden, Barbara 2017, *Introduction to Cosmology,* (Cambridge: Cambridge University Press)

Very recent and therefore up-to-date second edition of an award-winning text. The style is clear and lucid, the level is right, and the choice of topics is excellent. Less GR and more astrophysical in content but with a blend appropriate to the subject matter. Ryden is always very careful in her writing, making this a real pleasure to read. Warmly recommended.

A few other texts of interest:

Binney J, and Temaine, S. 2008, *Galactic Dynamics,* (Princeton: Princeton University Press) Masterful text on galaxies with an excellent cosmology treatment in the appendix. Very readable, given the high level of mathematics.

Longair, M., 2006, *The Cosmic Century,* (Cambridge: Cambridge University Press) Excellent blend of observations, theory and history of cosmology, as part of a more general study. Good general reference for anyone interested in astrophysics.

Landau, L., and Lifschitz, E. M. 1962, *Classical Theory of Fields,* (Oxford: Pergamon) Classic advanced text; original and interesting treatment of gavitational radiation. Dedicated students only!

Peebles, P. J. E. 1993, *Principles of Physical Cosmology,* (Princeton: Princeton University Press) Authoritative advanced treatment by the leading cosmologist of the 20th century, but in my view a difficult and sometimes frustrating read.

Shapiro, S., and Teukolsky S. 1983, *Black Holes, White Dwarfs, and Neutron Stars,* (Wiley: New York) Very clear text with a nice summary of applications of GR to compact objects and good physical discussions. Level is appropriate to this course.

Notational Conventions & Miscellany

Spacetime dimensions are labelled $0, 1, 2, 3$ or (Cartesian) ct, x, y, z or (spherical) ct, r, θ, ϕ. Time is always the 0-component. Beware of extraneous factors of c in 0-index quantities, present in e.g. $T^{00} = \rho c^2$, $dx^0 = cdt$, but absent in e.g. $g_{00} = -1$. (That is one reason why some like to set $c = 1$ from the start.)

Repeated indices are summed over, unless otherwise specified. (Einstein summation convention.)

The Greek indices $\kappa, \lambda, \mu, \nu$ etc. are used to represent arbitrary spacetime components in all general relativity calculations.

The Greek indices α, β, etc. are used to represent arbitrary spacetime components in special relativity calculations (Minkowski spacetime).

The Roman indices i, j, k are used to represent purely spatial components in any spacetime.

The Roman indices a, b, c, d are used to represent fiducial spacetime components for mnemonic aids, and in discussions of how to perform index-manipulations and/or permutations, where Greek indices may cause confusion.

$*$ is used to denote a generic dummy index, always summed over with another $*$.

The tensor $\eta^{\alpha\beta}$ is numerically identical to $\eta_{\alpha\beta}$ with $-1, 1, 1, 1$ corresponding to the $00, 11, 22, 33$ diagonal elements.

Viewed as matrices, the metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ are always inverses. The respective diagonal elements of *diagonal* $g_{\mu\nu}$ and $g^{\mu\nu}$ metric tensors are therefore reciprocals.

c almost always denotes the speed of light. It is occasionally used as an (obvious) tensor index. c as the velocity of light is only rarely set to unity in these notes, and if so it is explicitly stated. (Relativity texts often set $c = 1$ to avoid clutter.) Newton's G is *never* unity, no matter what. And don't you even think of setting 2π to unity.

Notice that it is "Lorentz invariance," but "Lorenz gauge." Not a typo, two different blokes.

Really Useful Numbers

$c = 2.99792458 \times 10^{8}$ m s^{-1} (Exact speed of light.)

$c^2 = 8.9875517873681764 \times 10^{16}$ m^2 s^{-2} (Exact!)

$a = 7.565723 \times 10^{-16}$ J m^{-3} K^{-4} (Blackbody radiation constant.)

$G = 6.67384 \times 10^{-11}$ m^3 kg^{-1} s^{-2} (Newton's G.)

$M_\odot = 1.98855 \times 10^{30}$ kg (Mass of the Sun.)

$r_\odot = 6.955 \times 10^{8}$ m (Radius of the Sun.)

$GM_\odot = 1.32712440018 \times 10^{20}$ m^3 s^{-2} (Solar gravitational parameter; much more accurate than either G or $M_\odot$ separately.)

$2GM_\odot/c^2 = 2.9532500765 \times 10^{3}$ m (Solar Schwarzschild radius.)

$GM_\odot/c^2 r_\odot = 2.1231 \times 10^{-6}$ (Solar relativity parameter.)

$M_\oplus = 5.97219 \times 10^{24}$ kg (Mass of the Earth)

$r_\oplus = 6.371 \times 10^{6}$ m (Mean Earth radius.)

$GM_\oplus = 3.986004418 \times 10^{14}$ m^3 s^{-2}(Earth gravitational parameter.)

$2GM_\oplus/c^2 = 8.87005608 \times 10^{-3}$ m (Earth Schwarzschild radius.)

$GM_\oplus/c^2 r_\oplus = 6.961 \times 10^{-10}$ (Earth relativity parameter.)

1 AU $= 1.495978707 \times 10^{11}$m (1 Astronomical Unit by definition.)

1pc $= 3.085678 \times 10^{16}$ m (1 parsec.)

$H_0 = 100h$ km s^{-1} Mpc^{-1} (Hubble constant. $h \simeq 0.7$. $H_0^{-1} = 3.085678 h^{-1} \times 10^{17}s=$ $9.778 h^{-1} \times 10^{9}$ yr.)

For diagonal g_{ab},

$$\Gamma^a_{ba} = \Gamma^a_{ab} = \frac{1}{2g_{aa}} \frac{\partial g_{aa}}{\partial x^b} \quad (a = b \text{ permitted, NO SUM})$$

$$\Gamma^a_{bb} = -\frac{1}{2g_{aa}} \frac{\partial g_{bb}}{\partial x^a} \quad (a \neq b, \quad \text{NO SUM})$$

$$\Gamma^a_{bc} = 0, \quad (a, b, c \quad \text{distinct})$$

Ricci tensor:

$$R_{\mu\kappa} = \frac{1}{2} \frac{\partial^2 \ln |g|}{\partial x^\kappa \partial x^\mu} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda} \Gamma^\lambda_{\kappa\eta} - \frac{\Gamma^\eta_{\mu\kappa}}{2} \frac{\partial \ln |g|}{\partial x^\eta} \quad \text{FULL SUMMATION, } g = \det g_{\mu\nu}$$

Contents

Most of the fundamental ideas of science are essentially simple, and may, as a rule, be expressed in a language comprehensible to everyone.

— Albert Einstein

1 An overview

1.1 The legacy of Maxwell

We are told by the historians that the greatest Roman generals would have their most important victories celebrated with a triumph. The streets would line with adoring crowds, cheering wildly in support of their hero as he passed by in a grand procession. But the Romans astutely realised the need for a counterpoise, so a slave would ride with the general, whispering in his ear, "All glory is fleeting."

All glory is fleeting. And never more so than in theoretical physics. No sooner is a triumph hailed, but unforseen puzzles emerge that couldn't possibly have been anticipated before the breakthrough. The mid-nineteenth century reduction of all electromagnetic phenomena to four equations, the "Maxwell Equations," is very much a case in point.

Maxwell's equations united electricity, magnetism, and optics, showing them to be different manifestations of the same field. The theory accounted for the existence of electromagnetic waves, explained how they propagate, and that the propagation velocity is $1/\sqrt{\epsilon_0\mu_0}$ (ϵ_0 is the permitivity, and μ_0 the permeability, of free space). This combination is numerically precisely equal to the speed of light. Light is electromagnetic radiation! The existence of electromagnetic raditation was then verified by brilliant experiments carried out by Heinrich Hertz in 1887, in which the radiation was directly generated and detected.

But Maxwell's theory, for all its success, had disquieting features when one probed. For one, there seemed to be no provision in the theory for allowing the velocity of light to change with the observer's velocity. The speed of light is aways $1/\sqrt{\epsilon_0\mu_0}$. A related point was that simple Galilean invariance was not obeyed, i.e. absolute velocities seemed to affect the physics, something that had not been seen before. Lorentz and Larmor in the late nineteenth century discovered that Maxwell's equations did have a simple mathematical velocity transformation that left them invariant, but it was *not* Galilean, and most bizarrely, it involved changing the *time.* The non-Galilean character of the transformation equation relative to the "aetherial medium" hosting the waves was put down, a bit vaguely, to electromagnetic interactions between charged particles that truly changed the length of the object. In other words, the non-Galilean transformation were somehow electrodynamical in origin. As to the time change...well, one would just have to put up with it as an aetherial formality.

All was resolved in 1905 when Einstein showed how, by adopting as a postulates (i) that the speed of light was constant in all frames (as had already been indicated by a body of irrefutable experiments, including the famous Michelson-Morley investigation); (ii) the abandonment of the increasingly problematic aether medium that supposedly hosted these waves; and (iii) reinstating the truly essential Galilean notion that relative uniform velocity cannot be detected by any physical experiment, that the "Lorentz transformations" (as they had become known) must follow. *All* equations of physics, not just electromagnetic phenomena, had to be invariant in form under these Lorentz transformations, even with its peculiar relative time variable. The non-Galilean transformations were purely kinematic in this view, having nothing in particular to do with electrodynamics: they were much more general. These ideas and the consequences that ensued collectively became known as *relativity theory*, in reference to the invariance of form with respect to relative velocities. The relativity theory stemming from Maxwell's equations is rightly regarded as one of the crown jewels of 20th century physics. In other words, a triumph.

1.2 The legacy of Newton

Another triumph, another problem. If indeed, all of physics had to be compatible with relativity, what of Newtonian gravity? It works incredibly well, yet it is manifestly *not* compatible with relativity, because Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho \tag{1}$$

implies instantaneous transmission of changes in the gravitational field from source to potential. (Here Φ is the Newtonian potential function, G the Newtonian gravitational constant, and ρ the mass density.) Wiggle the density locally, and throughout all of space there must instantaneously be a wiggle in Φ, as given by equaton (1).

In Maxwell's theory, the *electrostatic* potential satisfies its own Poisson equation, but the appropriate time-dependent potential obeys a wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \tag{2}$$

and solutions of this equation propagate signals at the speed of light c. In retrospect, this is rather simple. Mightn't it be the same for gravity?

No. The problem is that the source of the signals for the electric potential field, i.e. the charge density, behaves differently from the source for the gravity potential field, i.e. the mass density. The electrical charge of an individual bit of matter does not change when the matter is viewed in motion, but the mass does: the mass increases with velocity. This seemingly simple detail complicates everything. Moreover, in a relativisitic theory, energy, like matter, is a source of a gravitational field, including the distributed energy of the gravitational field itself! A relativisitic theory of gravity would have to be nonlinear. In such a time-dependent theory of gravity, it is not even clear a priori what the appropriate mathematical objects should be on either the right side or the left side of the wave equation. Come to think of it, should we be using a wave equation at all?

1.3 The need for a geometrical framework

In 1908, the mathematician Hermann Minkowski came along and argued that one should view the Lorentz transformations not merely as a set of rules for how coordinates (including a time coordinate) change from one constant-velocity reference frame to another, but that these coordinates should be regarded as living in their own sort of pseudo-Euclidian geometry—a *spacetime,* if you will: Minkowski spacetime.

To understand the motivation for this, start simply. We know that in ordinary Euclidian space we are free to choose any coordinates we like, and it can make no difference to the description of the space itself, for example, in measuring how far apart objects are. If (x, y) is a set of Cartesian coordinates for the plane, and (x', y') another coordinate set related to the first by a rotation, then

$$dx^2 + dy^2 = dx'^2 + dy'^2 \tag{3}$$

i.e., the distance between two closely spaced points is the same number, regardless of the coordinates used. $dx^2 + dy^2$ is said to be an "invariant."

Now, an abstraction. There is nothing special from a mathematical viewpoint about the use of $dx^2 + dy^2$ as our so-called metric. Imagine a space in which the metric invariant was $dy^2 - dx^2$. From a purely mathematical point of view, we needn't worry about the plus/minus sign. An invariant is an invariant. However, with $dy^2 - dx^2$ as our invariant, we are describing a Minkowski space, with $dy = cdt$ and dx an ordinary space interval, just as before. The fact that $c^2dt^2 - dx^2$ is an invariant quantity is precisely what we need in order to guarantee that the speed of light is always constant—an invariant! In this case, $c^2dt^2 - dx^2$ is always zero for light propagation along x, whatever coordinates (read "observers") are involved, and more generally,

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0 \tag{4}$$

will guarantee the same in any direction. We have thus taken a kinematical requirement—that the speed of light be a universal constant—and given it a geometrical interpretation in terms of an invariant quantity (a "quadratic form" as it is sometimes called) in Minkowski space. Rather, Minkowski's *spacetime.*

Pause. As the French would say, "Bof." And so what? Call it whatever you like. Who needs obfuscating mathematical pretence? Eschew obfuscation! The Lorentz transform stands on its own! That was very much Einstein's initial take on Minkowski's pesky little meddling with his theory.

However, it is the geometrical viewpoint that is the more fundamental. In Minkowski's 1908 paper, we find the first mention of 4-vectors, of relativistic tensors, of the Maxwell equations in manifestly covariant form, and the realisation that the magnetic and vector potentials combine to form a 4-vector. This is more than "überflüssige Gelehrsamkeit" (superfluous erudition), Einstein's dismissive term for the whole business. In 1912, Einstein changed his opinion. His great revelation, his big idea, was that *gravity arises because the effect of the presence of matter in the universe is to distort Minkowski's spacetime.* Minkowski spacetime is physical, and embedded spacetime distortions manifest themselves as what we view as the force of gravity. These same distortions must therefore become, in the limit of weak gravity, familiar Newtonian theory. Gravity itself is a purely geometrical phenomenon.

Now that is one big idea. It is an idea that will take the rest of this course—and beyond—to explain. How did Einstein make this leap? Why did he change his mind? Where did this notion of geometry come from?

From a simple observation. In a freely falling elevator, or more safely in an aircraft executing a ballistic parabolic arch, one feels "weightless." That is, the effect of gravity can be made to locally disappear in the appropriate reference frame—the right coordinates. This is because gravity has exactly the same effect on all types mass, regardless of composition, which is precisely what we would expect if objects were responding to background geometrical distortions instead of an applied force. In the effective absence of gravity, we locally return to the environment of undistorted ("flat," in mathematical parlance) Minkowski spacetime, much as a flat Euclidian tangent plane is an excellent local approximation to the surface of a curved sphere. This is why it is easy to be fooled into thinking that the earth is flat, if your view is local. "Tangent plane coordinates" on small scale road maps locally eliminate spherical geometry complications, but if we are flying to Hong Kong, the earth's curvature is important. Einstein's notion that the effect of gravity is to cause a geometrical distortion of an otherwise flat Minkowski spacetime, and therefore that it is always possible to find coordinates in which these local distortions may be eliminated to leading order, is the foundational insight of general relativity. It is known as the *Equivalence Principle.* We will have more to say on this topic.

Spacetime. *Spacetime.* Bringing in time, you see, is everything. Who would have thought of it? Non-Euclidean geometry as developed by the great mathematician Bernhard Riemann begins with just the notion we've been discussing, that any space looks *locally* flat. Riemannian geometry is the natural language of gravitational theory, and Riemann himself had the notion that gravity might arise from a non-Euclidian curvature in three-dimensional space. He got nowhere, because *time was not part of his geometry.* It was the (underrated) genius of Minkowski to incorporate time into a purely geometrical theory that allowed Einstein to take the crucial next step, freeing himself to think of gravity in geometrical terms, without having to ponder over whether it made any sense to have time as part of a geometrical framework. In fact, the Newtonian limit is reached *not* from the leading order curvature terms in the spatial part of the geometry, but from the leading order "curvature" (if that is the word) of the time dimension.

Riemann created the mathematics of non-Euclidian geometry. Minkoswki realised that natural language of the Lorentz transformations was neither electrodynamical, nor even really kinematic, it was geometrical. But you need to include time as a component of the geometrical interpretation! Einstein took the great leap of realising that gravity arises from the distortions of Minkowski's flat spacetime created by the existence of matter.

Well done. You now understand the conceptual framework of general relativity, and that is itself a giant leap. From here on, it is just a matter of the technical details. But then, you and I also can paint like Leonardo da Vinci. It is just a matter of the technical details.

From henceforth, space by itself and time by itself, have vanished into the merest shadows, and only a blend of the two exists in its own right.

— Hermann Minkowski

2 The toolbox of geometrical theory: special relativity

In what sense is general relativity "general?" In the sense that since we are dealing with an abstract spacetime geometry, the essential mathematical description must be the same in *any* coordinate system at all, not just those related by constant velocity reference frame shifts, nor even just those coordinate transformations that make tangible physical sense as belonging to some observer or another. *Any mathematically proper coordinates at all, however unusual.* Full stop.

We need the coordinates for our description of the structure of spacetime, but somehow the essential physics (and other mathematical properties) must not depend on which coordinates we use, and it is no easy business to formulate a theory which satisfies this restriction. We owe a great deal to Bernhard Riemann for coming up with a complete mathematical theory for these non-Euclidian geometries. The sort of geometry in which it is always possible to find coordinates in which the space looks locally smooth is known as a *Riemannian manifold.* Mathematicians would say that an n-dimensional manifold is *homeomorphic* to n-dimensional Euclidian space. Actually, since our invariant interval $c^2dt^2 - dx^2$ is not a simple sum of squares, but contains a minus sign, the manifold is said to be *pseudo-Riemannian.* Pseudo or no, the descriptive mathematical machinery is the same.

The objects that geometrical theories work with are scalars, vectors, and higher order tensors. You have certainly seen scalars and vectors before in your other physics courses, and you may have encountered tensors as well. We will need to be very careful how we define these objects, and very careful to distinguish them from objects that look like vectors and tensors (because they have the appropriate number of components) but actually are not.

To set the stage, we begin with the simplest geometrical objects of Minkowski spacetime that are not just simple scalars: the 4-vectors.

2.1 The 4-vector formalism

In their most elementary form, the familiar Lorentz transformations from "fixed" laboratory coordinates (t, x, y, z) to moving frame coordinates (t', x', y', z') take the form

$$ct' = \gamma(ct - vx/c) = \gamma(ct - \beta x) \tag{5}$$

$$x' = \gamma(x - vt) = \gamma(x - \beta ct) \tag{6}$$

$$y' = y \tag{7}$$

$$z' = z \tag{8}$$

where v is the relative velocity (taken along the x axis), c the speed of light, $\beta = v/c$ and

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \frac{1}{\sqrt{1 - \beta^2}} \tag{9}$$

is the *Lorentz factor*. The primed frame can be thought of as the frame moving with an object we are studying, that is to say the object's rest frame. To go backwards to find (x, t) as a function (x', t'), just interchange the primed and unprimed coordinates in the above equations, and then flip the sign of v. Do you understand why this works?

Exercise. Show that in a coordinate free representation, the Lorentz transformations are

$$ct' = \gamma(ct - \boldsymbol{\beta} \cdot \boldsymbol{x}) \tag{10}$$

$$\boldsymbol{x'} = \boldsymbol{x} + \frac{(\gamma - 1)}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{x})\boldsymbol{\beta} - \gamma ct\boldsymbol{\beta} \tag{11}$$

where $c\boldsymbol{\beta} = \boldsymbol{v}$ is the vector velocity and boldface x's are spatial vectors. (Hint: This is not *nearly* as scary as it looks! Note that $\boldsymbol{\beta}/\beta$ is just a unit vector in the direction of the velocity and sort out the components of the equation.)

Exercise. The Lorentz transformation can be made to look more rotation-like by using hyperbolic trigonometry. The idea is to place equations (5)–(8) on the same footing as the transformation of Cartesian position vector components under a simple rotation, say about the z axis:

$$x' = x\cos\theta + y\sin\theta \tag{12}$$

$$y' = -x\sin\theta + y\cos\theta \tag{13}$$

$$z' = z \tag{14}$$

Show that if we define

$$\beta \equiv \tanh\zeta, \tag{15}$$

then

$$\gamma = \cosh\zeta, \quad \gamma\beta = \sinh\zeta, \tag{16}$$

and

$$ct' = ct\cosh\zeta - x\sinh\zeta, \tag{17}$$

$$x' = -ct\sinh\zeta + x\cosh\zeta. \tag{18}$$

What happens if we apply this transformation twice, once with "angle" ζ from (x, t) to (x', t'), then with angle ξ from (x', t') to (x'', t'')? How is (x, t) related to (x'', t'')?

Following on, rotations can be made to look more Lorentz-like by introducing

$$\alpha \equiv \tan\theta, \quad \Gamma \equiv \frac{1}{\sqrt{1+\alpha^2}} \tag{19}$$

Then show that (12) and (13) become

$$x' = \Gamma(x + \alpha y) \tag{20}$$

$$y' = \Gamma(y - \alpha x) \tag{21}$$

Thus, while a having a different appearance, the Lorentz and rotational transformations have mathematical structures that are similar.

Of course lots of quantities besides position are vectors, and it is possible (indeed desirable) just to *define* a quantity as a vector if its individual components satisfy equations (12)–(14). Likewise, we find that many quantities in physics obey the transformation laws of equations (5–8), and it is therefore natural to give them a name and to probe their properties more deeply. We call these quantities 4-vectors. They consist of an ordinary vector $\boldsymbol{V}$, together with an extra component —a "time-like" component we will designate as V^0. (We use superscripts for a reason that will become clear later.) The"space-like" components are then V^1, V^2, V^3. The generic form for a 4-vector is written V^α, with α taking on the values 0 through 3. Symbolically,

$$V^\alpha = (V^0, \boldsymbol{V}) \tag{22}$$

We have seen that $(ct, \boldsymbol{x})$ is one 4-vector. Another, you may recall, is the *4-momentum,*

$$p^\alpha = (E/c, \boldsymbol{p}) \tag{23}$$

where $\boldsymbol{p}$ is the ordinary momentum vector and E is the total energy. Of course, we speak of relativisitic momentum and energy:

$$\boldsymbol{p} = \gamma m \boldsymbol{v}, \quad E = \gamma m c^2 \tag{24}$$

where m is a particle's rest mass. Just as

$$(ct)^2 - x^2 \tag{25}$$

is an invariant quantity under Lorentz transformations, so too is

$$E^2 - (pc)^2 = m^2 c^4 \tag{26}$$

A rather plain 4-vector is p^α without the coefficient of m. This is the 4-velocity U^α,

$$U^\alpha = \gamma(c, \boldsymbol{v}) \tag{27}$$

Note that in the rest frame of a particle, $U^0 = c$ (a constant) and the ordinary 3-velocity components $\boldsymbol{U} = 0$. To get to any other frame, just use ("boost with") the Lorentz transformation. (Be careful with the sign of v). We don't have to worry that we boost along one axis only, whereas the velocity has three components. If you wish, just rotate the axes, after we've boosted. This sorts out all the 3-vector components the way you'd like, and leaves the time ("0") component untouched.

Humble in appearance, the 4-velocity is a most important 4-vector. Via the simple trick of boosting, the 4-velocity may be used as the starting point for constructing many other important physical 4-vectors. Consider, for example, a charge density ρ_0 which is at rest. We may create a 4-vector which, in the rest frame, has only one component: $\rho_0 c$ is the lonely time component and the ordinary spatial vector components are all zero. It is just like U^α, only with a different normalisation constant. Now boost! The resulting 4-vector is denoted

$$J^\alpha = \gamma(c\rho_0, \boldsymbol{v}\rho_0) \tag{28}$$

The time component gives the charge density in any frame, and the 3- vector components are the corresponding standard current density $\boldsymbol{J}$! This 4-current is the fundamental 4-vector of Maxwell's theory. As the source of the fields, this 4-vector source current is the basis for Maxwell's electrodynamics being a fully relativistic theory. J^0 is the source of the electric field potential function Φ, and $\boldsymbol{J}$ is the source of the magnetic field vector potential $\boldsymbol{A}$. Moreover, as we will shortly see,

$$A^\alpha = (\Phi, \boldsymbol{A}/c) \tag{29}$$

is itself a 4-vector! From here, we can generate the electromagnetic fields themselves from the potentials by constructing a tensor...well, we are getting a bit ahead of ourselves.

2.2 More on 4-vectors

2.2.1 Transformation of gradients

We have seen how the Lorentz transformation express x'^α as a function of the x coordinates. It is a simple linear transformation, and the question naturally arises of how the partial derivatives, $\partial/\partial t$, $\partial/\partial x$ transform, and whether a 4-vector can be constructed from these components. This is a simple exercise. Using

$$ct = \gamma(ct' + \beta x') \tag{30}$$

$$x = \gamma(x' + \beta ct') \tag{31}$$

we find

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} = \gamma\frac{\partial}{\partial t} + \gamma\beta c\frac{\partial}{\partial x} \tag{32}$$

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = \gamma\frac{\partial}{\partial x} + \gamma\beta\frac{1}{c}\frac{\partial}{\partial t} \tag{33}$$

In other words,

$$\frac{1}{c}\frac{\partial}{\partial t'} = \gamma\left(\frac{1}{c}\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x}\right) \tag{34}$$

$$\frac{\partial}{\partial x'} = \gamma\left(\frac{\partial}{\partial x} + \beta\frac{1}{c}\frac{\partial}{\partial t}\right) \tag{35}$$

and for completeness,

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \tag{36}$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}. \tag{37}$$

This is *not* the Lorentz transformation (5)–(8); it differs by the sign of v. By contrast, *coordinate differentials* dx^α transform, of course, just like x^α:

$$cdt' = \gamma(cdt - \beta dx), \tag{38}$$

$$dx' = \gamma(dx - \beta cdt), \tag{39}$$

$$dy' = dy, \tag{40}$$

$$dz' = dz. \tag{41}$$

This has a very important consequence:

$$dt'\frac{\partial}{\partial t'} + dx'\frac{\partial}{\partial x'} = \gamma^2\left[(dt - \beta\frac{dx}{c})\left(\frac{\partial}{\partial t} + \beta c\frac{\partial}{\partial x}\right) + (dx - \beta cdt)\left(\frac{\partial}{\partial x} + \beta\frac{1}{c}\frac{\partial}{\partial t}\right)\right], \tag{42}$$

or simplifying,

$$dt'\frac{\partial}{\partial t'} + dx'\frac{\partial}{\partial x'} = \gamma^2(1-\beta^2)\left(dt\frac{\partial}{\partial t} + dx\frac{\partial}{\partial x}\right) = dt\frac{\partial}{\partial t} + dx\frac{\partial}{\partial x} \tag{43}$$

Adding y and z into the mixture changes nothing. Thus, a scalar product exists between dx^α and $\partial/\partial x^\alpha$ that yields a *Lorentz scalar,* much as $\boldsymbol{dx}\cdot\boldsymbol{\nabla}$, the ordinary complete differential, is a rotational scalar. It is the fact that only certain combinations of 4-vectors and 4-gradients appear in the equations of physics that allows these equations to remain invariant in form from one reference frame to another.

It is time to approach this topic, which is the mathematical foundation on which special and general relativity is built, on a firmer and more systematic footing.

2.2.2 Transformation matrix

We begin with a simple but critical notational convention: repeated indices are summed over, unless otherwise explicitly stated. This is known as the *Einstein summation convention,* invented to avoid tedious repeated summation Σ's. For example:

$$dx^\alpha \frac{\partial}{\partial x^\alpha} = dt\frac{\partial}{\partial t} + dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} \tag{44}$$

I will often further shorten this to $dx^\alpha \partial_\alpha$. This brings us to another important notational convention. I was careful to write ∂_α, not ∂^α. Superscripts will be reserved for vectors, like dx^α which transform like (5) through (8) from one frame to another (primed) frame moving a relative velocity v along the x axis. Subscripts will be used to indicate vectors that transfrom like the gradient components in equations (34)–(37). Superscipt vectors like dx^α are referred to as *contravariant* vectors; subscripted vectors as *covariant.* (The names will acquire significance later.) The co- contra- difference is an important distinction in general relativity, and we begin by respecting it here in special relativity.

Notice that we can write equations (38) and (39) as

$$[-cdt'] = \gamma([-cdt] + \beta dx) \tag{45}$$

$$dx' = \gamma(dx + \beta[-cdt]) \tag{46}$$

so that the 4-vector $(-cdt, dx, dy, dz)$ is covariant, like a gradient! We therefore have

$$dx^\alpha = (cdt, dx, dy, dz) \tag{47}$$

$$dx_\alpha = (-cdt, dx, dy, dz) \tag{48}$$

It is easy to go between covariant and contravariant forms by flipping the sign of the time component. We are motivated to formalise this by introducing a matrix $\eta_{\alpha\beta}$ defined as

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{49}$$

Then $dx_\alpha = \eta_{\alpha\beta} dx^\beta$ "lowers the index." We will write $\eta^{\alpha\beta}$ to raise the index, though it is a numerically identical matrix. Note that the invariant spacetime interval may be written

$$c^2 d\tau^2 \equiv c^2 dt^2 - dx^2 - dy^2 - dz^2 = -\eta_{\alpha\beta} dx^\alpha dx^\beta \tag{50}$$

The time interval $d\tau$ is just the "proper time," the time shown ticking on the clock in the rest frame moving with the object of interest (since in this frame all spatial differentials dx^i are zero). Though introduced as a bookkeeping device, $\eta_{\alpha\beta}$ is an important quantity: it goes from being a constant matrix in special relativity to a function of coordinates in general relativity, mathematically embodying the departures of spacetime from simple Minkowski form when matter is present.

The standard Lorentz transformation may now be written as a matrix equation, $dx'^\alpha = \Lambda^\alpha{}_\beta dx^\beta$, where

$$\Lambda^\alpha{}_\beta \, dx^\beta = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} \tag{51}$$

$\Lambda^{\alpha}{}_{\beta}$ is symmetric in α and β. (A possible notational ambiguity is difficult to avoid here: β and γ used as subscripts or superscripts are of course never velocity variables!) Direct matrix multiplication gives:

$$\Lambda^{\alpha}{}_{\beta}\Lambda^{\epsilon}{}_{\gamma}\eta_{\alpha\epsilon} = \eta_{\beta\gamma} \tag{52}$$

(Do it, and notice that the η matrix must go in the middle...why?) Then, if V^{α} is any contravariant vector and W_{α} any covariant vector, $V^{\alpha}W_{\alpha}$ must be an invariant (or "scalar") because

$$V'^{\alpha}W'_{\alpha} = V'^{\alpha}W'^{\beta}\eta_{\beta\alpha} = \Lambda^{\alpha}{}_{\gamma}V^{\gamma}\Lambda^{\beta}_{\epsilon}W^{\epsilon}\eta_{\beta\alpha} = V^{\gamma}W^{\epsilon}\eta_{\gamma\epsilon} = V^{\gamma}W_{\gamma} \tag{53}$$

For covariant vectors, for example ∂_{α}, the transformation is $\partial'_{\alpha} = \tilde{\Lambda}^{\beta}{}_{\alpha}\partial_{\beta}$, where $\tilde{\Lambda}^{\beta}{}_{\alpha}$ is the same as $\Lambda^{\beta}{}_{\alpha}$, but the sign of β reversed:

$$\tilde{\Lambda}^{\alpha}{}_{\beta} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{54}$$

Note that

$$\tilde{\Lambda}^{\alpha}{}_{\beta}\Lambda^{\beta}{}_{\gamma} = \delta^{\alpha}_{\gamma}, \tag{55}$$

where δ^{α}_{γ} is the Kronecker delta function. This leads immediately once again to $V'^{\alpha}W'_{\alpha} = V^{\alpha}W_{\alpha}$.

Notice that equation (38) says something rather interesting in terms of 4-vectors. The right side is just proportional to $-dx^{\alpha}U_{\alpha}$, where U_{α} is the (covariant) 4-vector corresponding to ordinary velocity v. Consider now the case $dt' = 0$, a surface in t, x, y, z, spacetime corresponding to simultaneity in the frame of an observer moving at velocity $\boldsymbol{v}$. The equations of constant time in this frame are given by the requirement that dx^{α} and U_{α} are orthogonal.

Exercise. Show that the general Lorentz transformation matrix is:

$$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\beta_x^2/\beta^2 & (\gamma-1)\beta_x\beta_y/\beta^2 & (\gamma-1)\beta_x\beta_z/\beta^2 \\ -\gamma\beta_y & (\gamma-1)\beta_x\beta_y/\beta^2 & 1+(\gamma-1)\beta_y^2/\beta^2 & (\gamma-1)\beta_y\beta_z/\beta^2 \\ -\gamma\beta_z & (\gamma-1)\beta_x\beta_z/\beta^2 & (\gamma-1)\beta_y\beta_z/\beta^2 & 1+(\gamma-1)\beta_z^2/\beta^2 \end{pmatrix} \tag{56}$$

Hint: Keep calm and use (10) and (11).

2.2.3 Tensors

There is more to relativistic life than vectors and scalars. There are objects called tensors, with more that one indexed component. But possessing indices isn't enough! All tensor components must transform in the appropriate way under a Lorentz transformation. Thus, a tensor $T^{\alpha\beta}$ transforms according to the rule

$$T'^{\alpha\beta} = \Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\epsilon}T^{\gamma\epsilon}, \tag{57}$$

while

$$T'_{\alpha\beta} = \tilde{\Lambda}^{\gamma}{}_{\alpha}\tilde{\Lambda}^{\epsilon}{}_{\beta}T_{\gamma\epsilon}, \tag{58}$$

and of course

$$T'^{\alpha}_{\beta} = \Lambda^{\alpha}{}_{\gamma}\tilde{\Lambda}^{\epsilon}{}_{\beta}T^{\gamma}_{\epsilon}, \tag{59}$$

You get the idea. Contravariant superscript use Λ, covariant subscript use $\tilde{\Lambda}$.

Tensors are not hard to find. Remember equation (52)? It works for $\tilde{\Lambda}^\alpha{}_\beta$ as well, since it doesn't depend on the sign of β (or its magnitude for that matter):

$$\tilde{\Lambda}^\alpha{}_\beta \tilde{\Lambda}^\epsilon{}_\gamma \eta_{\alpha\epsilon} = \eta_{\beta\gamma} \tag{60}$$

So $\eta_{\alpha\beta}$ is a tensor, with the same components in any frame! The same is true of δ^α_β, a *mixed* tensor (which is the reason for writing its indices as we have), that we must transform as follows:

$$\Lambda^\epsilon{}_\gamma \tilde{\Lambda}^\alpha{}_\beta \delta^\gamma_\alpha = \Lambda^\epsilon{}_\gamma \tilde{\Lambda}^\gamma{}_\beta = \delta^\epsilon_\beta. \tag{61}$$

Here is another tensor, slightly less trivial:

$$W^{\alpha\beta} = U^\alpha U^\beta \tag{62}$$

where the $U's$ are 4-velocities. This obviously transforms as tensor, since each U obeys its own vector transformation law. Consider next the tensor

$$T^{\alpha\beta} = \rho_r \langle u^\alpha u^\beta \rangle \tag{63}$$

where the $\langle\ \rangle$ notation indicates an average of all the 4-velocity products $u^\alpha u^\beta$ taken over a whole swarm of little particles, like a gas. (An average of 4-velocities is certainly itself a 4-velocity, and an average of all the little particle tensors is itself a tensor.) ρ_r is a local rest density, a scalar number. (Here, r is not an index.)

The component T^{00} is just ρc^2, the energy density of the swarm, where ρ (without the r) includes both a rest mass energy and a thermal contribution. (The latter comes from averaging the γ factors in the $u^0 = \gamma c$.) Moreover, if, as we shall assume, the particle velocities are isotropic, then $T^{\alpha\beta}$ vanishes if $\alpha \neq \beta$. Finally, when $\alpha = \beta \neq 0$, then T^{ii} (no sum!) is by definition the pressure P of the swarm. (Do you see why this works when the u^i are relativistic?) Hence, in the frame in which the swarm has no net bulk motion,

$$T^{\alpha\beta} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{64}$$

This is, in fact, the most general form for the so-called *energy-momentum stress tensor* for an isotropic fluid in the rest frame of the fluid.

To find $T^{\alpha\beta}$ in any frame with 4-velocity U^α we could adopt a brute force method and apply the Λ matrix twice to the rest frame form, but what a waste of effort that would be! If we can find *any* true tensor that reduces to our result in the rest frame, then that tensor is *the* unique stress tensor. Proof: if a tensor is zero in any frame, then it is zero in all frames, as a trivial consequence of the transformation law. Suppose the tensor I construct, which is designed to match the correct rest frame value, may not be (you claim) correct in all frames. Hand me your tensor, the one you think is the correct choice. Now, the two tensors by definition match in the rest frame. I'll subtract one from the other to form the difference between my tensor and your tensor. The difference is also a tensor, but it vanishes in the rest frame by construction. Hence this "difference tensor" must vanish in all frames, so your tensor and mine are identical after all! Corollary: if you can prove that the two tensors are the same in any one particular frame, then they are the same in all frames. This is a very useful ploy.

The only two tensors we have at our disposal to construct $T^{\alpha\beta}$ are $\eta^{\alpha\beta}$ and $U^\alpha U^\beta$, and there is only one linear superposition that matches the rest frame value and does the trick:

$$T^{\alpha\beta} = P\eta^{\alpha\beta} + (\rho + P/c^2)U^\alpha U^\beta \tag{65}$$

This is the general form of energy-momentum stress tensor appropriate to an ideal fluid.

2.2.4 Conservation of $T^{\alpha\beta}$

One of the most salient properties of $T^{\alpha\beta}$ is that it is *conserved,* in the sense of

$$\frac{\partial T^{\alpha\beta}}{\partial x^\alpha} = 0 \tag{66}$$

Since gradients of tensors transform as tensors, this must be true in all frames. What, exactly, are we conserving?

First, the time-like 0-component of this equation is

$$\frac{\partial}{\partial t}\left[\gamma^2\left(\rho + \frac{Pv^2}{c^4}\right)\right] + \boldsymbol{\nabla\cdot}\left[\gamma^2\left(\rho + \frac{P}{c^2}\right)\boldsymbol{v}\right] = 0 \tag{67}$$

which is the relativistic version of mass conservation,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla\cdot}(\rho\boldsymbol{v}) = 0. \tag{68}$$

Elevated in special relativity, it becomes a statement of *energy conservation.* So one of the things we are conserving is energy. (And not just rest mass energy by the way, thermal energy as well!) This is good.

The spatial part of the conservation equation reads

$$\frac{\partial}{\partial t}\left[\gamma^2\left(\rho + \frac{P}{c^2}\right)v_i\right] + \left(\frac{\partial}{\partial x^j}\right)\left[\gamma^2\left(\rho + \frac{P}{c^2}\right)v_i v_j\right] + \frac{\partial P}{\partial x^i} = 0 \tag{69}$$

You may recognise this as Euler's equation of motion, a statement of momentum conservation, upgraded to special relativity. Conserving momentum is also good.

What if there are other external forces? The idea is that these are included by expressing them in terms of the divergence of their own stress tensor. Then it is the total $T^{\alpha\beta}$ including, say, electromagnetic fields, that comes into play. What about the force of gravity? *That,* it will turn out, is on an all-together different footing.

You start now to gain a sense of the difficulty in constructing a theory of gravity compatible with relativity. The density ρ is part of the stress tensor, and it is the entire stress tensor in a relativistic theory that would have to be the source of the gravitational field, just as the entire 4-current J^α is the source of electromangetic fields. No fair just picking the component you want. Relativistic theories work with scalars, vectors and tensors to preserve their invariance properties from one frame to another. This insight is already an achievement: we can, for example, expect pressure to play a role in generating gravitational fields. Would you have guessed that? Our relativistic gravity equation maybe ought to look something like :

$$\nabla^2 G^{\mu\nu} - \frac{1}{c^2}\frac{\partial^2 G^{\mu\nu}}{\partial t^2} = T^{\mu\nu} \tag{70}$$

where $G^{\mu\nu}$ is some sort of, I don't know, conserved tensor guy for the...spacetime geometry and stuff? In Maxwell's theory we had a 4-vector (A^α) operated on by the so-called "d'Alembertian operator" $\nabla^2 - (1/c)^2\partial^2/\partial t^2$ on the left side of the equation and a source (J^α) on the right. So now we just need to find a $G^{\mu\nu}$ tensor to go with $T^{\mu\nu}$. Right?

Actually, this really is a pretty good guess. It is more-or-less correct for weak fields, and most of the time gravity *is* a weak field. But...well...patience. One step at a time.

Then there occurred to me the 'glücklichste Gedanke meines Lebens,' the happiest thought of my life, in the following form. The gravitational field has only a relative existence in a way similar to the electric field generated by magnetoelectric induction. Because [1] for an observer falling freely from the roof of a house there exists—*at least in his immediate surroundings*—no gravitational field.

— *Albert Einstein*

3 The effects of gravity

The central idea of general relativity is that presence of mass (more precisely the presence of any stress-energy tensor component) causes departures from flat Minkowski spacetime to appear, and that other matter (or radiation) responds to these distortions in some way. There are then really two questions: (i) How does the affected matter/radiation move in the presence of a distorted spacetime?; and (ii) How does the stress-energy tensor distort the spacetime in the first place? The first question is purely computational, and fairly straightforward to answer. It lays the groundwork for answering the much more difficult second question, so let us begin here.

3.1 The Principle of Equivalence

We have discussed the notion that by going into a frame of reference that is in free-fall, the effects of gravity disappear. In this era in which space travel is common, we are all familiar with astronauts in free-fall orbits, and the sense of weightlessness that is produced. This manifestation of the Equivalence Principle is so palpable that hearing total mishmashes like "In orbit there is no gravity" from an over-eager science correspondent is a common experience. (Our own BBC correspondent in Oxford Astrophysics, Prof. Christopher Lintott, would certainly *never* say such a thing.)

The idea behind the equivalence principle is that the m in $F = ma$ and the m in the force of gravity $F_g = mg$ are the same m and thus the acceleration caused by gravity, g, is invariant for any mass. We could imagine, for example, that $F = m_I a$ and $F_g = m_g g$, where m_g is some kind of "massy" property that might vary from one type of body to another with the same m_I. In this case, the acceleration a is $m_g g/m_I$, i.e., it varies with the ratio of inertial to gravitational mass from one body to another. How well can we actually measure this ratio, or what is more to the point, how well do we know that it is truly a universal constant for all types of matter?

The answer is very, very well indeed. We don't of course do anything as crude as directly measure the rate at which objects fall to the ground any more, à la Galileo and the tower of Pisa. As with all classic precision gravity experiments (including those of Galileo!) we

[1] With apologies to any readers who may actually have fallen off the roof of a house—safe space statement.

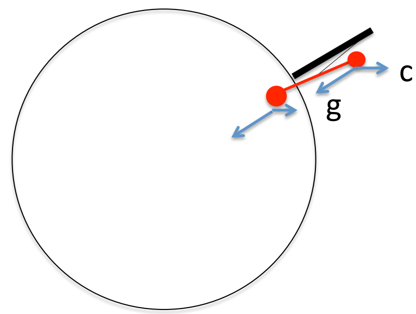

Figure 1: Schematic diagram of the Eötvös experiment. A barbell shape, the red object above, is hung from a pendulum on the Earth's surface (big circle) with two masses of two different types of material, say copper and lead. Each mass is affected by gravity pulling it to the centre of the earth (g) with a force proportional to a *gravitational mass* m_g, and a centrifugal force porportional to the *inertial* mass m_I, due to the earth's rotation (c). Forces are shown as blue arrows. Any difference between the inertial to gravitational mass ratio (in copper and lead here) will produce an unbalanced torque from the g and c forces about the axis of the suspending fibre of the barbell.

use a pendulum. The first direct measurement of the gravitational to inertial mass actually predates relativity, the so-called Eötvös experiment (after Baron Lorànd Eötvös, 1848-1919).

The idea is shown in schematic form in figure [1]. Hang a pendulum from a string, but instead of hanging a big mass, hang a rod, and put two masses of two different types of material at either end. There is a force of gravity toward the center of the earth (g in the figure), and a centrifugal force (c) due to the earth's rotation. The net force is the vector sum of these two, and if the components of the acceleration perpendicular to the string of each mass do not precisely balance, and they won't if m_g/m_I is not the same for both masses, there will be a net torque twisting the masses about the string (a quartz fibre in the actual experiment). The fact that no such twist is measured is an indication that the ratio m_g/m_I does not, in fact, vary. In practise, to achieve high accuracy, the pendulum rotates with a tightly controlled period, so that the masses would be sometimes hindered by any putative torque, sometimes pushed forward by this torque. This would imprint a frequency dependence onto the motion, and by using fourier signal processing, the resulting signal at a particular frequency can be tightly constrained. Experiment shows that the ratio between any difference in the twisting accelerations on either mass and the average acceleration must be less than a few parts in 10^{12} (Su et al. 1994, Phys Rev D, 50, 3614). With direct laser ranging experiments to track the Moon's orbit, it is possible, in effect, to use the Moon and

Earth as the masses on the pendulum as they rotate around the Sun! This gives an accuracy an order of magnitude better, a part in 10^{13} (Williams et al. 2012, Class. Quantum Grav., 29, 184004), an accuracy comparable to measuring the distance to the Sun to within the size of your thumbnail.

There are two senses in which the Equivalence Principle may be used, a strong sense and weak sense. The weak sense is that it is not possible to detect the effects of gravity locally in a freely falling coordinate system, that all matter behaves identically in a gravitational field independent of its composition. Experiments can test this form of the Principle directly. The strong, much more powerful sense, is that *all physical laws,* gravitational or not, behave in a freely falling coordinate system just as they do in Minkowski spacetime. In this sense, the Principle is a postulate which appears to be true.

If going into a freely falling frame eliminates gravity locally, then going from an inertial frame to an accelerating frame reverses the process and mimics the effect of gravity—again, locally. After all, if in an inertial frame

$$\frac{d^2x}{dt^2} = 0, \tag{71}$$

and we transform to the accelerating frame x' by $x = x' + gt^2/2$, where g is a constant, then

$$\frac{d^2x'}{dt^2} = -g, \tag{72}$$

which looks an awful lot like motion in a gravitational field.

One immediate consequence of this realisation is of profound importance: gravity affects light. In particular, if we are in an elevator of height h in a gravitational field of local strength g, *locally the physics is exactly the same as if we were accelerating upwards at g.* But the effect of this on light is then easily analysed: a photon released upwards reaches a detector at height h in a time h/c, at which point the detector is moving at a velocity gh/c relative to the bottom of the elevator (at the time of release). The photon is measured to be redshifted by an amount gh/c^2, or Φ/c^2 with Φ being the gravitational potential per unit mass at h. This is the classical gravitational redshift, the simplest nontrivial prediction of general relativity. The gravitational redshift has been measured accurately using changes in gamma ray energies (RV Pound & JL Snider 1965, *Phys. Rev.,* **140 B,** 788).

The gravitational redshift is the critical link between Newtonian theory and general relativity. It is not, after all, a distortion of space that gives rise to Newtonian gravity at the level we are familiar with, it is a distortion of the flow of *time.*

3.2 The geodesic equation

We denote by ξ^α our freely falling inertial coordinate frame in which the effects of gravity are locally absent. In this frame, the equation of motion for a particle is

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \tag{73}$$

with

$$c^2 d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{74}$$

being the invariant time interval. (If we are doing light, then $d\tau = 0$, but ultimately it doesn't really matter. Either take a limit from finite $d\tau$, or use any other parameter you fancy, like your wristwatch. In the end, we won't use τ or your watch. As for $d\xi^\alpha$, it is just the freely-falling guy's ruler and *his* wristwatch.) Next, write this equation in any other set of coordinates you like, and call them x^μ. Our inertial coordinates ξ^α will be some function or other of the x^μ so

$$0 = \frac{d^2\xi^\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) \tag{75}$$

where we have used the chain rule to express $d\xi^\alpha/d\tau$ in terms of $dx^\mu/d\tau$. Carrying out the differentiation,

$$0 = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{76}$$

where now the chain rule has been used on $\partial\xi^\alpha/\partial x^\mu$. This may not look very promising. But if we multiply this equation by $\partial x^\lambda/\partial\xi^\alpha$, and remember to sum over α now, then the chain rule in the form

$$\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\mu} = \delta^\lambda_\mu \tag{77}$$

rescues us. (We are using the chain rule repeatedly and will certainly continue to do so, again and again. Make sure you understand this, *and* that you understand what variables are being held constant when the partial derivatives are taken. Deciding what is constant is just as important as doing the differentiation!) Our equation becomes

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{78}$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \tag{79}$$

is known as the *affine connection*, and is a quantity of central importance in the study of Riemannian geometry and relativity theory in particular. You should be able to prove, using the chain rule of partial derivatives, an identity for the second derivatives of ξ^α that we will use shortly:

$$\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} = \frac{\partial \xi^\alpha}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu} \tag{80}$$

(How does this work out when used in equation [76]?)

No need to worry, despite the funny notation. (Early relativity texts liked to use **gothic font** $\mathfrak{G}^\lambda_{\mu\nu}$ for the affine connection, which must have imbued it with a nice steam-punk terror.) There is nothing especially mysterious about the affine connection. You use it all the time, probably without realising it. For example, in cylindrical (r, θ) coordinates, when you use the combinations $\ddot{r} - r\dot{\theta}^2$ or $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ for your radial and tangential accelerations, you are using the affine connection and the geodesic equation. In the first case, $\Gamma^r_{\theta\theta} = -r$; in the second, $\Gamma^\theta_{r\theta} = 1/r$. (What happened to the 2?)

Exercise. Prove the last statements using $\xi^x = r\cos\theta, \xi^y = r\sin\theta$.

Exercise. On the surface of a unit-radius sphere, choose any point as your North Pole, work in colatitude θ and azimuth ϕ coordinates, and show that *locally* near the North Pole $\xi^x = \theta\cos\phi$, $\xi^y = \theta\sin\phi$. It is in this sense that the ξ^α coordinates are tied to a local region of the *space* near the North Pole point. In our freely-falling coordinate system, the local coordinates are tied to a point in *spacetime.*

3.3 The metric tensor

In our locally inertial coordinates, the invariant spacetime interval is

$$c^2 d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta, \tag{81}$$

so that in any other coordinates, $d\xi^\alpha = (\partial \xi^\alpha / dx^\mu) dx^\mu$ and

$$c^2 d\tau^2 = -\eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} dx^\mu dx^\nu \equiv -g_{\mu\nu} dx^\mu dx^\nu \tag{82}$$

where

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \tag{83}$$

is known as the *metric tensor*. The metric tensor embodies the information of how coordinate differentials combine to form the invariant interval of our spacetime, and once we know $g_{\mu\nu}$, we know everything, including (as we shall see) the affine connections $\Gamma^\lambda_{\mu\nu}$. The object of general relativity theory is to compute $g_{\mu\nu}$ for a given distribution of mass (more precisely, a given stress energy tensor), and a key goal of this course is to find the field equations that enable us to do so.

3.4 The relationship between the metric tensor and affine connection

Because of their reliance of the *local* freely falling inertial coordinates ξ^α, the $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ quantities are awkward to use in their present formulation. Fortunately, there is a direct relationship between $\Gamma^\lambda_{\mu\nu}$ and the first derivatives of $g_{\mu\nu}$ that will allow us to become free of local bondage, permitting us to dispense with the ξ^α altogether. Though their *existence* is crucial to formulate the mathematical structure, the practical need of the ξ's to carry out calculations is minimal.

Differentiate equation (83):

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \eta_{\alpha\beta} \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} + \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu} \tag{84}$$

Now use (80) for the second derivatives of ξ:

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial \xi^\beta}{\partial x^\nu} \Gamma^\rho_{\lambda\mu} + \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\rho} \Gamma^\rho_{\lambda\nu} \tag{85}$$

All remaining ξ derivatives may be absorbed as part of the metric tensor, leading to

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = g_{\rho\nu} \Gamma^\rho_{\lambda\mu} + g_{\mu\rho} \Gamma^\rho_{\lambda\nu} \tag{86}$$

It remains only to unweave the Γ's from the cloth of indices. This is done by first adding $\partial g_{\lambda\nu} / \partial x^\mu$ to the above, then subtracting it with indices μ and ν reversed.

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} = g_{\rho\nu} \Gamma^\rho_{\lambda\mu} + \cancel{g_{\rho\mu} \Gamma^\rho_{\lambda\nu}} + g_{\rho\nu} \Gamma^\rho_{\mu\lambda} + \cancel{g_{\rho\lambda} \Gamma^\rho_{\mu\nu}} - \cancel{g_{\rho\mu} \Gamma^\rho_{\nu\lambda}} - \cancel{g_{\rho\lambda} \Gamma^\rho_{\nu\mu}} \tag{87}$$

Remembering that Γ is symmetric in its bottom indices, only the $g_{\rho\nu}$ terms survive, leaving

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} = 2g_{\rho\nu}\Gamma^\rho_{\mu\lambda} \tag{88}$$

Our last step is to mulitply by the inverse matrix $g^{\nu\sigma}$, defined by

$$g^{\nu\sigma}g_{\rho\nu} = \delta^\sigma_\rho, \tag{89}$$

leaving us with the pretty result

$$\Gamma^\sigma_{\mu\lambda} = \frac{g^{\nu\sigma}}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu}\right). \tag{90}$$

Notice that there is no mention of the ξ's. The affine connection is completely specified by $g^{\mu\nu}$ and the derivatives of $g_{\mu\nu}$ in whatever coordinates you like. In practise, the inverse matrix is not difficult to find, as we will usually work with metric tensors whose off diagonal terms vanish. (Gain confidence once again by practising the geodesic equation with cylindrical coordinates $g_{rr} = 1$, $g_{\theta\theta} = r^2$ and using [90.]) Note as well that with some very simple index relabeling, equation (88) leads directly to the mathematical identity

$$g_{\rho\nu}\Gamma^\rho_{\mu\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\lambda}{d\tau} = \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \frac{1}{2}\frac{\partial g_{\lambda\mu}}{\partial x^\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\lambda}{d\tau}. \tag{91}$$

We'll use this in a moment.

Exercise. Prove that $g^{\nu\sigma}$ is given explicitly by

$$g^{\nu\sigma} = \eta^{\alpha\beta}\frac{\partial x^\nu}{\partial \xi^\alpha}\frac{\partial x^\sigma}{\partial \xi^\beta}$$

Exercise. Prove the identities of page 6 of the notes for a diagonal metric g_{ab},

$$\Gamma^a_{ba} = \Gamma^a_{ab} = \frac{1}{2g_{aa}}\frac{\partial g_{aa}}{\partial x^b} \quad (a = b \text{ permitted, NO SUM})$$

$$\Gamma^a_{bb} = -\frac{1}{2g_{aa}}\frac{\partial g_{bb}}{\partial x^a} \quad (a \neq b, \quad \text{NO SUM})$$

$$\Gamma^a_{bc} = 0, \quad (a, b, c \quad \text{distinct})$$

3.5 Variational calculation of the geodesic equation

The physical significance of the relationship between the metric tensor and affine connection may be understood by a variational calculation. Off all possible paths in our spacetime from some point A to another B, which leaves the proper time an extremum (in this case, a maximum)? The motivation for this formulation is obvious: "The shortest distance between two points is a straight line," and the equations for this line-geodesic are $d^2\xi_i/ds^2 = 0$ in Cartesian coordinates. This is an elementary property of Euclidian space. We may ask what is the shortest distance between two points in a more general curved space as well, and this question naturally lends itself to a variational approach. What is less obvious is that this mathematical machinery, which was fashioned for generalising the spacelike straight line equation $d^2\xi^i/ds^2 = 0$ to more general non-Euclidian geometries, also works for generalising a *dynamical* equation of the form $d^2\xi^i/d\tau^2 = 0$, where now we are using invariant *timelike* intervals, to geodesics embedded in distorted Minkowski geometries.

We describe our path by some external parameter p, which could be anything really, perhaps the time on your very own wristwatch in your rest frame. (I don't want to start with τ, because $d\tau = 0$ for light.) Then the proper time from A to B is

$$T_{AB} = \int_A^B \frac{d\tau}{dp}\, dp = \frac{1}{c}\int_A^B \left(-g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\right)^{1/2} dp \tag{92}$$

Next, vary x^λ to $x^\lambda + \delta x^\lambda$ (we are regarding x^λ as a function of p remember), with δx^λ vanishing at the end points A and B. We find

$$\delta T_{AB} = \frac{1}{2c}\int_A^B \left(-g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\right)^{-1/2}\left(-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda \frac{dx^\mu}{dp}\frac{dx^\nu}{dp} - 2g_{\mu\nu}\frac{d\delta x^\mu}{dp}\frac{dx^\nu}{dp}\right) dp \tag{93}$$

(Do you understand the final term in the integral?)

Since the leading inverse square root in the integrand is just $dp/d\tau$, δT_{AB} simplifies to

$$\delta T_{AB} = \frac{1}{2c}\int_A^B \left(-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - 2g_{\mu\nu}\frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) d\tau, \tag{94}$$

and p has vanished from sight. We now integrate the second term by parts, noting that the contribution from the endpoints has been specified to vanish. Remembering that

$$\frac{dg_{\lambda\nu}}{d\tau} = \frac{dx^\sigma}{d\tau}\frac{\partial g_{\lambda\nu}}{\partial x^\sigma}, \tag{95}$$

we find

$$\delta T_{AB} = \frac{1}{c}\int_A^B \left(-\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{\partial g_{\lambda\nu}}{\partial x^\sigma}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} + g_{\lambda\nu}\frac{d^2x^\nu}{d\tau^2}\right)\delta x^\lambda\, d\tau \tag{96}$$

or

$$\delta T_{AB} = \frac{1}{c}\int_A^B \left[\left(-\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + g_{\lambda\nu}\frac{d^2x^\nu}{d\tau^2}\right]\delta x^\lambda\, d\tau \tag{97}$$

Finally, using equation (91), we obtain

$$\delta T_{AB} = \frac{1}{c}\int_A^B \left[\left(\frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau}\Gamma^\nu_{\mu\sigma} + \frac{d^2x^\nu}{d\tau^2}\right) g_{\lambda\nu}\right]\delta x^\lambda\, d\tau \tag{98}$$

Thus, if the geodesic equation (78) is satisfied, $\delta T_{AB} = 0$ is satisfied, and the proper time is an extremum. The name "geodesic" is used in geometry to describe the path of minimum distance between two points in a manifold, and it is therefore gratifying to see that there is a correspondence between a local "straight line" with zero curvature, and the local elimination of a gravitational field with the resulting zero acceleration, along the lines if the first paragraph of this section. In the first case, the proper choice of local coordinates results in the second derivative with respect to an invariant spatial interval vanishing; in the second case, the proper choice of coordinates means that the second derivative with respect to an invariant time interval vanishes, but *the essential mathematics is the same.*

There is often a very practical side to working with the variational method: it can be much easier to obtain the equations of motion for a given $g_{\mu\nu}$ this way than to construct them directly. For example, the method quickly produces all the non-vanishing affine connection components, just read them off as the coefficients of $(dx^\mu/d\tau)(dx^\nu/d\tau)$. You don't have to find them by trial and error. These quantities are then available for any variety of purposes (and they are needed for many).

Here is another trick. You should have little difficulty showing that if we apply the Euler-Lagrange variational method directly to the following functional $\mathcal{L}$,

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu,$$

where the dot is $d/d\tau$, the resulting Euler-Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\rho}\right) - \frac{\partial\mathcal{L}}{\partial x^\rho} = 0$$

is just the standard geodesic equation of motion! This is often the easiest way to proceed.

Indeed, in classical mechanics, we all know that the equations of motion may be derived from a Lagrangian variational principle of least action, an integral involving the difference between kinetic and potential energies. This doesn't seem geometrical at all. What is the connection with what we've just done? How do we make contact with Newtonian mechanics from the geodesic equation?

3.6 The Newtonian limit

We consider the case of a slowly moving mass ("slow" of course means relative to c, the speed of light) in a weak gravitational field ($GM/rc^2 \ll 1$). Since $cdt \gg |d\boldsymbol{x}|$, the geodesic equation greatly simplfies:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{cdt}{d\tau}\right)^2 = 0. \tag{99}$$

Now

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{0\nu}}{\partial(cdt)} + \frac{\partial g_{0\nu}}{\partial(cdt)} - \frac{\partial g_{00}}{\partial x^\nu}\right) \tag{100}$$

In the Newtonian limit, the largest of the g derivatives is the spatial gradient, hence

$$\Gamma^\mu_{00} \simeq -\frac{1}{2}g^{\mu\nu}\frac{\partial g_{00}}{\partial x^\nu} \tag{101}$$

Since the gravitational field is weak, $g_{\alpha\beta}$ differs very little from the Minkoswki value:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad h_{\alpha\beta} \ll 1, \tag{102}$$

and the $\mu = 0$ geodesic equation is

$$\frac{d^2t}{d\tau^2} + \frac{1}{2}\frac{\partial h_{00}}{\partial t}\left(\frac{dt}{d\tau}\right)^2 = 0 \tag{103}$$

Clearly, the second term is zero for a static field, and will prove to be tiny when the gravitational field changes with time under nonrelativistic conditions—we are, after all, calculating the difference between proper time and observer time! Dropping this term we find that t and τ are linearly related, so that the spatial components of the geodesic equation become

$$\frac{d^2\boldsymbol{x}}{dt^2} - \frac{c^2}{2}\boldsymbol{\nabla} h_{00} = 0 \tag{104}$$

Isaac Newton would say:

$$\frac{d^2\boldsymbol{x}}{dt^2} + \boldsymbol{\nabla}\Phi = 0, \tag{105}$$

with Φ being the classical gravitational potential. The two views are consistent if

$$h_{00} \simeq -\frac{2\Phi}{c^2}, \quad g_{00} \simeq -\left(1 + \frac{2\Phi}{c^2}\right) \tag{106}$$

In other words, the gravitational potential force emerges as a sort of centripital term, similar in structure to the centripital force in the standard radial equation of motion. *This is a remarkable result.* It is by no means obvious that a purely geometrical geodesic equation can serve the role of a Newtonian gravitational potential gradient force equation, but it can. Moreover, it teaches us that the Newtonian limit of general relativity is all in the time component, h_{00}. It is now possible to measure directly the differences in the rate at which clocks run at heights separated by 100 m or so on the Earth's surface.

The quantity h_{00} is a dimensionless number of order v^2/c^2, where v is a velocity typical of the system, an orbital speed or just the square root of a potential. Note that h_{00} is determined by the dynamical equations only up to an additive constant. Here we have chosen the constant to make the geometry Minkowskian at large distances from any matter. At the surface of a spherical object of mass M and radius R,

$$h_{00} \simeq 2 \times 10^{-6} \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right) \tag{107}$$

where $M_\odot$ is the mass of the sun (about 2×10^{30} kg) and $R_\odot$ is the radius of the sun (about 7×10^8 m). As an exercise, you may wish to look up masses of planets and other types of stars and evaluate h_{00}. What is its value at the surface of a white dwarf (mass of the sun, radius of the earth)? What about a neutron star (mass of the sun, radius of Oxford)? How many decimal points are needed to see the time difference in two digital clocks at a one meter separation in height on the earth?

We are now able to relate the geodesic equation to the principle of least action in classical mechanics. In the Newtonian limit, our variational integral becomes

$$\int \left[c^2(1 + 2\Phi/c^2)dt^2 - d|\boldsymbol{x}|^2\right]^{1/2} \tag{108}$$

(Remember our compact notation: $dt^2 \equiv (dt)^2$, $d|\boldsymbol{x}|^2 = (d|\boldsymbol{x}|)^2$.) Expanding the square root,

$$\int c\left(1 + \frac{\Phi}{c^2} - \frac{v^2}{2c^2} + ...\right) dt \tag{109}$$

where $v^2 \equiv (d|\boldsymbol{x}|/dt)^2$. Thus, minimising the Lagrangian (kinetic energy minus potential energy) is the same as maximising the proper time interval! What an unexpected and beautiful connection.

What we have calculated in this section is nothing more than our old friend the gravitational redshift, with which we began our formal study of general relativity. The invariant spacetime interval $d\tau$, the proper time, is given by

$$c^2 d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu \tag{110}$$

For an observer at rest at location x, the time interval registered on a clock will be

$$d\tau(x) = [-g_{00}(x)]^{1/2} dt \tag{111}$$

where dt is the time interval registered at infinity, where $-g_{00} \to 1$. (Compare: the "proper length" on the unit sphere for an interval at constant θ is $\sin\theta d\phi$, where $d\phi$ is the length registered by an equatorial observer.) If the interval between two wave crest crossings is found to be $d\tau(y)$ at location y, it will be $d\tau(x)$ when the light reaches x and it will be dt at infinity. In general,

$$\frac{d\tau(y)}{d\tau(x)} = \left[\frac{g_{00}(y)}{g_{00}(x)}\right]^{1/2}, \tag{112}$$

and in particular

$$\frac{d\tau(R)}{dt} = \frac{\nu(\infty)}{\nu} = [-g_{00}(R)]^{1/2} \tag{113}$$

where $\nu = 1/d\tau(R)$ is, for example, an atomic transition frequency measured at rest at the surface R of a body, and $\nu(\infty)$ the corresponding frequency measured a long distance away. Interestingly, the value of g_{00} that we have derived in the Newtonian limit is, in fact, the *exact* relativisitic value of g_{00} around a point mass M! (A black hole.) The precise redshift formula is

$$\nu_\infty = \left(1 - \frac{2GM}{Rc^2}\right)^{1/2} \nu \tag{114}$$

The redshift as measured by wavelength becomes infinite from light emerging from radius $R = 2GM/c^2$, the so-called Schwarzschild radius (about 3 km for a point with the mass of the sun!).

Historically, general relativity theory was supported in its infancy by the reported detection of a gravitational redshift in a spectral line observed from the surface of the white dwarf star Sirius B in 1925 by W.S. Adams. It "killed two birds with one stone," as the leading astronomer A.S. Eddington remarked. For it not only proved the existence of white dwarf stars (at the time controversial since the mechanism of pressure support was unknown), the measurement also confirmed an early and important prediction of general relativity theory: the redshift of light due to gravity.

Alas, the modern consensus is that the actual measurements were flawed! Adams knew what he was looking for and found it. Though he was premature, the activity this apparently positive observation imparted to the study of white dwarfs and relativity theory turned out to be very fruitful indeed. But we were lucky. Incorrect but well regarded single-investigator observations have in the past caused much confusion and needless wrangling, as well as years of wasted effort.

The first definitive test for gravitational redshift came much later, and it was terrestrial: the 1959 Pound and Rebka experiment performed at Harvard University's Jefferson Tower measured the frequency shift of a 14.4 keV gamma ray falling (if that is the word for a gamma ray) 22.6 m. Pound & Rebka were able to measure the shift in energy—just a few parts in 10^{14}—by what was at the time the new and novel technique of Mössbauer spectroscopy.

Exercise. A novel application of the gravitational redshift is provided by Bohr's refutation of an argument put forth by Einstein purportedly showing that an experiment could in principle be designed to bypass the quantum uncertainty relation $\Delta E\, \Delta t \geq h$. The idea is to hang a box containing a photon by a spring suspended in a gravitational field g. At some precise time a shutter is opened and the photon leaves. You weigh the box before and after the photon. There is in principle no interference between the arbitrarily accurate change in box weight and the arbitrarily accurate time at which the shutter is opened. Or is there?

1.) Show that box apparatus satisfies an equation of the form

$$M\ddot{x} = -Mg - kx$$

where M is the mass of the apparatus, x is the displacement, and k is the spring constant. Before release, the box is in equilibrium at $x = -gM/k$.

2.) Show that the momentum of the box apparatus after a short time interval Δt from when the photon escapes is

$$\delta p = -\frac{g\delta m}{\omega} \sin(\omega\Delta t) \simeq= -g\delta m\Delta t$$

where δm is the (uncertain!) photon mass and $\omega^2 = k/M$. With $\delta p \sim g\delta m\Delta t$, the uncertainty principle then dictates an uncertain location of the box position δx given by $g\delta m\, \delta x\Delta t \sim h$. But this is location uncertainty, not time uncertainty.

3.) Now the gravitational redshift comes in! Show that if there is an uncertainty in position δx, there is an uncertainty in the time of release: $\delta t \sim (g\delta x/c^2)\Delta t$.

4.) Finally use this in part (2) to establish $\delta E\, \delta t \sim h$ with $\delta E = \delta mc^2$.

Why does general relativity come into *nonrelativistic* quantum mechanics in such a fundamental way? Because the gravitational redshift is relativity theory's point-of-contact with classical Newtonian mechanics, and Newtonian mechanics when blended with the uncertainty principle is the start of nonrelativistic quantum mechanics.

A final thought

We Newtonian beings, with our natural mode of thinking in terms of forces and responses, would naturally say "How interesting, the force of gravity distorts the flow time." This is the way I have been describing the gravitational redshift throught this chapter. But Einstein has given us a more profound insight. It is not that gravity *distorts* the flow of time. An Einsteinian being, brought up from the cradle to be comfortable with a spacetime point-of-view, would, upon hearing this comment, cock their head and say: "What are you talking about? Newtonian gravity *is* the distortion of the flow of time. It is a simple geometric distortion that is brought about by the presence of matter." This is a better way to think of it. The nearby effect of weak gravity is indeed a distortion in the flow of time; the distant effect of weak gravity is gravitational radiation, and this, we shall see, is a distortion of *space*.

4 Tensor Analysis

Further, the dignity of the science seems to require that every possible means be explored itself for the solution of a problem so elegant and so celebrated.

— Carl Friedrich Gauss

A mathematical equation is valid in the presence of general gravitational fields when

i.) It is a valid equation in the absence of gravity and respects Lorentz invariance.

ii.) It preserves its form, not just under Lorentz transformations, but under any coordinate transformation, $x \to x'$.

What does "preserves its form" mean? It means that the equation must be written in terms of quantities that transform as scalars, vectors, and higher ranked tensors under general coordinate transformations. From (ii), we see that if we can find one coordinate system in which our equation holds, it will hold in any set of coordinates. But by (i), the equation *does* hold in locally freely falling coordinates, in which the effect of gravity is locally absent. The effect of gravity is strictly embodied in the two key quantities that emerge from the calculus of coordinate transformations: the metric tensor $g_{\mu\nu}$ and its first derivatives in $\Gamma^{\lambda}_{\mu\nu}$. This approach is known as the *Principle of General Covariance,* and it is a very powerful tool indeed.

4.1 Transformation laws

The simplest vector one can write down is the ordinary coordinate differential dx^{μ}. If $x'^{\mu} = x'^{\mu}(x)$, there is no doubt how the dx'^{μ} are related to the dx^{μ}. It is called the chain rule, and it is by now very familiar:

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\, dx^{\nu} \tag{115}$$

Be careful to distinguish between the coordinates x^{μ}, which can be pretty much anything, and their differentials dx^{μ}, which are true vectors. Indeed, any set of quantities V^{μ} that transforms in this way is known as a *contravariant vector*:

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\, V^{\nu} \tag{116}$$

The contravariant 4-velocity, which is a 4-vector, is simply $V^{\mu} = dx^{\mu}/d\tau$, a generalisation of the special relativistic $d\xi^{\alpha}/d\tau$. A *covariant vector,* by contrast, transforms as

$$V'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\, V_{\nu} \tag{117}$$

"CO LOW, PRIME BELOW." (Sorry. Maybe you can do better.) These definitions of contravariant and covariant vectors are consistent with those we first introduced in our discussions of the Lorentz matrices Λ^{α}_{β} and $\tilde{\Lambda}^{\beta}_{\alpha}$ in Chapter 2, but now generalised from specific linear transformations to arbitrary transformations.

The simplest covariant vector is the gradient $\partial/\partial x^\mu$ of a scalar Φ. Once again, the chain rule tells us how to transform from one set of coordinates to another—we've no choice:

$$\frac{\partial\Phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial\Phi}{\partial x^\nu} \tag{118}$$

The generalisation to tensor transformation laws is immediate. A contravariant tensor $T^{\mu\nu}$ transforms as

$$T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}T^{\rho\sigma} \tag{119}$$

a covariant tensor $T_{\mu\nu}$ as

$$T'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}T_{\rho\sigma} \tag{120}$$

and a mixed tensor T^μ_ν as

$$T'^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}T^\rho_\sigma \tag{121}$$

The generalisation to mixed tensors of arbitrary rank should be self-evident.

By this definition the metric tensor $g_{\mu\nu}$ really *is* a covariant tensor, just as its notation would lead you to believe, because

$$g'_{\mu\nu} \equiv \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x'^\mu}\frac{\partial\xi^\beta}{\partial x'^\nu} = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\lambda}\frac{\partial\xi^\beta}{\partial x^\rho}\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} \equiv g_{\lambda\rho}\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} \tag{122}$$

and the same for the *contravariant* $g^{\mu\nu}$. But the gradient of a vector is *not,* in general, a tensor or a vector:

$$\frac{\partial V'^\lambda}{\partial x'^\mu} = \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x'^\lambda}{\partial x^\nu}V^\nu\right) = \frac{\partial x'^\lambda}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial V^\nu}{\partial x^\rho} + \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\nu}\frac{\partial x^\rho}{\partial x'^\mu}V^\nu \tag{123}$$

The first term is just what we would have wanted if we were searching for a tensor transformation law. But oh those pesky second order derivatives—the final term spoils it all. This of course vanishes when the coordinate transformation is linear (as when we found that vector derivatives are perfectly good tensors under the Lorentz transformations), but not in general. We will show in the next section that while the gradient of a vector is in general not a tensor, there is an elegant solution around this problem.

Tensors can be created and manipulated in many ways. For example, direct products of tensors are tensors:

$$W^{\mu\nu}_{\rho\sigma} = T^{\mu\nu}S_{\rho\sigma}. \tag{124}$$

A linear combination of tensors of the same rank multiplied by scalars is obviously a tensor of unchanged rank. A tensor can lower its index by multiplying by $g_{\mu\nu}$ or raise it with $g^{\mu\nu}$:

$$T'^\rho_\mu \equiv g'_{\mu\nu}T'^{\nu\rho} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\kappa}\frac{\partial x'^\rho}{\partial x^\tau}g_{\sigma\lambda}T^{\kappa\tau} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x'^\rho}{\partial x^\tau}g_{\sigma\kappa}T^{\kappa\tau} \tag{125}$$

which indeed does transform as a tensor of mixed second rank, T^ρ_μ. To clarify: multiplying $T^{\mu\nu}$ by *any* covariant tensor $S_{\rho\mu}$ generates a mixed tensor M^ν_ρ, but we adopt the very useful convention of keeping the name $T_\rho^{\ \nu}$ when multiplying by $S_{\rho\mu} = g_{\rho\mu}$, and thinking of the index as "being lowered." (And of course index-raising for multiplication by $g^{\rho\mu}$.)

Mixed tensors can "contract" to scalars. Start with $T^\mu_{\ \nu}$. Then consider the transformation of $T^\mu_{\ \mu}$:

$$T'^\mu_{\ \mu} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\mu}T^\nu_{\ \rho} = \delta^\rho_\nu T^\nu_{\ \rho} = T^\nu_{\ \nu} \tag{126}$$

i.e., $T^\mu_{\ \mu}$ is a scalar T. Exactly the same type of calculation shows that $T^{\mu\nu}_{\ \ \ \mu}$ is a vector T^ν, and so on. Remember to contract "up–down:" $T^\mu_{\ \mu} = T$, *not* $T^{\mu\mu} = T$.

The generalisation of the familiar scalar dot product between vectors A^μ and B^μ is $A^\mu B_\mu = g_{\mu\nu}A^\mu B^\nu$. We are often interested in just the spatial part of 4-vectors, the 3-vector A^i. Then, in a non-Euclidian 3-space, the local angle between two vectors may be written as the ratio

$$\cos\Delta\theta = \frac{A^i B_i}{(A^j A_j\ B^k B_k)^{1/2}} = \frac{g_{ij}A^i B^j}{(g_{kl}A^k A^l\ g_{mn}B^m B^n)^{1/2}} \tag{127}$$

the analogue of $\boldsymbol{A}\cdot\boldsymbol{B}/(|\boldsymbol{A}||\boldsymbol{B}|)$. If we are given two parameterised curves, perhaps two orbits $x^i(p)$ and $y^i(p)$, and wish to know the angle between them at some particular point, this angle becomes

$$\cos\Delta\theta = \frac{\dot{x}^i\dot{y}_i}{(\dot{x}^j\dot{x}_j\ \dot{y}^k\dot{y}_k)^{1/2}} = \frac{g_{ij}\dot{x}^i\dot{y}^j}{(g_{kl}\dot{x}^k\dot{x}^l\ g_{mn}\dot{y}^m\dot{y}^n)^{1/2}}$$

where the dot notation denotes d/dp. Do you see why this is so?

4.2 The covariant derivative

Recall the geodesic equation

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{128}$$

The left hand side has one free index component, and the right hand side surely is a vector: the trivial zero vector. Since this equation is valid in any coordinates, the left side needs to transform as a vector. What is interesting is that neither of the two terms by itself is a vector, yet somehow their sum transforms as a vector.

Rewrite the geodesic equation as follows. Denote $dx^\lambda/d\tau$, a true vector, as V^λ. Then

$$V^\mu\left[\frac{\partial V^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu}V^\nu\right] = 0 \tag{129}$$

Ah ha! Since the left side must be a vector, the stuff in square brackets must be a tensor: it is contracted with a vector V^μ to produce a vector—namely zero. The square brackets must contain a mixed tensor of rank two. Now, $\Gamma^\lambda_{\mu\nu}$ vanishes in locally free falling coordinates, in which we know that simple partial derivatives of vectors are indeed tensors. So this prescription tells us how to upgrade the notion of a partial derivative to the status of a tensor: to make a tensor out of a plain old partial derivative, form the quantity

$$\frac{\partial V^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu}V^\nu \equiv V^\lambda_{;\mu} \tag{130}$$

the so called *covariant derivative.* Following convention, we use a semicolon to denote covariant differentiation. (Some authors get tired of writing out ordinary partial derivatives and so use a comma for this (e.g $V^{\nu}_{\ ,\mu}$), but it is more clear to use full partial derivative notation, and we shall abide by this in these notes, if not always in lecture.) The covariant derivative is a true tensor, taking on a plain partial derivative form only in local freely falling coordinates. We therefore have our partial derivative generalisation to tensor form!

You know, this is really too important a result not to check in detail. Perhaps you think there is something special about the geodesic equation, or something special about our V^{λ}. In addition to this concern, we need to understand how to construct the covariant derivative of covariant vectors, and of more general tensors. (Talk about confusing. Notice the use of the word "covariant" twice in that last statement in two very different senses. Apologies for this awkward, but completely standard, mathematical nomenclature.) If you are already convinced that the covariant derivative really is a tensor, just skip down to right after equation (137). You won't learn anything more than you already know in the next long paragraph, and there is a lot of calculation.

The first thing we need to do is to establish the transformation law for $\Gamma^{\lambda}_{\mu\nu}$. This is just repeated application of the chain rule:

$$\Gamma'^{\lambda}_{\mu\nu} \equiv \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right) \tag{131}$$

Carrying through the derivative,

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}} + \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right) \tag{132}$$

Cleaning up, and recognising an affine connection when we see one, helps to rid us of these meddlesome ξ's:

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} \tag{133}$$

This may also be written

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\sigma} \partial x^{\rho}} \tag{134}$$

Do you see why? (Hint: Either integrate $\partial/\partial x'^{\mu}$ by parts or differentiate the identity

$$\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} = \delta^{\lambda}_{\nu}.)$$

Hence

$$\Gamma'^{\lambda}_{\mu\nu} V'^{\nu} = \left(\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} \right) \frac{\partial x'^{\nu}}{\partial x^{\eta}} V^{\eta}, \tag{135}$$

and spotting some tricky sums over $\partial x'^{\nu}$ that turn into Kronecker delta functions,

$$\Gamma'^{\lambda}_{\mu\nu} V'^{\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \Gamma^{\rho}_{\tau\sigma} V^{\sigma} - \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} V^{\rho} \tag{136}$$

Finally, adding this to (123), the unwanted terms cancel just as they should. We thus obtain

$$\frac{\partial V'^{\lambda}}{\partial x'^{\mu}} + \Gamma'^{\lambda}_{\mu\nu} V'^{\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \left(\frac{\partial V^{\nu}}{\partial x^{\rho}} + \Gamma^{\nu}_{\rho\sigma} V^{\sigma} \right), \tag{137}$$

as desired. This combination really does transform as a tensor ought to.

It is now a one-step process to deduce how covariant derivatives work for covariant vectors. Consider

$$V_\lambda V^\lambda_{;\mu} = V_\lambda \frac{\partial V^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} V^\nu V_\lambda \tag{138}$$

which is a perfectly good covariant vector. Integrating by parts the first term on the right, and then switching dummy indices λ and ν in the final term, this expression is identical to

$$\frac{\partial (V^\lambda V_\lambda)}{\partial x^\mu} - V^\lambda \left[\frac{\partial V_\lambda}{\partial x^\mu} - \Gamma^\nu_{\mu\lambda} V_\nu \right]. \tag{139}$$

Since the first term is the covariant gradient of a scalar (zero actually, because $V^\lambda V_\lambda = -c^2$ in local inertial coordinates and it's a scalar, so its always c^2), and the entire expression must be a good covariant vector, the term in square brackets must be a purely covariant tensor of rank two. We have very quickly found our generalisation for the covariant derivative of a covariant vector:

$$V_{\lambda;\mu} = \frac{\partial V_\lambda}{\partial x^\mu} - \Gamma^\nu_{\mu\lambda} V_\nu \tag{140}$$

That this really *is* a vector can also be directly verified via a calculation exactly similar to our previous one for the covariant derivative of a contravariant vector.

Covariant derivatives of *tensors* are now simple to deduce. The tensor $T^{\lambda\kappa}$ must formally transform like a contravariant *vector* if we "freeze" one of its indices at some particular component and allow the other to take on all component values. Since the formula must be symmetric in the two indices,

$$T^{\lambda\kappa}_{;\mu} = \frac{\partial T^{\lambda\kappa}}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} T^{\nu\kappa} + \Gamma^\kappa_{\nu\mu} T^{\lambda\nu} \tag{141}$$

and then it should also follow

$$T_{\lambda\kappa;\mu} = \frac{\partial T_{\lambda\kappa}}{\partial x^\mu} - \Gamma^\nu_{\lambda\mu} T_{\nu\kappa} - \Gamma^\nu_{\kappa\mu} T_{\lambda\nu} \tag{142}$$

and of course

$$T^\lambda_{\kappa;\mu} = \frac{\partial T^\lambda_\kappa}{\partial x^\mu} + \Gamma^\lambda_{\nu\mu} T^\nu_\kappa - \Gamma^\nu_{\mu\kappa} T^\lambda_\nu \tag{143}$$

The generalisation to tensors of arbitrary rank should now be self-evident. To generate the affine connection terms, freeze all indices in your tensor, then unfreeze them one-by-one, treating each unfrozen index as either a covariant or contravariant vector, depending upon whether it is down or up. Practise this until it is second-nature.

We now can give a precise rule for how to take an equation that is valid in special relativity, and upgrade it to the general relativistic theory of gravity. *Work exclusively with 4-vectors and 4-tensors. Replace $\eta_{\alpha\beta}$ with $g_{\mu\nu}$. Take ordinary derivatives and turn them into covariant derivatives. Voilà: your equation is set for the presence of gravitational fields.*

It will not have escaped your attention, I am sure, that applying (142) to $g_{\mu\nu}$ produces

$$g_{\mu\nu;\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - g_{\rho\nu}\Gamma^\rho_{\mu\lambda} - g_{\mu\rho}\Gamma^\rho_{\nu\lambda} = 0 \tag{144}$$

where equation (86) has been used for the last equality. *The covariant derivatives of $g_{\mu\nu}$ vanish.* This is exactly what we would have predicted, since the ordinary derivatives of $\eta_{\alpha\beta}$

vanish in special relativity, and thus the covariant derivative of $g_{\mu\nu}$ should vanish in the presence of gravitational fields. It's just the general relativistic upgrade of $\partial\eta_{\alpha\beta}/\partial x^\gamma = 0$.

Here are two important technical points that are easily shown. (You should do so explicitly.)

i.) The covariant derivative obeys the Leibniz rule for products. For example:

$$(T^{\mu\nu}U_{\lambda\kappa})_{;\rho} = T^{\mu\nu}_{;\rho}U_{\lambda\kappa} + T^{\mu\nu}U_{\lambda\kappa;\rho},$$

$$(V^\mu V_\mu)_{;\nu} = V^\mu (V_\mu)_{;\nu} + V_\mu (V^\mu)_{;\nu} = V^\mu \frac{\partial V^\mu}{\partial x^\nu} + V_\mu \frac{\partial V_\mu}{\partial x^\nu} \quad (\Gamma'\text{s cancel!})$$

ii.) The operation of contracting two tensor indices commutes with covariant differentiation. It does not matter which you do first. Check it out in the second example above.

4.3 The affine connection and basis vectors

The reader may be wondering how this all relates to our notions of, say, spherical or polar geometry and their associated sets of unit vectors and coordinates. The answer is: very simply. Our discussion will be straightforward and intuitive, rather than rigorous.

A vector $\boldsymbol{V}$ may be expanded in a set of basis vectors,

$$\boldsymbol{V} = V^a \boldsymbol{e}_a \tag{145}$$

where we sum over the repeated a, but a here on a **bold-faced** vector refers to a particular vector in the basis set. The V^a are the usual vector contravariant components: old friends, just numbers. Note that the sum is *not* a scalar formed from a contraction! We've used roman letters here to help avoid that pitfall.

The covariant components are associated with what mathematicians are pleased to call a *dual* basis:

$$\boldsymbol{V} = V_b \boldsymbol{e}^b \tag{146}$$

Same $\boldsymbol{V}$ mind you, just different ways of representing its components. If the $\boldsymbol{e}$'s seem a little abstract, don't worry, just take them at a formal level for the moment. You've seen something very like them before in elementary treatments of vectors.

The basis and the dual basis are related by a dot product rule,

$$\boldsymbol{e}_a \cdot \boldsymbol{e}^{\boldsymbol{b}} = \delta^b_a. \tag{147}$$

This dot product rule relates the vectors of orthonormal bases. The basis vectors transform just as good old vectors should:

$$\boldsymbol{e'}_a = \frac{\partial x^b}{\partial x'^a}\boldsymbol{e_b}, \quad \boldsymbol{e'}^a = \frac{\partial x'^a}{\partial x^b}\boldsymbol{e^b}. \tag{148}$$

Note that the dot product rule gives

$$\boldsymbol{V}\cdot\boldsymbol{V} = V^a V_b \boldsymbol{e}_a \cdot \boldsymbol{e^b} = V^a V_b \delta^b_a = V^a V_a, \tag{149}$$

as we would expect. On the other hand, expanding the differential line element $\boldsymbol{ds}$,

$$ds^2 = \boldsymbol{e_a} dx^a \cdot \boldsymbol{e_b} dx^b = \boldsymbol{e_a}\cdot\boldsymbol{e_b} dx^a dx^b \tag{150}$$

so that we recover the metric tensor

$$g_{ab} = \boldsymbol{e_a} \cdot \boldsymbol{e_b} \tag{151}$$

Exactly the same style calculation gives

$$g^{ab} = \boldsymbol{e^a} \cdot \boldsymbol{e^b} \tag{152}$$

These last two equations tell us first that g_{ab} is the coefficient of $\boldsymbol{e^a}$ in an expansion of the vector $\boldsymbol{e}_b$ in the usual basis:

$$\boldsymbol{e}_b = g_{ab}\boldsymbol{e}^a, \tag{153}$$

and tell us second that g^{ab} is the coefficient of $\boldsymbol{e}_a$ in an expansion of the vector $\boldsymbol{e}^b$ in the dual basis:

$$\boldsymbol{e}^b = g^{ab}\boldsymbol{e_a} \tag{154}$$

We've recovered the rules for raising and lowering indices, in this case for the entire basis vector.

Basis vectors change with coordinate position, as pretty much all vectors do in general. We *define* an thrice-indexed object object Γ^b_{ac} by

$$\frac{\partial \boldsymbol{e}_a}{\partial x^c} = \Gamma^b_{ac}\boldsymbol{e}_b \tag{155}$$

so that

$$\Gamma^b_{ac} = \boldsymbol{e}^b \cdot \partial_c \boldsymbol{e}_a \equiv \partial_c(\boldsymbol{e}_a \cdot \boldsymbol{e}^b) - \boldsymbol{e}_a \cdot \partial_c \boldsymbol{e}^b = -\boldsymbol{e}_a \cdot \partial_c \boldsymbol{e}^b. \tag{156}$$

(Remember the shorthand notation $\partial/\partial x^c = \partial_c$.) The last equality gives the expansion

$$\frac{\partial \boldsymbol{e}^b}{\partial x^c} = -\Gamma^b_{ac}\boldsymbol{e}^a \tag{157}$$

Consider $\partial_c g_{ab} = \partial_c(\boldsymbol{e}_a \cdot \boldsymbol{e}_b)$. Using (155),

$$\partial_c g_{ab} = (\partial_c \boldsymbol{e}_a) \cdot \boldsymbol{e}_b + \boldsymbol{e}_a \cdot (\partial_c \boldsymbol{e}_b) = \Gamma^d_{ac}\boldsymbol{e}_d \cdot \boldsymbol{e}_b + \boldsymbol{e}_a \cdot \Gamma^d_{bc}\boldsymbol{e}_d, \tag{158}$$

or finally

$$\partial_c g_{ab} = \Gamma^d_{ac} g_{db} + \Gamma^d_{bc} g_{ad}, \tag{159}$$

exactly what we found in (86)! This leads, in turn, precisely to (90), the equation for the affine connection in terms of the g partial derivatives. We now have a more intuitive understanding of what the Γ's really represent: they are expansion coefficients for the derivatives of basis vectors, which is how we are used to thinking of the extra acceleration terms in non Cartesian coordinates when we first encounter them in our first mechanics courses. In Cartesian coordinates, the Γ^b_{ac} just go away.

Finally, consider

$$\partial_a(V^b\boldsymbol{e}_b) = (\partial_a V^b)\boldsymbol{e}_b + V^b\partial_a\boldsymbol{e}_b = (\partial_a V^b)\boldsymbol{e}_b + V^b\Gamma^c_{ab}\boldsymbol{e}_c \tag{160}$$

Taking the dot product with $\boldsymbol{e}^d$:

$$\boldsymbol{e}^d \cdot \partial_a(V^b\boldsymbol{e}_b) = \partial_a V^d + V^b\Gamma^d_{ab} \equiv V^d_{;a}, \tag{161}$$

just the familiar covariant derivative of a contravariant vector. This one you should be able to do yourself:

$$\boldsymbol{e}_d\boldsymbol{\cdot}\partial_a(V_b\boldsymbol{e}^b) = \partial_a V_d - V_b\Gamma^b_{ad} \equiv V_{d;a}, \tag{162}$$

the covariant derivative of a covariant vector. This gives us some understanding as to why the true tensors formed from the partial derivatives of a vector $\boldsymbol{V}$ are not simply $\partial_a V^d$ and $\partial_a V_d$, but rather $\boldsymbol{e}^d\boldsymbol{\cdot}\partial_a(V^b\boldsymbol{e}_b)$ and $\boldsymbol{e}_d\boldsymbol{\cdot}\partial_a(V_b\boldsymbol{e}^b)$ respectively. Our terse and purely coordinate notation avoids the use of the $\boldsymbol{e}$ bases, but at a cost of missing a deeper and ultimately simplifying mathematical structure. We can see an old maxim of mathematicians in action: good mathematics starts with good definitions.

4.4 Volume element

The transformation of the metric tensor $g_{\mu\nu}$ may be thought of as a matrix equation:

$$g'_{\mu\nu} = \frac{\partial x^\kappa}{\partial x'^\mu} g_{\kappa\lambda} \frac{\partial x^\lambda}{\partial x'^\nu} \tag{163}$$

Remembering that the determinant of the product of matrices is the product of the determinants, we find

$$g' = \left| \frac{\partial x}{\partial x'} \right|^2 g \tag{164}$$

where g is the determinant of $g_{\mu\nu}$ (just the product of the diagonal terms for the diagonal metrics we will be using), and the notation $|\partial x'/\partial x|$ indicates the Jacobian of the transformation $x \to x'$. The significance of this result is that there is another quantity that also transforms with a Jacobian factor: the volume element d^4x.

$$d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x. \tag{165}$$

This means

$$\sqrt{-g'}\, d^4x' = \sqrt{-g} \left| \frac{\partial x}{\partial x'} \right| \left| \frac{\partial x'}{\partial x} \right| d^4x = \sqrt{-g}\, d^4x. \tag{166}$$

In other words, $\sqrt{-g}\, d^4x$ is the invariant volume element of curved spacetime. The minus sign is used merely as an absolute value to keep the quantities positive. In flat Minkowski space time, d^4x is invariant by itself.

Euclidian example: in going from Cartesian ($g = 1$) to cylindrical polar ($g = R^2$) to spherical coordinates ($g = r^4 \sin^2\theta$), we have $dx\, dy\, dz = RdR\, dz\, d\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$. You knew that. For a diagonal $g_{\mu\nu}$, our formula gives a volume element of

$$\sqrt{|g_{11}g_{22}g_{33}g_{00}|}dx^1\ dx^2\ dx^3\ dx^0,$$

just the product of the *proper* differential intervals. That also makes sense.

4.5 Covariant div, grad, curl, and all that

The ordinary partial derivative of a scalar transforms generally as covariant vector, so in this case there is no distinction between a covariant and standard partial derivative. Another easy result is

$$V_{\mu;\nu} - V_{\nu;\mu} = \frac{\partial V_\mu}{\partial x^\nu} - \frac{\partial V_\nu}{\partial x^\mu}. \tag{167}$$

(The affine connection terms are symmetric in the two lower indices, so they cancel.) More interesting is

$$V^{\mu}_{;\mu} = \frac{\partial V^{\mu}}{\partial x^{\mu}} + \Gamma^{\mu}_{\mu\lambda} V^{\lambda} \tag{168}$$

where by definition

$$\Gamma^{\mu}_{\mu\lambda} = \frac{g^{\mu\rho}}{2}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\lambda}} + \frac{\partial g_{\rho\lambda}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\rho}}\right) \tag{169}$$

Now, $g^{\mu\rho}$ is symmetric in its indices, whereas the last two g derivatives combined are anti-symmetric in the same indices, so that combination disappears entirely. We are left with

$$\Gamma^{\mu}_{\mu\lambda} = \frac{g^{\mu\rho}}{2}\frac{\partial g_{\rho\mu}}{\partial x^{\lambda}} \tag{170}$$

In this course, we will be dealing entirely with diagonal metric tensors, in which $\mu = \rho$ for nonvanishing entries, and $g^{\mu\rho}$ is the reciprocal of $g_{\mu\rho}$. In this simple case,

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{2}\frac{\partial \ln|g|}{\partial x^{\lambda}} \tag{171}$$

where g is as usual the determinant of $g_{\mu\nu}$, here just the product of the diagonal elements. Though our result seems specific to diagonal $g_{\mu\nu}$, W72 pp. 106-7, shows that this result is true for *any* $g_{\mu\nu}$.[2]

The covariant divergence (168) becomes

$$V^{\mu}_{;\mu} = \frac{1}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}V^{\mu})}{\partial x^{\mu}} \tag{172}$$

a neat and tidy result. Note that

$$\int \sqrt{|g|} d^4x \, V^{\mu}_{;\mu} = 0 \tag{173}$$

if V^{μ} vanishes sufficiently rapidly) at infinity. (Why?)

We cannot leave the covariant derivative without discussing $T^{\mu\nu}_{;\mu}$, the covariant divergence of $T^{\mu\nu}$. (And similarly for the divergence of $T^{\mu}_{\ \nu}$.) Conserved stress tensors are, after all, general relativity's "coin of the realm." We have:

$$T^{\mu\nu}_{;\mu} = \frac{\partial T^{\mu\nu}}{\partial x^{\mu}} + \Gamma^{\mu}_{\mu\lambda}T^{\lambda\nu} + \Gamma^{\nu}_{\mu\lambda}T^{\mu\lambda}, \text{ or } \quad T^{\mu}_{\ \nu;\mu} = \frac{\partial T^{\mu}_{\ \nu}}{\partial x^{\mu}} + \Gamma^{\mu}_{\mu\lambda}T^{\lambda}_{\ \nu} - \Gamma^{\lambda}_{\mu\nu}T^{\mu}_{\ \lambda} \tag{174}$$

and using (171), we may condense this to

$$T^{\mu\nu}_{;\mu} = \frac{1}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}T^{\mu\nu})}{\partial x^{\mu}} + \Gamma^{\nu}_{\mu\lambda}T^{\mu\lambda}, \quad \text{or} \quad T^{\mu}_{\ \nu;\mu} = \frac{1}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}T^{\mu}_{\ \nu})}{\partial x^{\mu}} - \Gamma^{\lambda}_{\mu\nu}T^{\mu}_{\ \lambda}. \tag{175}$$

For an antisymmetric *contravariant* tensor, call it $A^{\mu\nu}$, the last term of the first equality drops out because Γ is symmetric in its lower indices:

$$A^{\mu\nu}_{;\mu} = \frac{1}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}A^{\mu\nu})}{\partial x^{\mu}} \quad \text{if } A^{\mu\nu} \text{ antisymmetric.} \tag{176}$$

[2]Sketchy proof for the mathematically inclined: For matrix M, trace Tr, differential δ, to first order in δ we have $\delta \ln\det M = \ln\det(M+\delta M) - \ln\det M = \ln\det M^{-1}(M+\delta M) = \ln\det(1+M^{-1}\delta M) = \ln(1+\mathrm{Tr}\, M^{-1}\delta M) = \mathrm{Tr}\, M^{-1}\delta M$. Can you supply the missing details?

4.6 Hydrostatic equilibrium

You have been patient and waded through a sea of indices, and it is time to be rewarded. We will do our first real physics problem in general relativity: hydrostatic equilibrium.

In Newtonian mechanics, you will recall that hydrostatic equilibrium represents a balance between a pressure gradient and the force of gravity. In general relativity this is completely encapsulated in the condition

$$T^{\mu\nu}_{;\mu} = 0$$

applied to the energy-momentum stress tensor (65), upgraded to covariant status:

$$T^{\mu\nu} = Pg^{\mu\nu} + (\rho + P/c^2)U^\mu U^\nu \tag{177}$$

Our conservation equation is

$$0 = T^{\mu\nu}_{;\mu} = g^{\mu\nu}\frac{\partial P}{\partial x^\mu} + \left[(\rho + P/c^2)U^\mu U^\nu\right]_{;\mu} \tag{178}$$

where we have made use of the Leibniz rule for the covariant derivative of a product, and the fact that the $g_{\mu\nu}$ covariant derivative vanishes. Using (175):

$$0 = g^{\mu\nu}\frac{\partial P}{\partial x^\mu} + \frac{1}{|g|^{1/2}}\frac{\partial}{\partial x^\mu}\left[|g|^{1/2}(\rho + P/c^2)U^\mu U^\nu\right] + \Gamma^\nu_{\mu\lambda}(\rho + P/c^2)U^\mu U^\lambda \tag{179}$$

In static equilibrium, all the U components vanish except U^0. To determine this, we use

$$g_{\mu\nu}U^\mu U^\nu = -c^2 \tag{180}$$

the upgraded version of special relativity's $\eta_{\alpha\beta}U^\alpha U^\beta = -c^2$. Thus,

$$(U^0)^2 = -\frac{c^2}{g_{00}}, \tag{181}$$

and with

$$\Gamma^\nu_{00} = -\frac{g^{\mu\nu}}{2}\frac{\partial g_{00}}{\partial x^\mu}, \tag{182}$$

our equation reduces to

$$0 = g^{\mu\nu}\left[\frac{\partial P}{\partial x^\mu} + \left(\rho c^2 + P\right)\frac{\partial \ln |g_{00}|^{1/2}}{\partial x^\mu}\right] \tag{183}$$

Since $g_{\mu\nu}$ has a perfectly good inverse, the term in square brackets must be zero:

$$\frac{\partial P}{\partial x^\mu} + \left(\rho c^2 + P\right)\frac{\partial \ln |g_{00}|^{1/2}}{\partial x^\mu} = 0 \tag{184}$$

This is the general relativistic equation of hydrostatic equilibrium. Compare this with the Newtonian counterpart:

$$\boldsymbol{\nabla} P + \rho \boldsymbol{\nabla}\Phi = 0 \tag{185}$$

The difference for a static problem is the replacement of ρ by $\rho + P/c^2$ for the inertial mass density, and the use of $\ln |g_{00}|^{1/2}$ for the potential (to which it reduces in the Newtonian limit).

If $P = P(\rho)$, $P' \equiv dP/d\rho$, equation (184) may be formally integrated:

$$\int \frac{P'(\rho)\, d\rho}{P(\rho) + \rho c^2} + \ln |g_{00}|^{1/2} = \text{constant.} \tag{186}$$

Exercise. Solve the GR equation of hydrostatic equilibrium exactly for the case $|g_{00}| = (1 - 2GM/rc^2)^{1/2}$ (e.g., near the surface of a neutron star) and $P = K\rho^\gamma$ for $\gamma \geq 1$.

4.7 Covariant differentiation and parallel transport

In this section, we view covariant differentiation in a different light. We make no new technical developments, rather we understand the content of the geodesic equation in a different way. Start with a by now old friend,

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{187}$$

Writing $dx^\lambda/d\tau$ as the vector it is, V^λ, to help our thinking a bit,

$$\frac{dV^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}V^\nu = 0, \tag{188}$$

a covariant formulation of the statement that the vector V^λ is conserved along a geodesic path. But the covariance property of this statement has nothing to do with the specific identity of V^λ with $dx^\lambda/d\tau$. The full left-side of this equation is a genuine vector for *any* V^λ as long as V^λ itself is a bona fide contravariant vector. The right side simply tells us that the fully covariant left side expression is zero. (In our particular example, because momentum is conserved.) Therefore, just as we "upgrade" from special to general relativity the partial derivative,

$$\frac{\partial V^\alpha}{\partial x^\beta} \to \frac{\partial V^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu}V^\nu \equiv V^\lambda_{;\mu} \tag{189}$$

we upgrade the derivative along a path $x(\tau)$ in the same way by multiplying by $dx^\mu/d\tau$ and summing over the index μ:

$$\frac{dV^\alpha}{d\tau} \to \frac{dV^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}V^\nu \equiv \frac{DV^\lambda}{D\tau} \tag{190}$$

$DV^\lambda/D\tau$ is a true vector; the transformation

$$\frac{DV'^\lambda}{D\tau} = \frac{\partial x'^\lambda}{\partial x^\mu}\frac{DV^\mu}{D\tau} \tag{191}$$

may be verified directly. (The inhomogeneous contributions from the Γ transformation and the derivatives of the derivatives of the coordinate transformation coefficients cancel in a manner exactly analogous to our original covariant partial derviative calculation.)

Exactly the same reasoning is used to define the covariant derivative for a covariant vector,

$$\frac{dV_\lambda}{d\tau} - \Gamma^\nu_{\mu\lambda}\frac{dx^\mu}{d\tau}V_\nu \equiv \frac{DV_\lambda}{D\tau}. \tag{192}$$

and for tensors, e.g.:

$$\frac{dT^{\sigma}_{\lambda}}{d\tau} + \Gamma^{\sigma}_{\mu\nu}\frac{dx^{\nu}}{d\tau}T^{\mu}_{\lambda} - \Gamma^{\mu}_{\lambda\nu}\frac{dx^{\nu}}{d\tau}T^{\sigma}_{\mu} \equiv \frac{DT^{\sigma}_{\lambda}}{D\tau}. \tag{193}$$

When a vector or tensor quantity is carried along a path that does not change in a locally inertially reference frame ($d/d\tau = 0$), this statment becomes in arbitrary coordinates $D/D\tau = 0$, the same physical result expressed in a covariant language. (Once again this works because of identical agreement in the inertial coordinates, and then zero is zero in any coordinate frame.) The condition $D/D\tau = 0$ is known as *parallel transport.* A steady vector, for example, may always point along the y axis as we move it around in the xy plane, but its r and θ components *will* have to change in order to keep this true! How those components change is the content of the parallel transport equation.

Now, if we do a round trip and come back to our exact starting point, does a vector have to have the same value it began with? You might think that the answer must be yes, but it turns out to be more complicated than that. Indeed, it is a most interesting question...

The stage is now set to introduce the key tensor embodying the gravitational distortion of spacetime.

5 The curvature tensor

The properties which distinguish space from other conceivable triply-extended magnitudes are only to be deduced from experience...At every point the three-directional measure of curvature can have an arbitrary value if only the effective curvature of every measurable region of space does not differ noticeably from zero.

— G. F. B. Riemann

5.1 Commutation rule for covariant derivatives

The covariant derivative shares many properties with the ordinary partial derivative: it is a linear operator, it obeys the Leibniz rule, and it allows true tensor status to be bestowed upon partial derivatives under any coordinate transformation. A natural question arises. Ordinary partial derivatives commute: the order in which they are taken does not matter, provided suitable smoothness conditions are present. Is the same true of covariant derivatives? Does $V^{\mu}_{;\sigma;\tau}$ equal $V^{\mu}_{;\tau;\sigma}$?

Just do it.

$$V^{\mu}_{;\sigma} = \frac{\partial V^{\mu}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\nu\sigma} V^{\nu} \equiv T^{\mu}_{\sigma} \tag{194}$$

Then

$$T^{\mu}_{\sigma;\tau} = \frac{\partial T^{\mu}_{\sigma}}{\partial x^{\tau}} + \Gamma^{\mu}_{\nu\tau} T^{\nu}_{\sigma} - \Gamma^{\nu}_{\sigma\tau} T^{\mu}_{\nu}, \tag{195}$$

or

$$T^{\mu}_{\sigma;\tau} = \frac{\partial^2 V^{\mu}}{\partial x^{\tau} \partial x^{\sigma}} + \frac{\partial}{\partial x^{\tau}} \left(\Gamma^{\mu}_{\lambda\sigma} V^{\lambda} \right) + \Gamma^{\mu}_{\nu\tau} \left(\frac{\partial V^{\nu}}{\partial x^{\sigma}} + \Gamma^{\nu}_{\lambda\sigma} V^{\lambda} \right) - \Gamma^{\nu}_{\sigma\tau} \left(\frac{\partial V^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\lambda\nu} V^{\lambda} \right) \tag{196}$$

The first term and the last group (proportional to $\Gamma^{\nu}_{\sigma\tau}$) are manifestly symmetric in σ and τ, and so will vanish when the same calculation is done with the indices reversed and then subtracted off. A bit of inspection shows that the same is true for all the remaining terms proportional to the partial derivatives of V^{μ}. The residual terms from taking the covariant derivative commutator are

$$T^{\mu}_{\sigma;\tau} - T^{\mu}_{\tau;\sigma} = \left[\frac{\partial \Gamma^{\mu}_{\lambda\sigma}}{\partial x^{\tau}} - \frac{\partial \Gamma^{\mu}_{\lambda\tau}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\nu\tau} \Gamma^{\nu}_{\lambda\sigma} - \Gamma^{\mu}_{\nu\sigma} \Gamma^{\nu}_{\lambda\tau} \right] V^{\lambda}, \tag{197}$$

which we may write as

$$T^{\mu}_{\sigma;\tau} - T^{\mu}_{\tau;\sigma} = R^{\mu}{}_{\lambda\sigma\tau} V^{\lambda} \tag{198}$$

Now the right side of this equation must be a tensor, and V^{λ} is an arbitrary vector, which means that $R^{\mu}_{\lambda\sigma\tau}$ needs to transform its coordinates as a tensor. That it does so may also be verified explicitly in a nasty calculation (if you want to see it spelt out in detail, see W72 pp.132-3). We conclude that

$$R^{\mu}{}_{\lambda\sigma\tau} = \frac{\partial \Gamma^{\mu}_{\lambda\sigma}}{\partial x^{\tau}} - \frac{\partial \Gamma^{\mu}_{\lambda\tau}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\nu\tau}\Gamma^{\nu}_{\lambda\sigma} - \Gamma^{\mu}_{\nu\sigma}\Gamma^{\nu}_{\lambda\tau} \tag{199}$$

is indeed a true tensor, and it is called the *curvature tensor.* In fact, it may be shown (W72 p. 134) that this is the only tensor that is *linear* in the second derivatives of $g_{\mu\nu}$ and contains only its first and second derivatives.

Why do we refer to this mixed tensor as the "curvature tensor?" Well, we begin to answer this by noting that it vanishes in ordinary flat Minkowski spacetime—we simply choose Cartesian coordinates to do our calculation. Then, because $R^{\mu}{}_{\lambda\sigma\tau}$ is a tensor, if it is zero in one set of coordinates, it is zero in all. Commuting covariant derivatives makes sense in this case, since they amount to ordinary derivatives. So distortions from Minkowski space are essential.

Exercise. What is the (much simpler) form of $R^{\mu}{}_{\lambda\sigma\tau}$ in local inertial coordinates? It is often convenient to work in such coordinates to prove a result, and then generalise it to arbitrary coordinates using the the fact that $R^{\mu}{}_{\lambda\sigma\tau}$ is a tensor.

5.2 Parallel transport

Our intuition sharpens with the yet more striking example of parallel transport. Consider a vector V_λ whose covariant derivative along a curve $x(\tau)$ vanishes. Then,

$$\frac{dV_\lambda}{d\tau} = \Gamma^{\mu}_{\lambda\nu}\frac{dx^\nu}{d\tau}V_\mu \tag{200}$$

Consider next a tiny round trip journey over a closed path in which V_λ is changing by the above prescription. If we remain in the neighbourhood of some point X^ρ, with x^ρ passing through X^ρ at some instant τ_0, $x^\rho(\tau_0) = X^\rho$, we Taylor expand as follows:

$$\Gamma^{\mu}_{\lambda\nu}(x) = \Gamma^{\mu}_{\lambda\nu}(X) + (x^\rho - X^\rho)\frac{\partial \Gamma^{\mu}_{\lambda\nu}}{\partial X^\rho} + ... \tag{201}$$

$$V_\mu[x(\tau)] = V_\mu(X) + dV_\mu + ... = V_\mu(X) + (x^\rho - X^\rho)\Gamma^{\sigma}_{\mu\rho}(X)V_\sigma(X) + ... \tag{202}$$

(where $x^\rho - X^\rho$ is dx^ρ from the parallel transport equation), whence

$$\Gamma^{\mu}_{\lambda\nu}(x)V_\mu(x) = \Gamma^{\mu}_{\lambda\nu}V_\mu + (x^\rho - X^\rho)V_\sigma\left(\frac{\partial \Gamma^{\sigma}_{\lambda\nu}}{\partial X^\rho} + \Gamma^{\sigma}_{\mu\rho}\Gamma^{\mu}_{\lambda\nu}\right) + ... \tag{203}$$

where all quantities on the right (except x!) are evaluated at X. Integrating

$$dV_\lambda = \Gamma^{\mu}_{\lambda\nu}(x)V_\mu(x)\,dx^\nu \tag{204}$$

around a tiny closed path $\oint$, and using (204) and (203), we find that there is a change in the starting value ΔV_λ arising from the term linear in x^ρ given by

$$\Delta V_\lambda = \left(\frac{\partial \Gamma^{\sigma}_{\lambda\nu}}{\partial X^\rho} + \Gamma^{\sigma}_{\mu\rho}\Gamma^{\mu}_{\lambda\nu}\right)V_\sigma \oint x^\rho\,dx^\nu \tag{205}$$

The integral $\oint x^\rho dx^\nu$ certainly doesn't vanish. (Try integrating it around a unit square in the xy plane.) But it *is* antisymmetric in ρ and ν. (Integrate by parts and note that the integrated term vanishes, being an exact differential.) That means the part of the $\Gamma\Gamma$ term that survives the $\rho\nu$ summation is the part that is antisymmetric in (ρ, ν). Since any object depending on two indices, say $A(\rho, \nu)$, can be written as a symmetric part plus an antisymmetric part,

$$\frac{1}{2}[A(\rho,\nu) + A(\nu,\rho)] + \frac{1}{2}[A(\rho,\nu) - A(\nu,\rho)],$$

we find

$$\Delta V_\lambda = \frac{1}{2} R^\sigma{}_{\lambda\nu\rho} V_\sigma \oint x^\rho \, dx^\nu \tag{206}$$

where

$$R^\sigma{}_{\lambda\nu\rho} = \left(\frac{\partial \Gamma^\sigma_{\lambda\nu}}{\partial X^\rho} - \frac{\partial \Gamma^\sigma_{\lambda\rho}}{\partial X^\nu} + \Gamma^\sigma_{\mu\rho}\Gamma^\mu_{\lambda\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\mu_{\lambda\rho} \right) \tag{207}$$

is precisely the curvature tensor. Parallel transport of a vector around a closed curve does not change the vector, unless the enclosed area has a nonvanishing curvature tensor. In fact, "the enclosed area" can be given a more intuitive if we think of integrating around a very tiny square in the $\rho\nu$ hyperplane. Then the closed loop integral is just the directed area $dx^\rho dx^\nu$:

$$\Delta V_\lambda = \frac{1}{2} R^\sigma{}_{\lambda\nu\rho} V_\sigma dx^\rho \, dx^\nu. \tag{208}$$

The conversion of a tiny closed loop integral to an enclosed surface area element reminds us a bit of Stokes theorem, and it will not be surprising to see that there is an analogy here to the identity "divergence of curl equals zero". We will see this shortly.

Exercise. **A laboratory demonstration.** Take a pencil and move it round the surface of a flat desktop without rotating the pencil. Moving the pencil around a closed path, *always parallel to itself,* will not change its orientation. Now do the same on the surface of a spherical globe. Take a small pencil, pointed poleward, and move it from the equator along the 0° meridian through Greenwich till you hit the north pole. Now, once again parallel to itself, move the pencil down the 90°E meridian till you come to the equator. Finally, once again parallel to itself, slide the pencil along the equator to return to the starting point at the prime meridian.

Has the pencil orientation changed from its initial one? Explain.

Curvature[3], or more precisely the departure of spacetime from Minkowski structure, reveals itself through the existence of the curvature tensor $R^\sigma{}_{\lambda\nu\rho}$. If spacetime is Minkowski-flat, every component of the curvature tensor vanishes. An important consequence is that parallel transport around a closed loop can result in a vector or tensor not returning to its orginal value, if the closed loop encompasses matter (or its energy equivalent). An experiment was proposed in the 1960's to measure the precession of a gyroscope orbiting the earth due to the effects of the spacetime curvature tensor. This eventually evolved into a satellite known as Gravity Probe B, a $750,000,000 mission, launched in 2004. Alas, it was plagued by technical problems for many years, and its results were controversial because of unexpectedly high noise levels (solar activity). A final publication of science results in 2011 claims to have verified the predictions of general relativity to high accuracy, including an even smaller effect known as "frame dragging" from the earth's rotation, but my sense is

[3] "Curvature" is one of these somewhat misleading mathematical labels that has stuck, like "imaginary" numbers. The name implies an external dimension into which the space is curved or embedded, an unnecessary complication. The space is simply *distorted.*

that there is lingering uneasiness in the physics community regarding the handling of the noise. Do an internet search on Gravity Probe B and judge for yourself!

When GPB was first proposed in the early 1960's, tests of general relativity were very few and far between. Any potentially observable result was novel and worth exploring. Since that time, experimental GR has evolved *tremendously*. We live in a world of gravitational lensing, exquisitely sensitive Shapiro time delays, and beautiful confirmations of gravitational radiation, first via the binary pulsar system PSR1913+16, and now the recent direct signal detection of GW150914 via advanced LIGO. All of these will be discussed in later chapters. At this point it borders on the ludicrous to entertain serious doubt that the crudest leading order general relativity parallel transport prediction is correct. (In fact, it looks like we have seen this effect directly in close binary pulsar systems.) Elaborately engineered artificial gyroscopes, precessing by teeny-tiny amounts in earth orbit don't seem very exciting any more to 21st century physicists.

5.3 Algebraic identities of $R^{\sigma}{}_{\nu\lambda\rho}$

5.3.1 Remembering the curvature tensor formula.

It is helpful to have a mnemonic for generating the curvature tensor. The hard part is keeping track of the indices. Remember that the tensor itself is just a sum of derivatives of Γ, and quadratic products of Γ. That part is easy to remember, since the curvature tensor has "dimensions" of $1/x^2$, where x represents a coordinate. To remember the coordinate juggling of $R^{a}{}_{bcd}$ start with:

$$\frac{\partial \Gamma^{a}_{bc}}{\partial x^{d}} + \Gamma^{*}_{bc}\Gamma^{a}_{d*}$$

where the first *abcd* ordering is simple to remember since it follows the same placement in $R^{a}{}_{bcd}$, and $*$ is a dummy variable. For the second $\Gamma\Gamma$ term, remember to just write out the lower *bcd* indices straight across, making the last unfilled space a dummy index $*$. The counterpart dummy index that is summed over must then be the upper slot on the other Γ, since there is no self-contracted Γ in the full curvature tensor. There is then only one place left for upper a. To finish off, just subtract the same thing with c and d reversed. Think of it as swapping your CD's. We arrive at:

$$R^{a}{}_{bcd} = \frac{\partial \Gamma^{a}_{bc}}{\partial x^{d}} - \frac{\partial \Gamma^{a}_{bd}}{\partial x^{c}} + \Gamma^{*}_{bc}\Gamma^{a}_{d*} - \Gamma^{*}_{bd}\Gamma^{a}_{c*} \tag{209}$$

5.4 $R_{\lambda\mu\nu\kappa}$: fully covariant form

The fully covariant form of the stress tensor can be written so that it involves only second-order derivatives of $g_{\mu\nu}$ and products of Γs, with no Γ partial derivatives. The second-order g-derivatives, which are linear terms, will be our point of contact with Newtonian theory from the full field equations. But hang on, we have a bit of heavy weather ahead.

We define

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\sigma} R^{\sigma}{}_{\mu\nu\kappa} \tag{210}$$

or

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\sigma}\left[\frac{\partial \Gamma^{\sigma}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\sigma}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\sigma}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\sigma}_{\nu\eta}\right] \tag{211}$$

Remembering the definition of the affine connection (90), the right side of (211) is

$$\frac{g_{\lambda\sigma}}{2}\frac{\partial}{\partial x^\kappa}\left[g^{\sigma\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\nu}+\frac{\partial g_{\rho\nu}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\rho}\right)\right]-\frac{g_{\lambda\sigma}}{2}\frac{\partial}{\partial x^\nu}\left[g^{\sigma\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\kappa}+\frac{\partial g_{\rho\kappa}}{\partial x^\mu}-\frac{\partial g_{\mu\kappa}}{\partial x^\rho}\right)\right]$$

$$+\, g_{\lambda\sigma}\left(\Gamma^\mu_{\nu\lambda}\Gamma^\sigma_{\kappa\eta}-\Gamma^\eta_{\mu\kappa}\Gamma^\sigma_{\nu\eta}\right) \tag{212}$$

The x^κ and x^ν partial derivatives will operate on the $g^{\sigma\rho}$ term and the g-derivative terms. Let us begin with the second group, the $\partial g/\partial x$ derivatives, as it is simpler. With $g_{\lambda\sigma}g^{\sigma\rho}=\delta^\lambda_\rho$, the terms that are linear in the second order g derivatives are

$$\frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa\partial x^\mu}-\frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa\partial x^\lambda}-\frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu\partial x^\mu}+\frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu\partial x^\lambda}\right) \tag{213}$$

If you can sense the beginnings of the classical wave equation lurking in these linear second order derivatives, the leading terms when $g_{\mu\nu}$ departs only a little from $\eta_{\mu\nu}$, then you are very much on the right track.

We are not done of course. We have the terms proportional to the κ and ν derivatives of $g^{\sigma\rho}$, which certainly do not vanish in general. But the covariant derivative of the metric tensor $g_{\lambda\sigma}$ *does* vanish, so invoke this sleight-of-hand integration by parts:

$$g_{\lambda\sigma}\frac{\partial g^{\sigma\rho}}{\partial x^\kappa}=-g^{\sigma\rho}\frac{\partial g_{\lambda\sigma}}{\partial x^\kappa}=-g^{\sigma\rho}\left(\Gamma^\eta_{\kappa\lambda}g_{\eta\sigma}+\Gamma^\eta_{\kappa\sigma}g_{\eta\lambda}\right) \tag{214}$$

where in the final equality, equation (142) has been used. By bringing $g^{\sigma\rho}$ out from the partial derivative, it recombines to form affine connections once again. All the remaining terms of $R_{\lambda\mu\nu\kappa}$ from (212) are now of the form $g\Gamma\Gamma$:

$$-\left(\Gamma^\eta_{\kappa\lambda}g_{\eta\sigma}+\cancel{\Gamma^\eta_{\kappa\sigma}g_{\eta\lambda}}\right)\Gamma^\sigma_{\mu\nu}+\left(\Gamma^\eta_{\nu\lambda}g_{\eta\sigma}+\cancel{\Gamma^\eta_{\nu\sigma}g_{\eta\lambda}}\right)\Gamma^\sigma_{\mu\kappa}+g_{\lambda\sigma}\left(\cancel{\Gamma^\eta_{\mu\nu}\Gamma^\sigma_{\kappa\eta}}-\cancel{\Gamma^\eta_{\mu\kappa}\Gamma^\sigma_{\nu\eta}}\right), \tag{215}$$

It is not obvious at first, but with a little colour coding and index agility to help, you should be able to see four of these six $g\Gamma\Gamma$ terms cancel out— the second group with the fifth, the fourth group with the sixth—leaving only the first and third terms:

$$g_{\eta\sigma}\left(\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa}-\Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right) \tag{216}$$

Adding together the terms in (213) and (216), we arrive at

$$R_{\lambda\mu\nu\kappa}=\frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa\partial x^\mu}-\frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa\partial x^\lambda}-\frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu\partial x^\mu}+\frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu\partial x^\lambda}\right)+g_{\eta\sigma}\left(\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa}-\Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right) \tag{217}$$

Exercise. What is $R_{\lambda\mu\nu\kappa}$ in local inertial coordinates?

Note the following important symmetry properties for the indices of $R_{\lambda\mu\nu\kappa}$. Because they may be expressed as vanishing tensor equations, they may be established in any coordinate frame, so we choose a local frame in which the Γ vanish. They are then easily verified from the terms linear in the g derivatives in (217):

$$R_{\lambda\mu\nu\kappa}=R_{\nu\kappa\lambda\mu}\qquad\text{(symmetry)} \tag{218}$$

$$R_{\lambda\mu\nu\kappa}=-R_{\mu\lambda\nu\kappa}=-R_{\lambda\mu\kappa\nu}=R_{\mu\lambda\kappa\nu}\qquad\text{(antisymmetry)} \tag{219}$$

$$R_{\lambda\mu\nu\kappa}+R_{\lambda\kappa\mu\nu}+R_{\lambda\nu\kappa\mu}=0\qquad\text{(cyclic)} \tag{220}$$

5.5 The Ricci Tensor

The Ricci tensor is the curvature tensor contracted on its (raised) first and third indices, ${R^a}_{bad}$. In terms of the covariant curvature tensor:

$$R_{\mu\kappa} = g^{\lambda\nu} R_{\lambda\mu\nu\kappa} = g^{\lambda\nu} R_{\nu\kappa\lambda\mu} \text{ (by symmetry) } = g^{\nu\lambda} R_{\nu\kappa\lambda\mu} = R_{\kappa\mu} \tag{221}$$

so that the Ricci tensor is symmetric.

The Ricci tensor is an extremely important tensor in general relativity. Indeed, we shall very soon see that $R_{\mu\nu} = 0$ is Einstein's Laplace equation. There is enough information here to calculate the deflection of light by a gravitating body or the advance of a planet's orbital perihelion! What is tricky is to guess the general relativistic version of the Poisson equation, and no, it is *not* $R_{\mu\nu}$ proportional to the stress energy tensor $T_{\mu\nu}$. (It wouldn't be very tricky then, would it?) Notice that while ${R^\lambda}_{\mu\nu\kappa} = 0$ implies that the Ricci tensor vanishes, the converse does not follow: $R_{\mu\nu} = 0$ does not necessarily mean that the full curvature tensor (covariant or otherwise) vanishes.

Exercise. **Fun with the Ricci tensor.** Prove first that

$$R_{\mu\kappa} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\lambda\eta}.$$

Next show that

$$R_{\mu\kappa} = -g^{\lambda\nu} R_{\mu\lambda\nu\kappa} = -g^{\lambda\nu} R_{\lambda\mu\kappa\nu} = g^{\lambda\nu} R_{\mu\lambda\kappa\nu},$$

and that $g^{\lambda\mu} R_{\lambda\mu\nu\kappa} = g^{\nu\kappa} R_{\lambda\mu\nu\kappa} = 0$. Why does this mean that $R_{\mu\kappa}$ is the only second rank covariant tensor that can be formed from contracting $R_{\lambda\mu\nu\kappa}$?

We are not quite through contracting. We may form the curvature scalar

$$R \equiv R^\mu_\mu \tag{222}$$

another very important quantity in general relativity.

Exercise. **The curvature scalar is unique.** Prove that

$$R = g^{\nu\lambda} g^{\mu\kappa} R_{\lambda\mu\nu\kappa} = -g^{\nu\lambda} g^{\mu\kappa} R_{\mu\lambda\nu\kappa}$$

and that

$$g^{\lambda\mu} g^{\nu\kappa} R_{\lambda\mu\nu\kappa} = 0.$$

Justify the title of this exercise.

5.6 The Bianchi Identities

The covariant curvature tensor obeys a very important differential identity, analogous to **div(curl)=0.** These are the Bianchi identities. We prove the Bianchi identities in our favourite freely falling inertial coordinates with $\Gamma = 0$, and since we will be showing that a tensor is zero in these coordinates, it is zero in all coordinates. In $\Gamma = 0$ coordinates,

$$R_{\lambda\mu\nu\kappa;\eta} = \frac{1}{2}\frac{\partial}{\partial x^\eta}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right) \tag{223}$$

The Bianchi identities follow from cycling κ goes to ν, ν goes to η, η goes to κ. Leave λ and μ alone. Repeat. Add the original $R_{\lambda\mu\nu\kappa;\eta}$ and the two cycled expressions together. You will find that this gives

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0 \tag{224}$$

An easy way to check the bookkeeping on this is just to pay attention to the g's: once you've picked a particular value of $\partial^2 g_{ab}$ in the numerator, the other ∂x^c indices downstairs are unambiguous, since as coordinate derivatives their order is immaterial. The first term in (224) is then just shown: $(g_{\lambda\nu}, -g_{\mu\nu}, -g_{\lambda\kappa}, g_{\mu\kappa})$. Cycle to get the second group for the second Bianchi term, $(g_{\lambda\eta}, -g_{\mu\eta}, -g_{\lambda\nu}, g_{\mu\nu})$. The final term then is $(g_{\lambda\kappa}, -g_{\mu\kappa}, -g_{\lambda\eta}, g_{\mu\eta})$. Look: every g has its opposite when you add these all up, so the sum is clearly zero.

We would like to get equation (224) into the form of a *single* vanishing covariant tensor divergence, for reasons that will soon become very clear. Toward this goal, contract λ with ν, remembering the symmetries in (219). (E.g.: in the second term on the left side of [224], swap ν and η before contracting, changing the sign.) We find,

$$R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}{}_{\mu\kappa\eta;\nu} = 0 \tag{225}$$

Next, contract μ with κ:

$$R_{;\eta} - R^{\mu}{}_{\eta;\mu} - R^{\nu}{}_{\eta;\nu} = 0 \tag{226}$$

(Did you understand the manipulations to get that final term on the left? First set things up with:

$$R^{\nu}{}_{\mu\kappa\eta;\nu} = g^{\nu\sigma} R_{\sigma\mu\kappa\eta;\nu} = -g^{\nu\sigma} R_{\mu\sigma\kappa\eta;\nu}$$

Now it is easy to raise μ and contract with κ:

$$-g^{\nu\sigma} R^{\mu}{}_{\sigma\mu\eta;\nu} = -g^{\nu\sigma} R_{\sigma\eta;\nu} = -R^{\nu}{}_{\eta;\nu})$$

Cleaning things up, our contracted identity (226) becomes:

$$(\delta^{\mu}_{\eta} R - 2R^{\mu}{}_{\eta})_{;\mu} = 0. \tag{227}$$

Raising η (we are allowed, of course, to bring $g^{\nu\eta}$ *inside* the covariant derivative to do this—why?), and dividing by -2 puts this identity into its classic "zero-divergence" form:

$$\left(R^{\mu\nu} - g^{\mu\nu} \frac{R}{2} \right)_{;\mu} = 0 \tag{228}$$

The generic tensor combination $A^{\mu\nu} - g^{\mu\nu}A/2$ will appear repeatedly in our study of gravitational radiation.

Einstein did not know equation (228) when he was struggling mightily with his theory, but to be fair neither did most mathematicians! The identities were actually first discovered by the German mathematician A. Voss in 1880, then independently in 1889 by Ricci. These results were then quickly forgotten, even, it seems, by Ricci himself. Bianchi then rediscovered them on his own in 1902, but they were still not widely known in the mathematics community in 1915. This was a pity, because the Bianchi identities have been called the "royal road to the Gravitational Field Equations " by Einstein's biographer A. Pais. It seems to have been the mathematician H. Weyl who in 1917 first recognised the importance of the Bianchi identitites for relativity, but the particular derivation we have followed was not formulated until 1922, by Harward.

The reason for the identities' importance is precisely analogous to Maxwell's understanding of the restrictions that the curl operator imposes on the field it generates, and to why the displacement current needs to be added to the equation $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$. Taking the divergence of this equation gives zero identically on the left—the divergence of the curl is zero—so the right hand source term must also have a vanishing divergence. In other words, it must become a statement of some sort of *physical* conservation law. Maxwell needed and invoked a physical "displacement current," $(1/c^2)\partial \boldsymbol{E}/\partial t$, and added it to the right side of the equation. The ensuing physical conservation law corresponded to the conservation of electric charge, now built into the fundamental formulation of Maxwell's Equations. Here, we shall use the Bianchi identities as an analogue (and it really is a precise mathematical analogue) of "the divergence of the curl is zero," a *geometrical* constraint that ensures that the Gravitational Field Equations have conservation of the stress energy tensor automatically built into *their* fundamental formulation, just as Maxwell's Field Equations have charge conservation built into their underlying structure. What is good for Maxwell is good for Einstein.

6 The Einstein Field Equations

In the spring of 1913, Planck and Nernst had come to Zürich for the purpose of sounding out Einstein about his possible interest in moving to Berlin...Planck [asked him] what he was working on, and Einstein described general relativity as it was then. Planck said 'As an older friend, I must advise you against it for in the first place you will not succeed; and even if you succeed, no one will believe you.'

— A. Pais, writing in 'Subtle is the Lord'

6.1 Formulation

We will now apply the principle of general covariance to the gravitational field itself. What is the relativistic analogue of $\nabla^2\Phi = 4\pi G\rho$? We have now built up a sufficiently strong mathematical arsenal from Riemannian geometry to be able to give a satisfactory answer to this question.

We know that we must work with vectors and tensors to maintain general covariance, and that the Newtonian-Poisson source, ρ, is a mere component of a more general stress-energy tensor $T_{\mu\nu}$ (in covariant tensor form) in relativity. We expect therefore that the gravitional field equations will take the form

$$G_{\mu\nu} = CT_{\mu\nu} \tag{229}$$

where $G_{\mu\nu}$ is a tensor comprised of $g_{\mu\nu}$ and its second derivatives, or products of the first derivatives of $g_{\mu\nu}$. We guess this since i) we know that in the Newtonian limit the largest component of $g_{\mu\nu}$ is the $g_{00} \simeq -1 - 2\Phi/c^2$ component; ii) we need to recover the Poisson equation; and iii) we assume that we are seeking a theory of gravity that does not change its character with scale: it has no characteristic length associated with it where the field changes fundamentally in character. The last condition may strike you as a bit too restrictive. *Who ordered that?* Well, umm...OK, we now know this is actually wrong. It is wrong when applied to the Universe at large! But it is the simplest assumption that we can make that will satisfy all the basic requirements of a good theory. We'll come back to the general relativity updates once we have operating system GR1.0 installed.

Next, we know that the stress energy tensor is conserved in the sense of $T^{\mu\nu}_{;\nu} = 0$. We also know from our work with the Bianchi identities of the previous section that this will automatically be satisfied if we take $G_{\mu\nu}$ to be proportional to the particular linear combination

$$G_{\mu\nu} \propto R_{\mu\nu} - \frac{g_{\mu\nu}R}{2}$$

(Notice that there is no difficulty shifting indices up or down as considerations demand: our index shifters $g_{\mu\nu}$ and $g^{\mu\nu}$ all have vanishing covariant derivatives and can moved inside and outside of semi-colons.) We have determined the field equations of gravity up to an overall normalisation:

$$R_{\mu\nu} - \frac{g_{\mu\nu}R}{2} = CT_{\mu\nu} \tag{230}$$

The final step is to recover the Newtonian limit. In this limit, $T_{\mu\nu}$ is dominated by T_{00}, and $g_{\mu\nu}$ can be replaced by $\eta_{\alpha\beta}$ when shifting indices. The leading order derivative of $g_{\mu\nu}$ that enters into the field equations comes from

$$g_{00} \simeq -1 - \frac{2\Phi}{c^2}$$

where Φ is the usual Newtonian potential. In what follows, we use i, j, k to indicate spatial indices, and 0 will always be reserved for time.

The trace of equation (230) reads (raise μ, contract with ν):

$$R - 4 \times \frac{1}{2}R = -R = CT. \tag{231}$$

Substituting this for R back in the original equation leads to

$$R_{\mu\nu} = C\left(T_{\mu\nu} - \frac{g_{\mu\nu}T}{2}\right) \equiv CS_{\mu\nu} \tag{232}$$

which defines the so-called source function, a convenient grouping we shall use later:

$$S_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu}T/2. \tag{233}$$

The 00 component of of (232) is

$$R_{00} = C\left(T_{00} - \frac{g_{00}T}{2}\right) \tag{234}$$

In the Newtonian limit, the trace $T \equiv T^{\mu}_{\ \mu}$ is dominated by the 0 term, $T^{0}_{\ 0}$, and raising and lowering of the indices is done by the $\eta_{\mu\nu}$ weak field limit of $g_{\mu\nu}$.

$$R_{00} = C\left(T_{00} - \frac{\eta_{00}T^{0}_{\ 0}}{2}\right) = C\left(T_{00} - \frac{T_{00}}{2}\right) = C\frac{T_{00}}{2} = C\frac{\rho c^2}{2}, \tag{235}$$

where ρ is the Newtonian mass density. Calculating R_{00} explicitly,

$$R_{00} = R^{\nu}_{\ 0\nu 0} = \eta^{\lambda\nu}R_{\lambda 0\nu 0} \tag{236}$$

We need only the linear part of $R_{\lambda\mu\nu\kappa}$ in the weak field limit:

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}}\right), \tag{237}$$

and in the static limit with $\mu = \kappa = 0$, only the final term on the right side of this equation survives:

$$R_{\lambda 0\nu 0} = \frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^{\nu}\partial x^{\lambda}}. \tag{238}$$

Finally,

$$R_{00} = \eta^{\lambda\nu} R_{\lambda 0 \nu 0} = \frac{1}{2}\eta^{\lambda\nu}\frac{\partial^2 g_{00}}{\partial x^\lambda \partial x^\nu} = \frac{1}{2}\nabla^2 g_{00} = -\frac{1}{c^2}\nabla^2\Phi = \frac{C\rho c^2}{2} \tag{239}$$

This happily agrees with the Poisson equation if $C = -8\pi G/c^4$. Hello Isaac Newton. As Einstein himself put it: "No fairer destiny could be allotted to any physical theory, than that it should of itself point out the way to the introduction of a more comprehensive theory, in which it lives on as a limiting case." We therefore arrive at the *Einstein Field Equations*:

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu}} \tag{240}$$

The Field Equations first appeared in Einstein's notes on 25 November 1915, just over a hundred years ago, after an inadvertent competition with the mathematician David Hilbert, triggered by an Einstein colloquium at Göttingen. (Talk about being scooped! Hilbert actually derived the Field Equations first, by a variational method, but rightly insisted on giving Einstein full credit for the physical theory. Incidentally, in common with Einstein, Hilbert didn't know the Bianchi identities.)

It is useful to also exhibit these equations explicitly in source function form. Contracting μ and ν,

$$R = \frac{8\pi G}{c^4}T, \tag{241}$$

and the field equations become

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \equiv -\frac{8\pi G}{c^4}S_{\mu\nu} \tag{242}$$

where as before,

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T, \tag{243}$$

a "Bianchified form" of the stress tensor. In vacuo, the Field Equations reduce to the analogue of the Laplace Equation:

$$R_{\mu\nu} = 0. \tag{244}$$

One final point. If we allow the possibility that gravity could change its form on different scales, it is always possible to add a term of the form $\pm\Lambda g_{\mu\nu}$ to $G_{\mu\nu}$, where Λ is a constant (positive by convention), without violating the conservation of $T_{\mu\nu}$ condition. This is because the covariant derivatives of $g_{\mu\nu}$ vanish identically, so that $T_{\mu\nu}$ is still conserved. Einstein, pursuing the consequences of his theory for cosmology, realised that his Field Equations did not produce a static universe. This is bad, he thought, everyone knows the Universe is static. So he sought a source of static stabilisation, adding an offsetting, positive Λ term to the right side of the Field Equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{245}$$

and dubbed Λ the *cosmological constant.* Had he not done so, he could have made a spectacular prediction: the Universe is dynamic, a player in its own game, and must be either expanding or contracting.[4] With the historical discovery of an expanding univese, Einstein retracted the Λ term, calling it "the biggest mistake of my life."

[4]Even within the context of straight Euclidian geometry and Newtonian dynamics, uniform expansion of an infinite space avoids the self-consistency problems associated with a static model. I've never understood why this simple point is not emphasised more.

Surprise. We now know that this term is, in fact, present on the largest cosmological scales, and on these scales it is not a small effect. It mimics (and may well be) an energy density of the vacuum itself. It is measured to be 70% of the effective energy density in the Universe. It is to be emphasised that Λ must be taken into account only on the largest scales, over which the locally much higher baryon and dark matter inhomogeneities are lowered by effective smoothing; Λ is otherwise quite negligible. The so-called biggest mistake of Einstein's life was therefore quadratic in amplitude: one factor of error for introducing Λ for the wrong reason, the second factor for retracting Λ for the wrong reason!

Except for cosmological problems, we will always assume $\Lambda = 0$.

6.2 Coordinate ambiguities

There is no unique solution to the Field Equation because of the fact that they have been constructed to admit a new solution by a transformation of coordinates. To make this point as clear as possible, imagine that we have worked hard, solved for the metric $g_{\mu\nu}$, and in turns out to be plain old Minkowski space.[5] Denote the coordinates as t for the time dimension and α, β, γ for the spatial dimensions. Even if we restrict ourselves to diagonal $g_{\mu\nu}$, we might have found that the diagonal entries are $(-1, 1, 1, 1)$ or $(-1, 1, \alpha^2, 1)$ or $(-1, 1, \alpha^2, \alpha^2 \sin^2 \beta)$ depending upon whether we happen to be using Cartesian (x, y, z), cylindrical (R, ϕ, z), or spherical (r, θ, ϕ) spatial coordinate systems. Thus, we always have the freedom to work with coordinates that simplify our equations or that make physical properties of our solutions more transparent.

This is particularly useful for gravitational radiation. You may remember when you studied electromagnetic radiation that the equations for the potentials (both $\boldsymbol{A}$ and Φ) simplified considerably when a particular gauge was used—the Lorenz gauge. A different gauge could have been used and the potential would have looked different, but the fields would have been the same. The same is true for gravitational radiation, in which coordinate transformations play this role, but in a very peculiar way: we change the components of $g_{\mu\nu}$ as though a coordinate transformation were taking place, but we actually keep our working coordinates the same! What seems like an elementary blunder is actually perfectly correct, and will be explained more fully in Chapter 7.

For the problem of determining $g_{\mu\nu}$ around a point mass—the Schwarzschild black hole—we will choose to work with coordinates that look as much as possible like standard spherical coordinates.

6.3 The Schwarzschild Solution

We wish to determine the form of the metric tensor $g_{\mu\nu}$ for the spacetime surrounding a point mass M by solving the equation $R_{\mu\nu} = 0$, subject to the appropriate boundary conditions.

Because the spacetime is static and spherically symmetric, we expect the invariant line element to take the form

$$-c^2 d\tau^2 = -B\, c^2 dt^2 + A\, dr^2 + C\, d\Omega^2 \tag{246}$$

where $d\Omega$ is the (undistorted) solid angle,

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

[5]Don't smirk. If we're using awkward coordinates, it can be *very* hard to tell. You'll see.

and A, B, and C are all functions of the radial variable. We may choose our coordinates so that C is defined to be r^2 (if it is not already, do a coordinate transformation $r'^2 = C(r)$ and then drop the $'$). A and B will then be some unknown functions of r to be determined. Our metric is now in "standard form:"

$$-c^2 d\tau^2 = -B(r)\, c^2 dt^2 + A(r)\, dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{247}$$

We may now read the components of $g_{\mu\nu}$:

$$g_{tt} = -B(r) \qquad g_{rr} = A(r) \qquad g_{\theta\theta} = r^2 \qquad g_{\phi\phi} = r^2 \sin^2\theta \tag{248}$$

and its inverse $g^{\mu\nu}$,

$$g^{tt} = -B^{-1}(r) \qquad g^{rr} = A^{-1}(r) \qquad g^{\theta\theta} = r^{-2} \qquad g^{\phi\phi} = r^{-2}(\sin\theta)^{-2} \tag{249}$$

The determinant of $g_{\mu\nu}$ is $-g$, where

$$g = r^4 AB \sin^2\theta \tag{250}$$

We have seen that the affine connection for a *diagonal* metric tensor will be of the form

$$\Gamma^a_{ab} = \Gamma^a_{ba} = \frac{1}{2g_{aa}} \frac{\partial g_{aa}}{\partial x^b}$$

no sum on a, with $a = b$ permitted; or

$$\Gamma^a_{bb} = -\frac{1}{2g_{aa}} \frac{\partial g_{bb}}{\partial x^a}$$

no sum on a or b, with a and b distinct. The nonvanishing components follow straightforwardly:

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{B'}{2B}$$

$$\Gamma^r_{tt} = \frac{B'}{2A} \quad \Gamma^r_{rr} = \frac{A'}{2A} \quad \Gamma^r_{\theta\theta} = -\frac{r}{A} \quad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{A}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\,\cos\theta$$

$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r} \qquad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta \tag{251}$$

where $A' = dA/dr$, $B' = dB/dr$. We will also make use of this table to compute the orbits in a Schwarzschild geometry.

Next, we need the Ricci Tensor:

$$R_{\mu\kappa} \equiv R^\lambda{}_{\mu\lambda\kappa} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\lambda\eta} \tag{252}$$

Remembering equation (171), this may be written

$$R_{\mu\kappa} = \frac{1}{2}\frac{\partial^2 \ln g}{\partial x^\kappa \partial x^\mu} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \frac{\Gamma^\eta_{\mu\kappa}}{2}\frac{\partial \ln g}{\partial x^\eta} \tag{253}$$

Right. First R_{tt}. Remember, static fields.

$$R_{tt} = -\frac{\partial \Gamma^r_{tt}}{\partial r} + \Gamma^\eta_{t\lambda}\Gamma^\lambda_{t\eta} - \Gamma^\eta_{tt}\Gamma^\lambda_{\lambda\eta}$$

$$= -\frac{\partial}{\partial r}\left(\frac{B'}{2A}\right) + \Gamma^t_{t\lambda}\Gamma^\lambda_{tt} + \Gamma^r_{t\lambda}\Gamma^\lambda_{tr} - \Gamma^r_{tt}\Gamma^\lambda_{\lambda r}$$

$$= -\frac{\partial}{\partial r}\left(\frac{B'}{2A}\right) + \Gamma^t_{tr}\Gamma^r_{tt} + \Gamma^r_{tt}\Gamma^t_{tr} - \frac{\Gamma^r_{tt}}{2}\frac{\partial \ln g}{\partial r}$$

$$= -\left(\frac{B''}{2A}\right) + \frac{B'A'}{2A^2} + \frac{B'^2}{4AB} + \cancel{\frac{B'^2}{4AB}} - \frac{B'}{4A}\left(\frac{A'}{A} + \cancel{\frac{B'}{B}} + \frac{4}{r}\right)$$

This gives

$$R_{tt} = -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{B'}{rA} \tag{254}$$

Next, R_{rr}:

$$R_{rr} = \frac{1}{2}\frac{\partial^2 \ln g}{\partial r^2} - \frac{\partial \Gamma^r_{rr}}{\partial r} + \Gamma^\eta_{r\lambda}\Gamma^\lambda_{r\eta} - \frac{\Gamma^r_{rr}}{2}\frac{\partial \ln g}{\partial r}$$

$$= \frac{1}{2}\frac{\partial}{\partial r}\left(\cancel{\frac{A'}{A}} + \frac{B'}{B} + \frac{4}{r}\right) - \cancel{\frac{\partial}{\partial r}\left(\frac{A'}{2A}\right)} + \Gamma^\eta_{r\lambda}\Gamma^\lambda_{r\eta} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right)$$

$$= \frac{B''}{2B} - \frac{1}{2}\left(\frac{B'}{B}\right)^2 - \frac{2}{r^2} + \left(\Gamma^t_{rt}\right)^2 + \left(\Gamma^r_{rr}\right)^2 + \left(\Gamma^\theta_{r\theta}\right)^2 + \left(\Gamma^\phi_{r\phi}\right)^2 - \frac{1}{4}\left(\frac{A'}{A}\right)^2 - \frac{A'B'}{4AB} - \frac{A'}{rA}$$

$$= \frac{B''}{2B} - \frac{1}{2}\left(\frac{B'}{B}\right)^2 - \cancel{\frac{2}{r^2}} + \frac{B'^2}{4B^2} + \cancel{\frac{A'^2}{4A^2}} + \cancel{\frac{1}{r^2}} + \cancel{\frac{1}{r^2}} - \cancel{\frac{1}{4}\left(\frac{A'}{A}\right)^2} - \frac{A'B'}{4AB} - \frac{A'}{rA}$$

So that finally

$$R_{rr} = \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rA} \tag{255}$$

Tired? Well, here is a spoiler: all we will need for the problem at hand is R_{tt} and R_{rr}, so you can now skip to the end of the section. For the true fanatics, we are just getting warmed up! On to $R_{\theta\theta}$:

$$R_{\theta\theta} = \frac{\partial \Gamma^\lambda_{\theta\lambda}}{\partial \theta} - \frac{\partial \Gamma^\lambda_{\theta\theta}}{\partial x^\lambda} + \Gamma^\eta_{\theta\lambda}\Gamma^\lambda_{\theta\eta} - \Gamma^\eta_{\theta\theta}\Gamma^\lambda_{\lambda\eta}$$

$$= \frac{1}{2}\frac{\partial^2 \ln g}{\partial \theta^2} - \frac{\partial \Gamma^r_{\theta\theta}}{\partial r} + \Gamma^\eta_{\theta\lambda}\Gamma^\lambda_{\theta\eta} - \Gamma^r_{\theta\theta}\Gamma^\lambda_{\lambda r}$$

$$= \frac{d(\cot\theta)}{d\theta} + \frac{d}{dr}\left(\frac{r}{A}\right) + \Gamma^\eta_{\theta\lambda}\Gamma^\lambda_{\theta\eta} + \frac{r}{2A}\frac{\partial \ln g}{\partial r}$$

$$= -\frac{1}{\sin^2\theta} + \frac{1}{A} - \frac{rA'}{A^2} + \Gamma^r_{\theta\lambda}\Gamma^\lambda_{\theta r} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{\theta\theta} + \Gamma^\phi_{\theta\lambda}\Gamma^\lambda_{\theta\phi} + \frac{r}{2A}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right)$$

$$= -\frac{1}{\sin^2\theta} + \frac{3}{A} - \frac{rA'}{2A^2} + \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r} + \Gamma^\theta_{\theta r}\Gamma^r_{\theta\theta} + \left(\Gamma^\phi_{\theta\phi}\right)^2 + \frac{rB'}{2AB}$$

$$= -\frac{1}{\sin^2\theta} + \frac{3}{A} - \frac{rA'}{2A^2} - \frac{2}{A} + \cot^2\theta + \frac{rB'}{2AB}$$

The trigonometric terms add to -1. We finally obtain

$$R_{\theta\theta} = -1 + \frac{1}{A} + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) \tag{256}$$

$R_{\phi\phi}$ is the last nonvanishing Ricci component. No whining now! The first term in (252) vanishes, since nothing in the metric depends on ϕ. Then,

$$R_{\phi\phi} = -\frac{\partial \Gamma^{\lambda}_{\phi\phi}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\phi\lambda}\Gamma^{\lambda}_{\phi\eta} - \frac{\Gamma^{\eta}_{\phi\phi}}{2}\frac{\partial \ln|g|}{\partial x^{\eta}}$$

$$= -\frac{\partial \Gamma^{r}_{\phi\phi}}{\partial r} - \frac{\partial \Gamma^{\theta}_{\phi\phi}}{\partial \theta} + \Gamma^{r}_{\phi\lambda}\Gamma^{\lambda}_{\phi r} + \Gamma^{\theta}_{\phi\lambda}\Gamma^{\lambda}_{\phi\theta} + \Gamma^{\phi}_{\phi\lambda}\Gamma^{\lambda}_{\phi\phi} - \frac{1}{2}\Gamma^{r}_{\phi\phi}\frac{\partial \ln|g|}{\partial r} - \frac{1}{2}\Gamma^{\theta}_{\phi\phi}\frac{\partial \ln|g|}{\partial \theta}$$

$$= \frac{\partial}{\partial r}\left(\frac{r\sin^2\theta}{A}\right) + \frac{\partial}{\partial\theta}(\sin\theta\cos\theta) + \Gamma^{r}_{\phi\phi}\Gamma^{\phi}_{\phi r} + \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta} + \Gamma^{\phi}_{\phi r}\Gamma^{r}_{\phi\phi} + \Gamma^{\phi}_{\phi\theta}\Gamma^{\theta}_{\phi\phi}$$

$$+\frac{1}{2}\sin\theta\cos\theta\frac{\partial \ln\sin^2\theta}{\partial\theta} + \frac{1}{2}\left(\frac{r\sin^2\theta}{A}\right)\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right)$$

$$= \cancel{\frac{\sin^2\theta}{A}} - \frac{rA'\sin^2\theta}{A^2} \cancel{+\cos^2\theta} - \sin^2\theta - \cancel{\frac{\sin^2\theta}{A}} \cancel{-\cos^2\theta} - \frac{\sin^2\theta}{A} \cancel{-\cos^2\theta} \cancel{+\cos^2\theta} + \frac{r\sin^2\theta}{2A}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right)$$

$$= \sin^2\theta\left[\frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} - 1\right] = \sin^2\theta R_{\theta\theta}$$

The fact that $R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$ and that $R_{\mu\nu} = 0$ if μ and ν are not equal are consequences of the spherical symmetry and time reversal symmetry of the problem repsectively. If the first relation did not hold, or if R_{ij} did not vanish when i and j were different spatial coordinates, then an ordinary rotation of the axes would change the relative form of the tensor components, despite the spherical symmetry. This is impossible. If $R_{ti} \equiv R_{it}$ were non-vanishing (i is again a spatial index), the coordinate transformation $t' = -t$ would change the components of the Ricci tensor. But a static $R_{\mu\nu}$ must be invariant to this form of time reversal coordinate change. (Why?) Note that this argument is *not* true for R_{tt}. (Why not?)

Learn to think like a mathematical physicist in this kind of a calculation, taking into account the symmetries that are present, and you will save a lot of work.

Exercise. **Self-gravitating masses in general relativity.** We are solving in this section the vacuum equations $R_{\mu\nu} = 0$, but it is of great interest for stellar structure and cosmology to have a set of equations for a self-gravitating spherical mass. Toward that end, we recall equation (242):

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}S_{\mu\nu} \equiv -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{g_{\mu\nu}}{2}T^{\lambda}_{\lambda}\right)$$

Let us evaluate $S_{\mu\nu}$ for the case of an isotropic stress energy tensor of an ideal gas in its rest frame. With

$$g_{tt} = -B, \quad g_{rr} = A, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2\sin^2\theta,$$

the stress-energy tensor

$$T_{\mu\nu} = Pg_{\mu\nu} + (\rho + P/c^2)U_{\mu}U_{\nu},$$

where U_μ is the 4-velocity, show that, in addition to the trivial condition

$$U_r = U_\theta = U_\phi = 0,$$

we must have $U_t = -c\sqrt{B}$ (remember equation [180]) and that

$$S_{tt} = \frac{B}{2}(3P + \rho c^2), \quad S_{rr} = \frac{A}{2}(\rho c^2 - P), \quad S_{\theta\theta} = \frac{r^2}{2}(\rho c^2 - P)$$

We will develop the solutions of $R_{\mu\nu} = -8\pi G S_{\mu\nu}/c^4$ shortly.

Enough. We have more than we need to solve the problem at hand. To solve the equations $R_{\mu\nu} = 0$ is now a rather easy task. Two components will suffice (we have only A and B to solve for after all), all others then vanish identically. In particular, work with R_{rr} and R_{tt}, both of which must separately vanish, so

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \tag{257}$$

whence we find

$$AB = \text{constant} = 1 \tag{258}$$

where the constant must be unity since A and B go over to their Minkowski values at large distances. The condition that $R_{tt} = 0$ is now from (254) simply

$$B'' + \frac{2B'}{r} = 0, \tag{259}$$

which means that B is a linear superposition of a constant plus another constant times $1/r$. But B must approach unity at large r, so the first constant is one, and we know from long ago that the next order term at large distances must be $2\Phi/c^2$ in order to recover the Newtonian limit. Hence,

$$B = 1 - \frac{2GM}{rc^2}, \qquad A = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \tag{260}$$

The *Schwarzschild Metric* for the spacetime around a point mass is exactly

$$\boxed{-c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2} \tag{261}$$

This remarkable, simple and critically important exact solution of the Einstein Field Equation was obtained in 1916 by Karl Schwarzschild from the trenches of World War I. Tragically, Schwarzschild did not survive the war,[6] dying from a skin infection five months after finding his marvelous solution. He managed to communicate his result fully in a letter to Einstein. His last letter to Einstein was dated December 22, 1915, some 28 days after the formulation of the Field Equations.

Exercise. **The Tolman-Oppenheimer-Volkoff Equation.** Let us strike again while the iron is hot. Referring back to the previous exercise, we repeat part of our Schwarzschild

[6] The senseless WWI deaths of Karl Schwarzschild for the Germans and of Henry Moseley (of Oxford) for the British were incalculable losses for science. Schwarzschild's son Martin, a 4-year-old at the time of his father's death, also became a great astrophysicist, developing much of the modern theory of stellar evolution.

calculation, but with the source terms $S_{\mu\nu}$ retained. Form a familiar combination once again:

$$\frac{R_{rr}}{A}+\frac{R_{tt}}{B}=-\frac{1}{rA}\left(\frac{A'}{A}+\frac{B'}{B}\right)=-\frac{8\pi G}{c^4}\left(\frac{S_{tt}}{B}+\frac{S_{rr}}{A}\right)=-\frac{8\pi G}{c^4}(P+\rho c^2)$$

Show now that adding $2R_{\theta\theta}/r^2$ eliminates the B dependence:

$$\frac{R_{rr}}{A}+\frac{R_{tt}}{B}+\frac{2R_{\theta\theta}}{r^2}=-\frac{2A'}{rA^2}-\frac{2}{r^2}+\frac{2}{Ar^2}=-\frac{16\pi G\rho}{c^2}.$$

Solve this equation for A and show that the solution with finite $A(0)$ is

$$A(r)=\left(1-\frac{2G\mathcal{M}(r)}{r}\right)^{-1},\quad \mathcal{M}(r)=\int_0^r 4\pi\rho(r')\,r'^2\,dr'$$

Finally, use the equation $R_{\theta\theta}=-8G\pi S_{\theta\theta}/c^4$ together with hydrostatic equilibrium (184) (for the term B'/B in $R_{\theta\theta}$) to obtain the celebrated Tolman-Oppenheimer-Volkoff equation for the interior structure of general relativistic stars:

$$\frac{dP}{dr}=-\frac{G\mathcal{M}(r)\,\rho}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi r^3 P}{\mathcal{M}(r)\,c^2}\right)\left(1-\frac{2G\mathcal{M}(r)}{rc^2}\right)^{-1}$$

This is a rather long, but completely straightforward, exercise.

Students of stellar structure will recognise the classical equation hydrostatic equilibrium equation for a Newtonian star, with three correction terms. The final factor on the right is purely geometrical, the radial curvature term A from the metric. The corrective replacement of ρ by $\rho+P/c^2$ arises even in the special relativistic equations of motion for the inertial density; for inertial purposes P/c^2 is an effective density. Finally the modification of the gravitating $\mathcal{M}(r)$ term (to $\mathcal{M}(r)+4\pi r^3 P/c^2$) also includes a contribution from the pressure, as though an additional effective mass density $3P(r)/c^2$ were spread throughout the interior spherical volume within r, even though $P(r)$ is just the local pressure. Note that in massive stars, this pressure could be *radiative.*

6.4 The Schwarzschild Radius

It will not have escaped the reader's attention that at

$$r=\frac{2GM}{c^2}\equiv R_S \tag{262}$$

the metric becomes singular in appearance. R_S is known as the Schwarzschild radius. Numerically, normalising M to one solar mass $M_\odot$,

$$R_S=2.95\,(M/M_\odot)\text{ km}, \tag{263}$$

which is *well* inside any normal star! The Schwarzschild radius is part of the external vacuum spacetime only for black holes. Indeed, it is what makes black holes black. At least it was *thought* to be the feature that made black holes truly black, until Hawking came along in

1974 and showed us that quantum field theory changes the behaviour of black holes. But as usual, we are getting ahead of ourselves. Let us stick to classical theory.

I have been careful to write "singular in appearance" because in fact, the spacetime is perfectly well behaved at $r = R_S$. It is only the coordinates that become strained at this point, and these coordinates have been introduced, you will recall, so that they would be familiar to us, we few, we happy band of observers at infinity, as ordinary spherical coordinates. The curvature scalar R, for example, remains zero without so much as a ripple as we pass through $r = R_S$. We can see this coordinate effect staring at us if we start with the ordinary metric on the unit sphere,

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2,$$

and change coordinates to $x = \sin\theta$:

$$ds^2 = \frac{dx^2}{1 - x^2} + x^2 d\phi^2.$$

This looks horrible at $x = 1$, but in reality nothing is happening. Since x is just the distance from the z-axis to spherical surface (i.e. cylindrical radius), the "singularity" simply reflects the fact that at the equator x has reached its maximum value 1. So, dx must be zero at this point. x is just a bad coordinate at the equator; ϕ is a bad coordinate at the pole. Bad coordinates happen to good spacetimes. Get over it.

The physical interpretation of the first two terms of the metric (261) is that the proper time interval at a fixed spatial location is given by

$$dt\left(1 - \frac{2GM}{rc^2}\right)^{1/2} \qquad \text{(proper time interval at fixed location).} \tag{264}$$

The proper radial distance interval at a fixed angular location and time is

$$dr\left(1 - \frac{2GM}{rc^2}\right)^{-1/2} \qquad \text{(proper radial distance interval at fixed time \& angle).} \tag{265}$$

Exercise. **Getting rid of the Schwarzschild coordinate singularity. A challenge problem for the adventurous student only. Make sure you want to do this before you start.** Consider the rather unusual coordinate transformation found by Martin Kruskal. Start with our standard spherical coordinates t, r, θ, ϕ and introduce new r' and t' coordinates:

$$r'^2 - c^2t'^2 = c^2T^2\left(\frac{rc^2}{2GM} - 1\right)\exp\left(\frac{rc^2}{2GM}\right)$$

$$\frac{2r'ct'}{r'^2 + c^2t'^2} = \tanh\left(\frac{c^3t}{2GM}\right)$$

where T is an arbitrary constant. Show that the Schwarzschild metric transforms to

$$-c^2d\tau^2 = \left(\frac{32G^3M^3}{c^8rT^2}\right)\exp\left(\frac{-rc^2}{2GM}\right)(c^2dt'^2 - dr'^2) - r^2d\Omega^2$$

where T is arbitrary constant with dimensions of time, and r is the implicit solution of our first equation for $r'^2 - c^2t'^2$. The right side of this equation has a minimum of $-c^2T^2$ at $r = 0$, hence we must have

$$r'^2 > c^2(t'^2 - T^2)$$

always. When $t' < T$ there is no problem. But when $t' > T$ there are two distinct regions: $r' = \pm c\sqrt{t'^2 - T^2}$! Then the metric has a real singularity at either of these values of r' (which is just $r = 0$), but still no singularity at $r' = \pm ct'$, the value $r = R_S$.

6.5 Schwarzschild spacetime.

6.5.1 Radial photon geodesic

This doesn't mean that there is nothing of interest happening at $r = R_S$.

For starters, the gravitational redshift recorded by an observer at infinity relative to someone at rest at location r in the Schwarzschild spacetime is given (we now know) *precisely* by

$$dt = \frac{d\tau}{(1 - 2GM/rc^2)^{1/2}} \quad \text{(Exact.)} \tag{266}$$

so that at $r \to R_S$, signals arrive at a distant observer's post infinitely redshifted. What does this mean?

Comfortably sitting in the Clarendon Labs, monitoring the radio signals my hardworking graduate student is sending me whilst engaged on a perfectly reasonable thesis mission to take measurements of the $r = R_S$ tidal forces in a nearby black hole, I grow increasingly impatient. Not only are the incessant complaints becoming progressively more torpid and drawn out, the transmission frequency keeps shifting to longer and longer wavelengths, slipping out of my receiver's bandpass. Most irritating. Eventually, all contact is lost. (Typical.) I never receive any signal of any kind from within R_S. R_S is said to be the location of the *event horizon.* The singularity at $r = 0$ is present, but completely hidden from the outside world at $R = R_S$ within an event horizon. It is what Roger Penrose has aptly named "cosmic censorship."

The time coordinate change for light to travel from r_A to r_B following its geodesic path is given by setting

$$-(1 - 2GM/rc^2)c^2dt^2 + dr^2/(1 - 2GM/rc^2) = 0$$

and then computing

$$t_{AB} = \int_A^B dt = \frac{1}{c}\int_{r_A}^{r_B} \frac{dr}{(1 - 2GM/rc^2)} = \frac{r_B - r_A}{c} + \frac{R_S}{c}\ln\left(\frac{r_B - R_S}{r_A - R_S}\right) \tag{267}$$

which will be recognised as the Newtonian time interval plus a logarithmic correction proportional to the Schwarzschild radius R_S. Note that our expression becomes infinite when a path endpoint includes R_S. When R_S may be considered small over the entire integration path, to leading order

$$t_{AB} \simeq \frac{r_B - r_A}{c} + \frac{R_S}{c}\ln\left(\frac{r_A}{r_B}\right) = \frac{r_B - r_A}{c}\left(1 + \frac{R_S \ln(r_A/r_B)}{r_B - r_A}\right) \tag{268}$$

A GPS satellite orbits at an altitude of 20,200 km, and the radius of the earth is 6370 km. R_S for the earth is only 9mm! (Make a fist. Squeeze the entire earth inside it. You're not even close to making a black hole.) Then, the general relativisitic correction factor is

$$\frac{R_S}{r_B - r_A} \simeq \frac{9 \times 10^{-3}}{(20,200 - 6370) \times 10^3} = 6.5 \times 10^{-10}$$

This level of accuracy, about a part in 10^9, is needed for determining positions on the surface of the earth to a precision of a few meters (as when your GPS intones "Turn right onto the Lon-don Road."). How does the gravitational effect compare with the *second order* kinematic time dilation due to the satellite's motion? You should find them comparable.

6.5.2 Orbital equations

Start with the geodesic equation, written in terms of an arbitrary time parameter p:

$$\frac{d^2x^\lambda}{dp^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp} = 0 \tag{269}$$

It doesn't matter what p is, just use your watch. Using the table of equation (251), it is very easy to write down the equations for the orbits in a Schwarzschild geometry:

$$\frac{d^2(ct)}{dp^2} + \frac{B'}{B}\frac{dr}{dp}\frac{d(ct)}{dp} = 0, \tag{270}$$

$$\frac{d^2r}{dp^2} + \frac{B'}{2A}\left(\frac{cdt}{dp}\right)^2 + \frac{A'}{2A}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\theta}{dp}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{d\phi}{dp}\right)^2 = 0, \tag{271}$$

$$\frac{d^2\theta}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\theta}{dp} - \sin\theta\cos\theta\left(\frac{d\phi}{dp}\right)^2 = 0, \tag{272}$$

$$\frac{d^2\phi}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\phi}{dp} + 2\cot\theta\frac{d\theta}{dp}\frac{d\phi}{dp} = 0. \tag{273}$$

Obviously, it is silly to keep θ as a variable. The orbit may be set to the $\theta = \pi/2$ plane. Then, our equations become:

$$\frac{d^2(ct)}{dp^2} + \frac{B'}{B}\frac{dr}{dp}\frac{d(ct)}{dp} = 0, \tag{274}$$

$$\frac{d^2r}{dp^2} + \frac{B'}{2A}\left(\frac{cdt}{dp}\right)^2 + \frac{A'}{2A}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\phi}{dp}\right)^2 = 0, \tag{275}$$

$$\frac{d^2\phi}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\phi}{dp} = 0. \tag{276}$$

Exercise. Derive the last three equations very simply by applying the Euler-Lagrange Equations on the Lagrangian

$$-B(r)c^2\dot{t}^2 + A(r)\dot{r}^2 + r^2\dot{\phi}^2$$

where the dot represents d/dp. Which method do you prefer?

Remember that A and B depend explicitly on r, and only implicity on p via $r = r(p)$. Then, the first and last of these equations are particularly simple:

$$\frac{d}{dp}\left(B\,\frac{cdt}{dp}\right) = 0 \tag{277}$$

$$\frac{d}{dp}\left(r^2\frac{d\phi}{dp}\right) = 0 \tag{278}$$

It is convenient to choose our parameter p to be close to the time:

$$\frac{dt}{dp} = B^{-1}, \tag{279}$$

and of course general relativity conserves angular momentum for a spherical geometry:

$$r^2 \frac{d\phi}{dp} = J \qquad \text{(constant)} \tag{280}$$

Finally, just as we may form an energy integration constant from the radial motion equation in Newtonian theory, so too in Schwarzschild geometry. Multiplying (275) by $2A dr/dp$, and using our results for dt/dp and $d\phi/dp$, we find:

$$\frac{d}{dp}\left[A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{c^2}{B}\right] = 0 \tag{281}$$

or

$$A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{c^2}{B} = -E \quad \text{(constant.)} \tag{282}$$

Fixing $\theta = \pi/2$ and using our results for dt/dp, dr/dp and $d\phi/dp$,

$$-c^2\left(\frac{d\tau}{dp}\right)^2 = -B\,c^2\left(\frac{dt}{dp}\right)^2 + A\left(\frac{dr}{dp}\right)^2 + r^2\left(\frac{d\phi}{dp}\right)^2 = -\frac{c^2}{B} + A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} = -E. \tag{283}$$

Hence $d\tau^2 = dp^2(E/c^2)$, i.e. p and τ differ only by a proportionality constant. For matter, $E > 0$, while $E = 0$ for photons. To leading Newtonian order $E \simeq c^2$, i.e. the rest mass energy per unit mass. Substituting for B in (282), we find that extremal values of orbital r locations correspond to

$$\left(1 - \frac{2GM}{rc^2}\right)\left(\frac{J^2}{r^2} + E\right) - c^2 = 0 \tag{284}$$

for matter, and thus to

$$\left(1 - \frac{2GM}{rc^2}\right)\frac{J^2}{r^2} - c^2 = 0 \tag{285}$$

for photons.

The radial equation of motion may be written for $dr/d\tau$, dr/dt, or $dr/d\phi$ respectively (we use $AB = 1$):

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{c^2}{A}\left(1 + \frac{J^2}{Er^2}\right) = \frac{c^4}{E} \tag{286}$$

$$\left(\frac{dr}{dt}\right)^2 + \frac{B^2}{A}\left(E + \frac{J^2}{r^2}\right) = \frac{Bc^2}{A} \tag{287}$$

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^2}{A}\left(1 + \frac{Er^2}{J^2}\right) = \frac{c^2 r^4}{J^2} \tag{288}$$

From here, it is simply a matter of evaluating a (perhaps complicated) integral over r to obtain a solution.

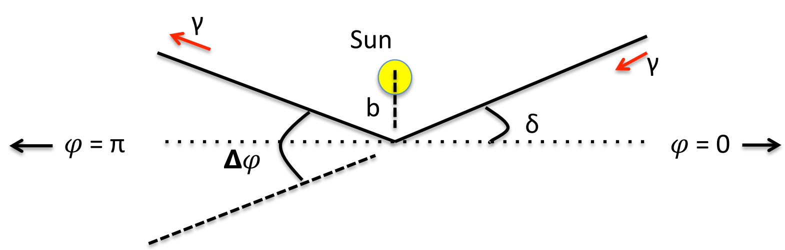

Figure 2: Bending of light by the gravitational field of the sun. In flat spacetime the photon γ travels the straight line from $\varphi = 0$ to $\varphi = \pi$ along the path $r \sin\varphi = b$. The presence of spacetime curvature starts the photon at $\varphi = -\delta$ and finishes its passage at $\varphi = \pi + \delta$. The deflection angle is $\Delta\varphi = 2\delta$.

6.6 The deflection of light by an intervening body.

The first prediction made by General Relativity Theory that could be tested was that starlight passing by the limb of the sun would be slightly but measurably deflected by the gravitational field. This type of measurement can only be done, of course, when the sun is completely eclipsed by the moon. Fortunately, the timing of the appearance of the theory with an eclipse was ideal. One of the longest total solar eclipses of the century occured on 29 May 1919. The path of totality extended from a strip in South America to central Africa. An expedition headed by A.S. Eddington observed the eclipse from the island of Principe, off the west coast of Africa. Measurements of thirteen stars confirmed not only that gravity affected the propagation of light, but that it did so by an amount in much better accord with general relativity theory than with a Newtonian "corpuscular theory," with the test mass velocity set equal to c. (The latter gives a deflection angle half as large as GR, in essence because the $2GM/rc^2$ terms in both the dt and dx metric coefficients contribute equally to the photon deflection, whereas in the Newtonian limit only the dt modification is retained—as we know.) This success earned Einstein press coverage that today is normally reserved for rock stars. *Everybody* knew who Albert Einstein was!

Today, not only mere deflection, but "gravitational lensing" and image formation across the electromagnetic spectrum are standard astronomical techniques to probe matter in all its forms: from small planets to huge, diffuse cosmological agglomerations of dark matter.

Let us return to the classic test. As in Newtonian dynamics, it turns out to be easier to work with $u \equiv 1/r$, in which case

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2. \tag{289}$$

Equation (288) with $B = 1/A$ and $E = 0$ for a photon may be written

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{B}{r^2} = \frac{c^2}{J^2} = \text{constant} \tag{290}$$

In terms of u:

$$\left(\frac{du}{d\phi}\right)^2 + u^2\left(1 - \frac{2GMu}{c^2}\right) = \frac{c^2}{J^2} \tag{291}$$

Differentiating with respect to ϕ ($du/d\phi \equiv u'$) leads quickly to

$$u'' + u = \frac{3GM}{c^2}u^2 \equiv 3\epsilon u^2. \tag{292}$$

We treat $\epsilon \equiv GM/c^2$ as a small parameter. We expand u as $u = u_0 + u_1$, with $u_1 = \mathrm{O}(\epsilon u_0) \ll u_0$ (read "u_1 is of order ϵ times u_0 and much smaller than u_0"). Then, terms of order unity must obey the equation

$$u_0'' + u_0 = 0, \tag{293}$$

and the terms of order ϵ must obey the equation

$$u_1'' + u_1 = 3\epsilon u_0^2. \tag{294}$$

To leading order ($u = u_0$), nothing happens: the photon moves in a straight line. If the point of closest approach is the impact parameter b, then the equation for a straight line is $r\sin\phi = b$, or

$$u_0 = \frac{\sin\phi}{b} \tag{295}$$

which is the unique solution to equation (293) with boundary conditions $r = \infty$ at $\phi = 0$ and $\phi = \pi$.

At order ϵ, there is a deflection from a straight line due to the presence of u_1:

$$u_1'' + u_1 = 3\epsilon u_0^2 = \frac{3\epsilon}{b^2}\sin^2\phi = \frac{3\epsilon}{2b^2}(1 - \cos 2\phi) \tag{296}$$

Clearly, we need to search for solutions of the form $u_1 = U + V\cos 2\phi$, where U and V are constants. Substituting this into (296), we easily find find $U = 3\epsilon/2b^2$ and $V = \epsilon/2b^2$. Our solution is then

$$\frac{1}{r} = u_0 + u_1 = \frac{\sin\phi}{b} + \frac{3\epsilon}{2b^2} + \frac{\epsilon\cos 2\phi}{2b^2} \tag{297}$$

With $\epsilon = 0$, the solution describes a straight line, $r\sin\phi = b$. The first order effects of including ϵ incorporate the tiny deflections from this straight line. The $\epsilon = 0$ solution sends r off to infinity at $\phi = 0$ and $\phi = \pi$. We may compute the leading order small changes to these two "infinity angles" by using $\phi = 0$ and $\phi = \pi$ in the correction $\epsilon\cos 2\phi$ term. Then we find that r goes off to infinity not at $\phi = 0$ and π, but at the slightly corrected values $\phi = -\delta$ and $\phi = \pi + \delta$ where

$$\delta = \frac{2\epsilon}{b} \tag{298}$$

In other words, there is now a total deflection angle $\Delta\phi$ from a straight line of 2δ, or

$$\Delta\phi = \frac{4GM}{bc^2} = 1.75 \text{ arcseconds for the Sun.} \tag{299}$$

Happily, arcsecond deflections were just at the limit of reliable photographic methods of measurement in 1919. Those arcsecond deflections unleashed a truly revolutionary paradigm shift. For once, the word is not an exaggeration.

6.7 The advance of the perihelion of Mercury

For Einstein personally, the revolution had started earlier, even before he had his Field Equations. The vacuum form of the Field Equations is, as we know, sufficient to describe the spacetime outside the gravitational source bodies themselves. Working with the equation $R_{\mu\nu} = 0$, Einstein found, and on 18 November 1915 presented, the explanation of a 60-year-old astronomical puzzle: what was the cause of Mercury's excess perihelion advance of $43''$ per century? The directly measured perihelion advance is actually much larger than this, but after the interactions from all the planets are taken into account, the excess $43''$ per century is an unexplained residual of 7.5% of the total. According to Einstein's biographer A. Pais, the discovery that this precise perihelion advance emerged from general relativity was

"...by far the strongest emotional experience in Einstein's scientific life, perhaps in all his life. Nature had spoken to him. He had to be right."

6.7.1 Newtonian orbits

Interestingly, the perihelion first-order GR calculation is not much more difficult than straight Newtonian. GR introduces a $1/r^2$ term in the effective gravitational potential, but there is already a $1/r^2$ term from the centrifugal term! Other corrections do not add substantively

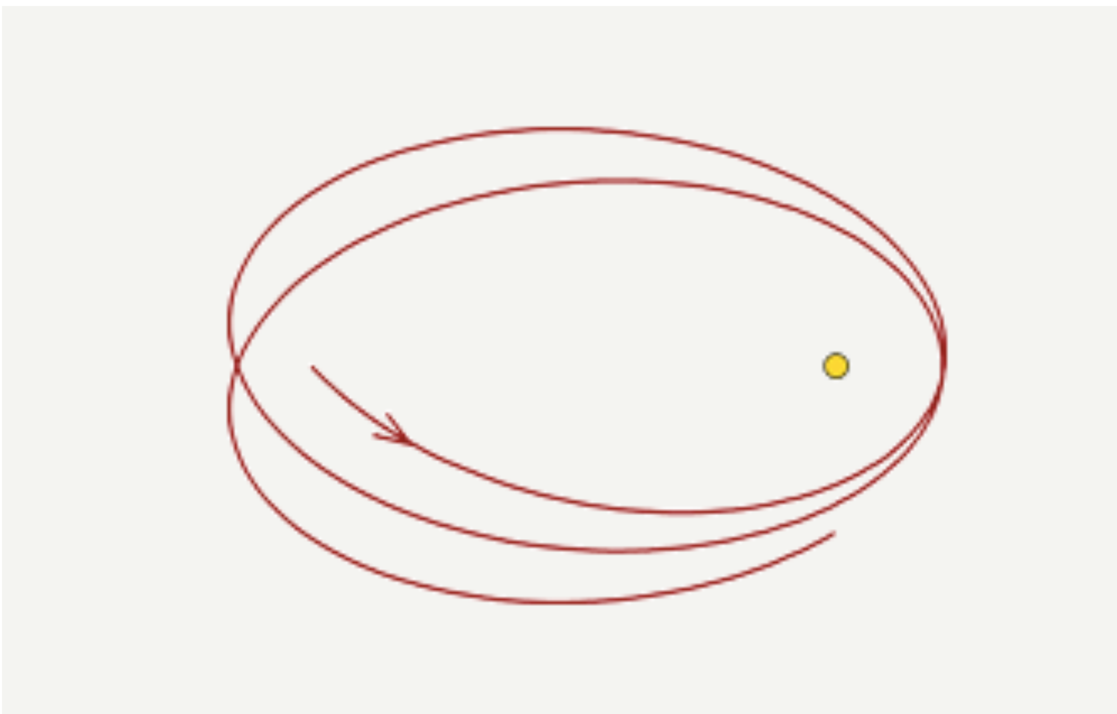

Figure 3: Departures from a $1/r$ gravitational potential cause elliptical orbits not to close. In the case of Mercury, the perihelion advances by 43 seconds of arc per century. The effect is shown here, *greatly* exaggerated.

to the difficulty. We thus begin with a detailed review of the Newtonian problem, and we will play off this solution for the GR perihelion advance.

Conservation of energy is

$$\frac{v_r^2}{2} + \frac{J^2}{2r^2} - \frac{GM}{r} = \mathcal{E} \tag{300}$$

where J is the (constant) specific angular momentum $r^2 d\phi/dt$ and $\mathcal{E}$ is the constant energy per unit mass. (In this Newtonian case, when the two bodies have comparable masses, M is actually the sum of the individual masses, and r the relative separation of the two bodies.) This is just the low energy limit of (286), whose exact form we may write as

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{c^2}{E}\left(\frac{J^2}{2r^2}\right) - \frac{GM}{r}\left(1 + \frac{J^2}{r^2 E}\right) = \left(\frac{c^2 - E}{2E}\right)c^2. \tag{301}$$

We now identify E with c^2 to leading order, and to next order $(c^2 - E)/2$ with $\mathcal{E}$ (i.e. the mechanical energy above and beyond the rest mass energy). The Newtonian equation may be written

$$v_r = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{J}{r^2}\frac{dr}{d\phi} = \pm\left(2\mathcal{E} + \frac{2GM}{r} - \frac{J^2}{r^2}\right)^{1/2} \tag{302}$$

and thence separated:

$$\int \frac{J\,dr}{r^2\left(2\mathcal{E} + \dfrac{2GM}{r} - \dfrac{J^2}{r^2}\right)^{1/2}} = \pm\phi \tag{303}$$

With $u = 1/r$,

$$\int \frac{du}{\left(\dfrac{2\mathcal{E}}{J^2} + \dfrac{2GMu}{J^2} - u^2\right)^{1/2}} = \mp\phi \tag{304}$$

or

$$\int \frac{du}{\left[\dfrac{2\mathcal{E}}{J^2} + \dfrac{G^2M^2}{J^4} - \left(u - \dfrac{GM}{J^2}\right)^2\right]^{1/2}} = \mp\phi \tag{305}$$

Don't be put off by all the fluff. The integral is standard trigonometric:

$$\cos^{-1}\left[\frac{u - \dfrac{GM}{J^2}}{\left(\dfrac{2\mathcal{E}}{J^2} + \dfrac{G^2M^2}{J^4}\right)^{1/2}}\right] = \pm\phi \tag{306}$$

In terms of $r = 1/u$ this equation unfolds and simplifies to

$$r = \frac{J^2/GM}{1 + \epsilon\cos\phi}, \qquad \epsilon^2 \equiv 1 + \frac{2\mathcal{E}J^2}{G^2M^2} \tag{307}$$

With $\mathcal{E} < 0$ we find that $\epsilon < 1$, and that (307) is just the equation for a classical elliptical orbit of eccentricity ϵ. We identify the *semi-latus rectum*,

$$L = J^2/GM \tag{308}$$

the perihelion (radius of closest approach) r_- and the aphelion (radius of farthest extent) r_+,

$$r_- = \frac{L}{1+\epsilon}, \qquad r_+ = \frac{L}{1-\epsilon}, \qquad \frac{1}{L} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) \tag{309}$$

and the semi-major axis

$$a = \frac{1}{2}(r_+ + r_-), \qquad \text{whence } \ L = a(1-\epsilon^2) \tag{310}$$

Notice that the zeros of the denominator in the integral (305) occur at $u_- = 1/r_-$ and $u_+ = 1/r_+$, corresponding in our arccosine function to ϕ equals 0 and π respectively.

Exercise.) **The Shows must go on.** Show that the semi-minor axis of an ellipse is $b = a\sqrt{1-\epsilon^2}$. Show that the area of an ellipse is πab. Show that the total energy of a two-body bound system (masses m_1 and m_2) is $-Gm_1m_2/2a$, independent of ϵ. With $M = m_1 + m_2$, show that the period of a two-body bound system is $2\pi\sqrt{a^3/GM}$, independent of ϵ. (There is a very simple way to do the latter!)

6.7.2 The perihelion advance of Mercury

Equation (288) may be written in terms of $u = 1/r$ as

$$\left(\frac{du}{d\phi}\right)^2 + \left(1 - \frac{2GMu}{c^2}\right)\left(u^2 + \frac{E}{J^2}\right) = \frac{c^2}{J^2}. \tag{311}$$

Now differentiate with respect to ϕ and simplify. The resulting equation is:

$$u'' + u = \frac{GME}{c^2J^2} + \frac{3GMu^2}{c^2} \simeq \frac{GM}{J^2} + \frac{3GMu^2}{c^2}, \tag{312}$$

since E is very close to c^2 for a nonrelativistic Mercury, and the difference here is immaterial. The Newtonian limit corresponds to dropping the final term on the right side of the equation; the resulting solution is

$$u \equiv u_N = \frac{GM}{J^2}(1 + \epsilon\cos\phi) \ \text{ or } \ r = \frac{J^2/GM}{1+\epsilon\cos\phi} \tag{313}$$

where ϵ is an arbitrary constant. This is just the classic equation for a conic section, with hyperbolic ($\epsilon > 1$), parabolic ($\epsilon = 1$) and ellipsoidal ($\epsilon < 1$) solutions. For ellipses, ϵ is the eccentricity.

As the general relativistic term $3GMu^2/c^2$ is tiny, we are entirely justified in using the Newtonian solution for u^2 in this higher order term. Writing $u = u_N + \delta u$ with u_N given by (313), the differential equation becomes

$$\frac{d^2\delta u}{d\phi^2} + \delta u = \frac{3GM}{c^2}u_N^2 = \frac{3(GM)^3}{c^2J^4}(1 + 2\epsilon\cos\phi + \epsilon^2\cos^2\phi). \tag{314}$$

In Problem Set 2, you will be asked to solve this equation. The resulting solution for $u = u_N + \delta u$ may be written

$$u \simeq \frac{GM}{J^2}\left(1 + \epsilon\cos[\phi(1-\alpha)]\right) \tag{315}$$

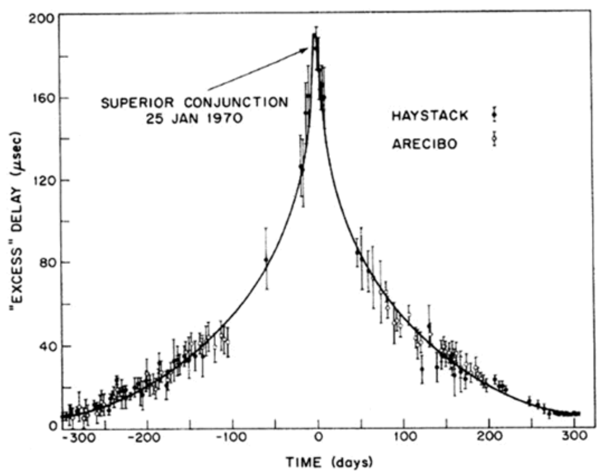

Figure 4: Radar echo delay from Venus as a function of time, fit with general relativistic prediction.

where $\alpha = 3(GM/Jc)^2$. Thus, the perihelion occurs not with a ϕ-period of 2π, but with a slightly longer period of

$$\frac{2\pi}{1-\alpha} \simeq 2\pi + 2\pi\alpha, \tag{316}$$

i.e. an advance of the perihelion by an amount

$$\Delta\phi = 2\pi\alpha = 6\pi\left(\frac{GM}{Jc}\right)^2 = 6\pi\left(\frac{GM}{c^2L}\right) = 2.783\times10^{-6}\ \left(\frac{10^{10}\text{m}}{L}\right) \tag{317}$$

in units of radians per orbit. With $L = 5.546\times10^{10}$ m, the measured semi-latus rectum for Mercury's orbit, this value of $\Delta\phi$ works out to be precisely 43 seconds of arc per century.

From its discovery in 1915 until the stunning gravitational radiation measurement in 1982 of the binary pulsar 1913+16, the precision perihelion advance of Mercury was general relativity's greatest observational success.

6.8 Shapiro delay: the fourth protocol

For many years, the experimental foundation of general relativity consisted of the three tests we have described that were first proposed by Einstein: the gravitational red shift, the bending of light by gravitational fields, and the advance of Mercury's perihelion. In 1964, nearly a decade after Einstein's passing, a fourth test was proposed: the time delay by radio signals when crossing near the sun in the inner solar system. The idea, proposed and carried out by Irwin Shapiro, is that a radio signal is sent from earth, bounces off Mercury, and returns. One does the experiment when Mercury is at its closest point to the earth, then repeats the experiment when the planet is on the far side of orbit. There should be an additional delay of the pulses when Mercury is on the far side of the sun because of the traversal of the radio waves across the sun's Schwarzschild geometry. It is this delay that is measured.

Recall equation (287), using the "ordinary" time parameter t for an observer at infinity, with $E = 0$ for radio waves:

$$\left(\frac{dr}{dt}\right)^2 + \frac{B^2}{A}\frac{J^2}{r^2} = \frac{Bc^2}{A} \tag{318}$$

It is convenient to evaluate the constant J in terms of r_0, the point of closest approach to the sun. With $dr/dt = 0$, we easily find

$$J^2 = \frac{r_0^2 c^2}{B_0} \tag{319}$$

where $B_0 \equiv B(r_0)$. The differential equation then separates and we find that the time $t(r, r_0)$ to traverse from r_0 to r (or vice-versa) is

$$t(r, r_0) = \frac{1}{c}\int_{r_0}^{r} \frac{A\, dr}{\left(1 - \dfrac{B}{B_0}\dfrac{r_0^2}{r^2}\right)^{1/2}}, \tag{320}$$

where we have made use of $AB = 1$. Expanding to first order in GM/c^2r with $B = 1 - 2GM/c^2r$:

$$1 - \frac{B}{B_0}\frac{r_0^2}{r^2} \simeq 1 - \left[1 + \frac{2GM}{c^2}\left(\frac{1}{r_0} - \frac{1}{r}\right)\right]\frac{r_0^2}{r^2}. \tag{321}$$

This may now be rewritten as:

$$1 - \frac{B}{B_0}\frac{r_0^2}{r^2} \simeq \left(1 - \frac{r_0^2}{r^2}\right)\left(1 - \frac{2GMr_0}{c^2 r(r + r_0)}\right) \tag{322}$$

Using this in our time integral for $t(r_0, r)$ and expanding,

$$t(r_0, r) = \frac{1}{c}\int_{r_0}^{r} dr \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2}\left(1 + \frac{2GM}{rc^2} + \frac{GMr_0}{c^2 r(r + r_0)}\right) \tag{323}$$

The required integrals are

$$\frac{1}{c}\int_{r_0}^{r} \frac{r\, dr}{(r^2 - r_0^2)^{1/2}} = \frac{1}{c}(r^2 - r_0^2)^{1/2} \tag{324}$$

$$\frac{2GM}{c^3}\int_{r_0}^{r} \frac{dr}{(r^2 - r_0^2)^{1/2}} = \frac{2GM}{c^3}\cosh^{-1}\left(\frac{r}{r_0}\right) = \frac{2GM}{c^3}\ln\left(\frac{r}{r_0} + \sqrt{\frac{r^2}{r_0^2} - 1}\right) \tag{325}$$

$$\frac{GMr_0}{c^3}\int_{r_0}^{r} \frac{dr}{(r + r_0)(r^2 - r_0^2)^{1/2}} = \frac{GM}{c^3}\sqrt{\frac{r - r_0}{r + r_0}} \tag{326}$$

Thus,

$$t(r, r_0) = \frac{1}{c}(r^2 - r_0^2)^{1/2} + \frac{2GM}{c^3}\ln\left(\frac{r}{r_0} + \sqrt{\frac{r^2}{r_0^2} - 1}\right) + \frac{GM}{c^3}\sqrt{\frac{r - r_0}{r + r_0}} \tag{327}$$

We are interested in $2t(r_1, r_0) \pm 2t(r_2, r_0)$ for the path from the earth at r_1, reflected from the planet (at r_2), and back. The $\pm$ sign depends upon whether the signal passes through

r_0 while enroute to the planet, i.e. on whether the planet is on the far side or the near side of the sun.

It may seem straightforward to plug in values appropriate to the earth's radial location and the planet's (either Mercury or Venus, in fact), compute the "expected Newtonian time" for transit (a sum of the first terms) and then measure the actual time for comparison with our formula. In practise, to know what the delay is, we have to know what the Newtonian transit time is to fantastic accuracy! In fact, the way this is done is to treat the problem not as a measurement of a single delay time, but as an entire function of time given by our solution (327) with $r = r(t)$. Figure (3) shows such a fit near the passage of superior conjunction (i.e. the far side orbital near the sun in sky projection), in excellent agreement with theory. Exactly how the parameterisation is carried out would take us too far afield; there is some discussion in W72 pp. 202–207, and an abundance of topical information on the internet under "Shapiro delay."

Modern applications of the Shaprio delay use pulsars as signal probes, whose time passage properties are altered by the presence of gravitational waves, a topic for the next chapter.

They are not objective, and (like absolute velocity) are not detectable by any conceivable experiment. They are merely sinuosities in the co-ordinate system, and the only speed of propagation relevant to them is "the speed of thought."

— A. S. Eddington writing in 1922 of Einstein's suspicions.

> *On September 14, 2015, at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational Wave Observatory simultaneously observed a transient gravitational wave signal. The signal sweeps upwards from 35 to 250 Hz with a peak gravitational wave strain of* 1×10^{-21}*. It matches the waveform predicted by general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole.*
>
> *— B. P. Abbott et al., 2016, Physical Review Letters, 116, 061102*

7 Gravitational Radiation

Gravity is spoken in the three languages. First, there is traditional Newtonian potential theory, the language used by most practising astrophysicists. Then, there is the language of Einstein's General Relativity Theory, the language of Riemannian geometry that we have been studying. Finally, there is the language of quantum field theory: gravity is a theory of the exchange of spin 2 particles, gravitons, much as electromagnetism is a theory arising from the exchange of spin 1 photons. Just as the starting point of quantum electrodynamics is the radiation theory of Maxwell, the starting point of quantum gravity must be a classical radiation theory of gravity. Unlike quantum electrodynamics, the most accurate physical theory ever created, there is no quantum theory of gravity at present, and there is not even a consensus approach. Quantum gravity is therefore very much an active area of ongoing research. For the theorist, this is reason enough to study the theory of gravitational radiation in general relativity. But there are good reasons for the practical astrophysicist to get involved. In Februrary 2016, the first detection of gravitational waves was announced. The event signal had been received and recorded on September 14, 2015, and is denoted G[ravitational]W[ave]150914. The detection was so clean, and matched the wave form predictions of general relativity in such detail, there can be no doubt that the detection was genuine. A new way to probe the most impenetrable parts of the Universe is at hand.

The theory of general relativity in the limit when $g_{\mu\nu}$ is very close to $\eta_{\mu\nu}$ is a classical theory of gravitational radiation (and not just Newtonian theory), in the same way that Maxwellian Electrodynamics is a classical electromagnetic radiation theory. The field equations for the small difference tensor $g_{\mu\nu} - \eta_{\mu\nu}$ become, in the weak field limit, a set of rather ordinary looking wave equations with source terms—much like Maxwell's Equations. The principal difference is that electrodynamics is sourced by a vector quantity (the usual vector potential $\boldsymbol{A}$ with the potential Φ combine to form a 4-vector), whereas gravitational fields in general relativity are sourced by a tensor quantity $T_{\mu\nu}$. This becomes a major difference when we relax the condition that the gravity field be weak: the gravitational radiation itself makes a contribution to its own source, something electromagnetic radiation cannot do. But this is not completely unprecedented in wave theories. We have seen this sort of thing before, in a purely classical context: sound waves can themselves generate acoustical disturbances, and one of the consequences is a shock wave, or sonic boom. While a few somewhat pathological mathematical solutions for exact gravitational radiation waves are known, in general people either work in the weak field limit or resort to numerical solutions of the field equations. Even with powerful computers, however, precise numerical solutions of the field equations for astrophysically interesting problems—like merging black holes—have long been a major technical challenge. In the last decade, a practical mathematical breakthrough has occurred, and it is now possible to compute highly accurate wave forms for these kinds of problems, with important predictions for the new generation of gravitational wave detectors.

As we have noted, astrophysicists now have perhaps the most important reason of all to understand gravitational radiation: we are on the verge of what will surely be a golden age of gravitational wave astronomy. That gravitational radiation truly exists was established in 1974, when a close binary system (7.75 hour period) with a neutron star and a pulsar (PSR 1913+16) was discovered by Hulse and Taylor. So much orbital information could be extracted from this remarkable system that it was possible to predict, then measure, the rate of orbital decay (more precisely, the gradual speed-up of the decaying orbit's period) caused by the energy carried off by gravitational radiation. The resulting period shortening, though tiny in any practical sense, was large enough to be cleanly measured. General relativity turned out to be exactly correct (Taylor & Weisberg, ApJ, 1982, 253, 908), and the 1993 Nobel Prize in Physics was duly awarded to Hulse and Taylor for this historical achievement.

The September 2015 gravitational wave detection pushed back the envelope dramatically. It established that i) the reception and analysis of gravitational waves is technically feasible and will soon become a widely-used probe of the universe; ii) black holes exist beyond any doubt whatsoever, this truly is the proverbial "smoking-gun"; iii) the full dynamical content of strong field general relativity on time and length scales characteristic of stellar systems is correct. This achievement is an historical milestone in physics. Some have speculated that its impact on astronomy will rival Galileo's introduction of the telescope. Perhaps Hertz's 1887 detection of electromagnetic radiation in the lab is another, more apt, comparison. (Commercial exploitation of gravity waves is probably some ways off. Maybe it will be licenced someday as a revenue source.)

There may be more to come. In the near future, it is anticipated that extremely delicate pulsar timing experiments, in which arrival times of pulses are measured to fantastic precision, will come on line. In essence, this is a measure of the Shapiro delay. It is caused neither by the Sun nor by a star, but by the passage of a gravitational wave between us and the pulsar source!

The subject of gravitational radiation is complicated and computationally intensive. Even the basics will involve some effort on your part. I hope you will agree that the effort is well rewarded.

7.1 The linearised gravitational wave equation

We assume that the metric is close to Minkowski space. Let us introduce the quantity $h_{\mu\nu}$, the (small) departure in the metric tensor $g_{\mu\nu}$ from its Minkowski $\eta_{\mu\nu}$ limit:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{328}$$

To leading order, when we raise and lower indices we may do so with $\eta_{\mu\nu}$. But be careful with $g^{\mu\nu}$ itself. Don't just lower the indices in the above equation willy-nilly! Instead, note that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{329}$$

to ensure $g_{\mu\nu}g^{\nu\kappa} = \delta^\kappa_\mu$. (You can raise the index of g with η only when approximating $g^{\mu\nu}$ as its leading order value, which is $\eta^{\mu\nu}$.) Note that

$$\eta^{\mu\nu} h_{\nu\kappa} = h^\mu_\kappa, \quad \eta^{\mu\nu}\frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x_\mu} \tag{330}$$

and that we can slide dummy indices "up-down" sometimes:

$$\frac{\partial h_{\mu\nu}}{\partial x_\mu} = \eta_{\mu\rho}\frac{\partial h^\rho_\nu}{\partial x_\mu} = \frac{\partial h^\rho_\nu}{\partial x^\rho} \equiv \frac{\partial h^\mu_\nu}{\partial x^\mu} \tag{331}$$

The story begins with the Einstein Field Equations cast in a form in which the "linearised Ricci tensor" is isolated on the left side of our working equation. Specifically, we write

$$R_{\mu\nu} = R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \text{...etc.} \tag{332}$$

and

$$G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \eta_{\mu\nu}\frac{R^{(1)}}{2} \tag{333}$$

where $R^{(1)}_{\mu\nu}$ is all the Ricci tensor terms linear in $h_{\mu\nu}$, $R^{(2)}_{\mu\nu}$ all terms quadratic in $h_{\mu\nu}$, and so forth. The linearised affine connection is

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}\left(\frac{\partial h_{\rho\nu}}{\partial x^\mu} + \frac{\partial h_{\rho\mu}}{\partial x^\nu} - \frac{\partial h_{\mu\nu}}{\partial x^\rho}\right) = \frac{1}{2}\left(\frac{\partial h^\lambda_\nu}{\partial x^\mu} + \frac{\partial h^\lambda_\mu}{\partial x^\nu} - \frac{\partial h_{\mu\nu}}{\partial x_\lambda}\right). \tag{334}$$

In terms of $h_{\mu\nu}$ and $h = h^\mu_\mu$, from equation (213) on page 50, we explicitly find

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial^2 h}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h^\lambda_\mu}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 h^\lambda_\nu}{\partial x^\mu \partial x^\lambda} + \Box h_{\mu\nu}\right) \tag{335}$$

where

$$\Box \equiv \frac{\partial^2}{\partial x^\lambda \partial x_\lambda} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{336}$$

is the d'Alembertian (clearly a Lorentz invariant), making a most welcome appearance into the proceedings. Contracting μ with ν, we find that

$$R^{(1)} = \Box h - \frac{\partial^2 h^{\mu\nu}}{\partial x^\mu \partial x^\nu} \tag{337}$$

where we have made use of

$$\frac{\partial h^\lambda_\mu}{\partial x_\mu} = \frac{\partial h^{\lambda\mu}}{\partial x^\mu}.$$

Assembling $G^{(1)}_{\mu\nu}$, we find

$$G^{(1)}_{\mu\nu} = \frac{1}{2}\left[\frac{\partial^2 h}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h^\lambda_\mu}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 h^\lambda_\nu}{\partial x^\lambda \partial x^\mu} + \Box h_{\mu\nu} - \eta_{\mu\nu}\left(\Box h - \frac{\partial^2 h^{\lambda\rho}}{\partial x^\lambda \partial x^\rho}\right)\right]. \tag{338}$$

The full, nonlinear Field Equations may then formally be written

$$G^{(1)}_{\mu\nu} = -\left(\frac{8\pi G T_{\mu\nu}}{c^4} + G_{\mu\nu} - G^{(1)}_{\mu\nu}\right) \equiv -\frac{8\pi G(T_{\mu\nu} + \tau_{\mu\nu})}{c^4}, \tag{339}$$

where

$$\tau_{\mu\nu} = \frac{c^4}{8\pi G}(G_{\mu\nu} - G^{(1)}_{\mu\nu}) \simeq \frac{c^4}{8\pi G}\left(R^{(2)}_{\mu\nu} - \eta_{\mu\nu}\frac{R^{(2)}}{2}\right) \tag{340}$$

Though composed of geometrical terms, the quantity $\tau_{\mu\nu}$ is written on the right side of the equation with the stress energy tensor $T_{\mu\nu}$, and is interpreted as the *stress energy contribution of the gravitational radiation itself.* We shall have more to say on this in section 7.4. In linear theory, $\tau_{\mu\nu}$ is neglected in comparison with the ordinary matter $T_{\mu\nu}$.

This is a bit disappointing to behold. Even the linearised Field Equations look to be a mess! But then, you may have forgotten that the raw Maxwell wave equations for the potentials are no present, either. You will permit me to remind you. Here are the equations for the scalar potential Φ and vector potential $\boldsymbol{A}$:

$$\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla\cdot A}) = -4\pi\rho \tag{341}$$

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla\cdot A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) = -\frac{4\pi}{c}\boldsymbol{J} \tag{342}$$

(Note: I have used esu units, which are much more natural for relativity. Here ρ is the electric charge density.) Do the following exercise!

Exercise. In covariant notation, with $A^\alpha = (\Phi, \boldsymbol{A})$ and $J^\alpha = (\rho, \boldsymbol{J}/c)$ representing respectively the potential and source term 4-vectors, the original general equations look a bit more presentable. The only contravariant 4-vectors that we can form which are second order in the derivatives of A^α are $\Box A^\alpha$ and $\partial^\alpha\partial_\beta A^\beta$. Show that if $\partial_\alpha J^\alpha = 0$ identically, then our equation relating A^α to J^α must be of the form

$$\Box A^\alpha - \partial^\alpha\partial_\beta A^\beta = CJ^\alpha$$

where C is a constant to be determined, and that this equation remains unchanged when the transformation $A^\alpha \to A^\alpha + \partial^\alpha\Lambda$ in made. This property is known as gauge-invariance. We will shortly see something very analogous in general relativity. In the meantime, how do we determine C?

Remember the story here. Work in the "Lorenz gauge," which we are always free to do:

$$\boldsymbol{\nabla\cdot A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0 \tag{343}$$

In covariant 4-vector language, this is just $\partial_\alpha A^\alpha = 0$. Then, the dynamical equations simplify:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial\Phi}{\partial t^2} = \Box\Phi = -4\pi\rho \tag{344}$$

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = \Box\boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J} \tag{345}$$

This is nicer. Physically transparent Lorentz-invariant wave equations emerge. Might something similar happen for the Einstein Field Equations?

That the answer might be YES is supported by noticing that $G^{(1)}_{\mu\nu}$ can be written entirely in terms of the "Bianchi-like" quantity

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{\eta_{\mu\nu}h}{2}, \quad \text{or} \quad \bar{h}^\mu_\nu = h^\mu_\nu - \frac{\delta^\mu_\nu h}{2}. \tag{346}$$

Using this in (338), the *linearised* Field Equation becomes

$$2G^{(1)}_{\mu\nu} = \Box\bar{h}_{\mu\nu} - \frac{\partial^2\bar{h}^\lambda_\mu}{\partial x^\nu\partial x^\lambda} - \frac{\partial^2\bar{h}^\lambda_\nu}{\partial x^\mu\partial x^\lambda} + \eta_{\mu\nu}\frac{\partial^2\bar{h}^{\lambda\rho}}{\partial x^\lambda\partial x^\rho} = -\frac{16\pi G T_{\mu\nu}}{c^4}. \tag{347}$$

(It is easiest to verify this by starting with (347), substituting with (346), and showing that this leads to (338).)

Interesting. Except for $\Box\bar{h}_{\mu\nu}$, every term in this equation involves the divergence of $\bar{h}^\mu_\nu$ or $\bar{h}^{\mu\nu}$. Hmmm. Shades of Maxwell's $\partial A^\alpha/\partial x^\alpha$. In the Maxwell case, the freedom of gauge invariance allowed us to pick the gauge in which $\partial A^\alpha/\partial x^\alpha = 0$. Does our equation have a gauge invariance that will allow us to do the same for gravitational radiation so that we can set these $\bar{h}$-divergence derivatives to zero?

It does. Go back to equation (338) and on the right side, change $h_{\mu\nu}$ to $h'_{\mu\nu}$, where

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial\xi_\nu}{\partial x^\mu} - \frac{\partial\xi_\mu}{\partial x^\nu}, \tag{348}$$

and the ξ_μ represent any vector function. You will find that the form of the equation is completely unchanged, i.e. the ξ_μ terms cancel out identically! This is a true gauge invariance.

In this case, what is happening is that an infinitesimal coordinate transformation itself is *acting* as a gauge transformation. If

$$x'^\mu = x^\mu + \xi^\mu(x), \quad \text{or} \quad x^\mu = x'^\mu - \xi^\mu(x') \qquad \text{to lead order.} \tag{349}$$

then

$$g'_{\mu\nu} = \eta'_{\mu\nu} + h'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma} = \left(\delta^\rho_\mu - \frac{\partial\xi^\rho}{\partial x^\mu}\right)\left(\delta^\sigma_\nu - \frac{\partial\xi^\sigma}{\partial x^\nu}\right)(\eta_{\rho\sigma} + h_{\rho\sigma}) \tag{350}$$

With η' identical to η, we must have

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial\xi_\nu}{\partial x^\mu} - \frac{\partial\xi_\mu}{\partial x^\nu} \tag{351}$$

as before. Though closely related, don't confuse *general covariance* under coordinate transformations with this gauge transformation. Unlike general covariance, the gauge transformation works without actually changing the coordinates! We keep the same x's and add a group of certain functional derivatives to the $h_{\mu\nu}$, analogous to adding a gradient $\boldsymbol{\nabla}\Phi$ to $\boldsymbol{A}$ in Maxwell's theory. We find that the equations remain identical, just as we would find if we took $\boldsymbol{\nabla}\times(\boldsymbol{A}+\boldsymbol{\nabla}\Phi)$ in the Maxwell case.

Pause for a moment. In general relativity, don't we actually need to change the coordinates when we...well, when we change the coordinates? What is going on here? Keeping the coordinates is not an option, is it? Change the $h^{\mu\nu}$ tensor components but leave the coordinates untouched? Why should that work?

Let me try to clarify what has always struck me as a genuinely confusing point. (If it is all clear to you already, or you willing to take this as it comes, feel free to skip this paragraph.) If we did a *full* coordinate transformation, we would of course find that the *full* Einstein tensor wave equation would also have the (nonlinear) solution $h'_{\mu\nu}$, in x' coordinates. The tensorial form of the field equations is built in just that way. Here, however, we are working only with the part linear in h, and linear in ξ, assuming these are comparable. So imagine doing the *full* transformation, but approaching it *order by order* in h or ξ. Every order in h has to independently cooperate: $\bar{h}_{\mu\nu}$ must be a solution to the equations when we keep only the linear terms by themselves. Then we must find that it is still a solution when we work with the quadratic terms, which cancel amongst themselves, and so on. We start first with all the terms linear in h, the largest terms to worry about. In the linearised equation we change the h's by adding the ξ derivatives following the equation (351) prescription. The additional terms generated are of order $\partial\xi/\partial x$. Okay, noted, very good. Now that we've modifed the h's, continue on with the same infinitesimal coordinate transformation, next applied to the $\partial/\partial x^\mu$ derivatives, to get those transformed as well to linear order. Ah. Interesting. The new ξ-terms generated are of order $(\partial\xi/\partial x')(\partial h'/\partial x)$:

$$\frac{\partial h'^{\mu\nu}}{\partial x'^{\mu}} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial h'^{\mu\nu}}{\partial x^\rho} = \frac{\partial h'^{\mu\nu}}{\partial x^\mu} - \frac{\partial \xi^\rho}{\partial x'^\mu}\frac{\partial h'^{\mu\nu}}{\partial x^\rho},$$

and remember that ξ and h' are *both* supposed to be small. By contrast, the change of h to h' via (351) gave us additional terms in our equation which are an order larger: $\partial\xi/\partial x$, not the product of $\partial_x\xi$ with $\partial_x h'$. If we had to additively combine $\partial_x\xi$ terms with the product terms $\partial_x\xi\,\partial_x h$, we would be blending orders in h and ξ that don't match! Do you see what this means? Since $h'(x')$ is a solution of the full tensor wave equation, it must also be of the more restricted linearised equation, when h is small. Changing h as per equation (351) generates relatively big linear terms, and then continuing our duty and changing x to x' in the derivatives actually generates only little stuff. The little stuff cannot cancel out the the big stuff, the $\partial_x\xi$ terms that we have generated from (351). So how do we get rid of those much bigger $\partial_x\xi$ terms, as we must in order to ensure that h' really is a solution of the linear equation? The answer is that we don't have to actively get rid of those terms. The equation kills those terms for us, all by itself when we add them all up. Miracle? No. *This is exactly how a coordinate transformation must behave.* That is the beauty of it: it reduces to a gauge-invariant theory in the linear regime. Even without transforming the partial derivatives explicitly, the largest $\partial_x\xi$ terms in the gauge transformation cancel one another. In the full theory, members of the "quadratic club," terms of order $(\partial_x\xi)\partial_x h$, will ultimately cancel out too. But they do so amongst *themselves,* thank you very much. Quadratic members only please. We are a higher order than you linear fellows.

Understanding the gauge properties of the gravitational wave equation was very challenging in the early days of the subject. The opening "speed-of-thought" quotation of this chapter by Eddington is taken somewhat out of context. What he really said in his famous paper (Eddington A.S. 1922 *Proc. Roy. Soc. A,* **102,** 716, 268) is the following:

"Weyl has classified plane gravitational waves into three types, viz.: (1) longitudinal-longitudinal; (2) longitudinal-transverse; (3) transverse-transverse. The present investigation leads to the conclusion that transverse-transverse waves are propagated with the speed of light *in all systems of co-ordindates*. Waves of the first and second types have no fixed velocity—a result which rouses suspicion as to their objective existence. Einstein had also become suspicious of these waves (in so far as they occur in his special co-ordinate system) for another reason, because he found they convey no energy. They are not objective and (like absolute velocity) are not detectable by any conceivable experiment. They are merely sinuosities in the co-ordinate system, and the only speed of propagation relevant to them is the 'speed of thought.' "

The quotation is often taken to be dismissive of the entire notion of gravitational radiation, which it clearly is not. Rather, it is directed toward those solutions which we would now say are gauge-dependent (either of the first two types of waves, which involve at least one longitudinal component) and those which are gauge-independent (the third, completely transverse, type). Physical solutions must ultimately be gauge independent. Matters would have been clear to someone who bothered to examine the components of the Riemann curvature tensor. The first two types of waves would have produced an identically zero $R^{\lambda}{}_{\mu\nu\kappa}$. They produce no curvature; they are indeed "merely sinuosities in the co-ordinate system," and they are are unphysical.

Back to our problem. Just as the Lorenz gauge $\partial_\alpha A^\alpha = 0$ was useful in the case of Maxwell's equations, so now is the so-called *harmonic gauge*:

$$\frac{\partial \bar{h}^{\mu}_{\nu}}{\partial x^{\mu}} = \frac{\partial h^{\mu}_{\nu}}{\partial x^{\mu}} - \frac{1}{2}\frac{\partial h}{\partial x^{\nu}} = 0 \tag{352}$$

In this gauge, the Field Equations (347) take the "wave-equation" form

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G T_{\mu\nu}}{c^4}} \tag{353}$$

How we can be sure that, even with our gauge freedom, we can find the right ξ^μ to get into a harmonic gauge and ensure the emergence of (353)? Well, if we have been unfortunate enough to be working in a gauge in which equation (352) is not satisfied, then form $h'_{\mu\nu}$ à la equation (351) and demand that $\partial h'^{\mu}_{\nu}/\partial x^\mu = (1/2)\partial h'/\partial x^\nu$. We find that this implies

$$\Box \xi_\nu = \frac{\partial \bar{h}^{\mu}_{\nu}}{\partial x^{\mu}}, \tag{354}$$

a wave equation for ξ_ν identical in form to (353). For this equation, a solution certainly exists. Indeed, our experience with electrodynamics has taught us that the solution to the fundamental radiation equation (353) takes the form

$$\bar{h}_{\mu\nu}(\boldsymbol{r}, t) = \frac{4G}{c^4}\int \frac{T_{\mu\nu}(\boldsymbol{r'}, t - R/c)}{R}\, d^3r', \qquad R \equiv |\boldsymbol{r} - \boldsymbol{r'}| \tag{355}$$

and hence a similar solution exisits for (354). The $\bar{h}_{\mu\nu}$, like their electrodynamic counterparts, are determined at time t and location $\boldsymbol{r}$ by a source intergration over $\boldsymbol{r'}$ taken at the retarded times $t' \equiv t - R/c$. In other words, disturbances in the gravitational field travel at a finite speed, the speed of light c.

Exercise. Show that for a source with motions near the speed of light, like merging black holes, $\bar{h}_{\mu\nu}$ (or $h_{\mu\nu}$ for that matter) is of order R_S/r, where R_S is the Schwarzschild radius based on the total mass of the system in question and r is the distance to the source. You want to know how big $h_{\mu\nu}$ is going to be in your detector when black holes merge? Count the number of expected Schwarzschild radii to the source and take the reciprocal. With $M_\odot^{tot}$ equal to the total mass measured in solar masses, show that $h_{\mu\nu} \sim 3M_\odot^{tot}/r_{km}$, measuring r in km. We are pushing our weak field approximation here, but to this order it works fine. We'll give a sharper estimate shortly.

7.1.1 Come to think of it...

You may not have actually seen the solution (355) before, or maybe, you know, you just need a little reminding. It is important. Let's derive it.

Consider the equation

$$-\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} + \nabla^2\Psi = -4\pi f(\boldsymbol{r}, t) \tag{356}$$

We specialise to the Green's function solution

$$-\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} + \nabla^2 G = -4\pi\delta(\boldsymbol{r})\delta(t) \tag{357}$$

Of course, our particular choice of origin is immaterial, as is our zero of time, so that we shall replace $\boldsymbol{r}$ by $\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{r'}$ ($R \equiv |\boldsymbol{R}|$), and t by $\tau \equiv t - t'$ at the end of the calculation, with the primed values being fiducial reference points. The form of the solution we find here will still be valid with the shifts of space and time origins.

Fourier transform (357) by integrating over $\int e^{i\omega t} dt$ and denote the fourier transform of G by $\tilde{G}$:

$$k^2\tilde{G} + \nabla^2\tilde{G} = -4\pi\delta(\boldsymbol{r}) \tag{358}$$

where $k^2 = \omega^2/c^2$. Clearly $\tilde{G}$ is a function only of r, hence the solution to the homogeneous equation away from the origin,

$$\frac{d^2(r\tilde{G})}{dr^2} + k^2(r\tilde{G}) = 0,$$

is easily found to be $\tilde{G} = e^{\pm ikr}/r$. The delta function behaviour is actually *already* included here, as can be seen by taking the limit $k \to 0$, in which we recover the correct potential of a point charge, with the proper normalisation already in place. The back transform gives

$$G = \frac{1}{2\pi r}\int_{-\infty}^{\infty} e^{\pm ikr - i\omega t}\, d\omega = \frac{1}{2\pi r}\int_{-\infty}^{\infty} e^{-i\omega(t\mp r/c)}\, d\omega \tag{359}$$

which we recognise as a Dirac delta function (remember $\omega/k = c$):

$$G = \frac{\delta(t \mp r/c)}{r} \to \frac{\delta(t - r/c)}{r} \to \frac{\delta(\tau - R/c)}{R} \tag{360}$$

where we have selected the *retarded* time solution $t - r/c$ as a nod to causality, and moved thence to $(\tau,\, R)$ variables for an arbitary time and space origin. We see that a flash at $t = t'$

located at $\boldsymbol{r} = \boldsymbol{r'}$ produces an effect at a time R/c later, at a distance R from the flash. The general solution constructed from our Green's function is

$$\Psi = \int \frac{f(\boldsymbol{r'}, t')}{R} \delta(t - t' - R/c) dt' \boldsymbol{dr'} = \int \frac{f(\boldsymbol{r'}, t')}{R} \boldsymbol{dr'} \tag{361}$$

where in the final integral we have set $t' = t - R/c$, the retarded time. Remember that t' depends on both $\boldsymbol{r}$ and $\boldsymbol{r'}$.

7.2 Plane waves

To understand more fully the solution (355), consider the problem in which $T_{\mu\nu}$ has an oscillatory time dependence, $e^{-i\omega t'}$. Since we are dealing with a linear theory, this isn't particularly restrictive, since any well-behaved time dependence can be represented by a Fourier sum. The source, say a binary star system, occupies a finite volume. We seek the solution for $\bar{h}_{\mu\nu}$ at distances huge compared with the scale of the source itself, i.e. $r \gg r'$. Then,

$$R \simeq r - \boldsymbol{e_r} \cdot \boldsymbol{r'} \tag{362}$$

where $\boldsymbol{e_r}$ is a unit vector in the $\boldsymbol{r}$ direction, and

$$\bar{h}_{\mu\nu}(\boldsymbol{r}, t) \simeq \exp[i(kr - \omega t)] \, \frac{4G}{rc^4} \int T_{\mu\nu}(\boldsymbol{r'}) \exp(-i\boldsymbol{k} \cdot \boldsymbol{r'}) \, d^3r \tag{363}$$

with $\boldsymbol{k} = (\omega/c)\boldsymbol{e_r}$ the wavenumber in the radial direction. Since r is huge, this has the asymptotic form of a plane wave. Hence, $\bar{h}_{\mu\nu}$ and thus $h_{\mu\nu}$ itself have the form of simple plane waves, travelling in the radial direction, at large distances from the source generating them. These waves turn out to have some remarkable polarisation properties, which we now discuss.

7.2.1 The transverse-traceless (TT) gauge

Consider a traveling plane wave for $h_{\mu\nu}$, orienting our z axis along $\boldsymbol{k}$, so that

$$k^0 = \omega/c, \ k^1 = 0, \ k^2 = 0, \ k^3 = \omega/c \qquad \text{and} \qquad k_0 = -\omega/c, \ k_i = k^i \tag{364}$$

where as usual we raise and lower indices with $\eta_{\mu\nu}$or its numerical identical dual $\eta^{\mu\nu}$.

Then $h_{\mu\nu}$ takes the form

$$h_{\mu\nu} = e_{\mu\nu} a \, \exp(ik_\rho x^\rho) \tag{365}$$

where a is an amplitude and $e_{\mu\nu} = e_{\nu\mu}$ a polarisation tensor, again with the η's raising and lowering subscripts. Thus

$$e_{ij} = e^i_j = e^{ij} \tag{366}$$

$$e^{0i} = -e^i_0 = e^0_i = -e_{0i} \tag{367}$$

$$e^{00} = e_{00} = -e^0_0 \tag{368}$$

The harmonic constraint

$$\frac{\partial h^\mu_\nu}{\partial x^\mu} = \frac{1}{2} \frac{\partial h^\mu_\mu}{\partial x^\nu} \tag{369}$$

implies

$$k_\mu e^\mu_\nu = k_\nu e^\mu_\mu/2 \tag{370}$$

For $\nu = 0$ this means

$$k_0 e^0_0 + k_3 e^3_0 = k_0(e^i_i + e^0_0)/2, \tag{371}$$

or

$$-(e_{00} + e_{30}) = (e_{ii} - e_{00})/2. \tag{372}$$

When $\nu = j$ (a spatial index),

$$k_0 e^0_j + k_3 e^3_j = k_j(e_{ii} - e_{00})/2 \tag{373}$$

The $j = 1$ and $j = 2$ cases reduce to

$$e_{01} + e_{31} = e_{02} + e_{32} = 0, \tag{374}$$

while $j = 3$ yields

$$e_{03} + e_{33} = (e_{ii} - e_{00})/2 = -(e_{00} + e_{03}) \tag{375}$$

Equations (374) and the first=last equality of (375) yield

$$e_{01} = -e_{31},\ e_{02} = -e_{32},\ e_{03} = -(e_{00} + e_{33})/2 \tag{376}$$

Using the above expression for e_{03} in the first=second equality of (375) then gives

$$e_{22} = -e_{11} \tag{377}$$

Of the 10 independent components of the symmetric $e_{\mu\nu}$ the harmonic condition (369) thus enables us to express e_{0i} and e_{22} in terms of e_{3i}, e_{00}, and e_{11}. These latter 5 components plus a sixth, e_{12}, remain unconstrained for the moment.

But wait! We have not yet used the gauge freedom of equation (351) *within* the harmonic constraint. We can still continue to eliminate components of $e_{\mu\nu}$. In particular, let us choose

$$\xi_\mu(x) = i\epsilon_\mu \exp(ik_\rho x^\rho) \tag{378}$$

where the ϵ_μ are four constants to be chosen. This satisfies $\Box\xi_\mu$=0, and therefore does not change the harmonic coordinate condition, $\partial_\mu \bar{h}^\mu_\nu = 0$. Then following the prescription of (351), we generate a new, but physically equivalent polarisation tensor,

$$e'_{\mu\nu} = e_{\mu\nu} + k_\mu \epsilon_\nu + k_\nu \epsilon_\mu \tag{379}$$

and by choosing the ϵ_μ appropriately, we can eliminate all of the $e'_{\mu\nu}$ except for e'_{11}, $e'_{22} = -e'_{11}$, and e'_{12}. In particular, using (379),

$$e'_{11} = e_{11}, \qquad e'_{12} = e_{12} \tag{380}$$

unchanged. But with $k = \omega/c$,

$$e'_{13} = e_{13} + k\epsilon_1, \quad e'_{23} = e_{23} + k\epsilon_2, \quad e'_{33} = e_{33} + 2k\epsilon_3, \quad e'_{00} = e_{00} - 2k\epsilon_0, \tag{381}$$

so that these four components may be set to zero by a simple choice of the ϵ_μ. When working with plane waves we may always choose this gauge, which is *transverse* (since the only e_{ij} components that are present are transverse to the z direction of propagation) and traceless (since $e_{11} = -e_{22}$). Oddly enough, this gauge is named the transverse-traceless (TT) gauge. *Notice that in the TT gauge, $h_{\mu\nu}$ vanishes if any of its indices are 0, whether raised or lowered.*

7.3 The quadrupole formula

In the limit of large r ("compact source approximation"), equation (355) is:

$$\bar{h}^{\mu\nu}(\boldsymbol{r}, t) = \frac{4G}{rc^4} \int T^{\mu\nu}(\boldsymbol{r}', t') d^3r', \tag{382}$$

where $t' = t - r/c$ is the retarded time. Moreover, for the TT gauge, we are interested in the spatial ij components of this equation, since all time indices vanish. (Also, because $\bar{h}_{\mu\nu}$ is traceless, we need not distinguish between h and $\bar{h}$.) The integral over T_{ij} may be cast in a very convenient form as follows.

$$0 = \int \frac{\partial(x'^j T^{ik})}{\partial x'^k} \, d^3r' = \int \left(\frac{\partial T^{ik}}{\partial x'^k}\right) x'^j \, d^3r' + \int T^{ij} \, d^3r', \tag{383}$$

where the first equality follows because the first integral reduces to a surface integration of T^{ik} at infinity, where it is presumed to vanish. Thus

$$\int T^{ij} \, d^3r' = -\int \left(\frac{\partial T^{ik}}{\partial x'^k}\right) x'^j \, d^3r' = \int \left(\frac{\partial T^{i0}}{\partial x'^0}\right) x'^j \, d^3r' = \frac{1}{c}\frac{d}{dt'} \int T^{i0} x'^j \, d^3r' \tag{384}$$

where the second equality uses the conservation of $T^{\mu\nu}$. Remember that t' is the retarded time. As T_{ij} is symmetric in its indices,

$$\frac{d}{dt'} \int T^{i0} x'^j \, d^3r' = \frac{d}{dt'} \int T^{j0} x'^i \, d^3r' \tag{385}$$

Continuing in this same spirit,

$$0 = \int \frac{\partial(T^{0k} x'^i x'^j)}{\partial x'^k} \, d^3r' = \int \left(\frac{\partial T^{0k}}{\partial x'^k}\right) x'^i x'^j \, d^3r' + \int (T^{0i} x'^j + T^{0j} x'^i) \, d^3r' \tag{386}$$

Using exactly the same reasoning as before,

$$\int (T^{0i} x'^j + T^{0j} x'^i) \, d^3r' = \frac{1}{c}\frac{d}{dt'} \int T^{00} x'^i x'^j \, d^3r' \tag{387}$$

Therefore,

$$\int T^{ij} \, d^3r' = \frac{1}{2c^2}\frac{d^2}{dt'^2} \int T^{00} x'^i x'^j \, d^3r' \tag{388}$$

Inserting this in (382), we obtain the *quadrupole formula* for gravitational radiation:

$$\boxed{\bar{h}^{ij} = \frac{2G}{c^6 r}\frac{d^2 I^{ij}}{dt'^2}} \tag{389}$$

where I^{ij} is the *quadrupole-moment tensor* of the energy density:

$$I^{ij} = \int T^{00} x'^i x'^j \, d^3r' \tag{390}$$

To estimate this numerically, we write

$$\frac{d^2 I^{ij}}{dt'^2} \sim Ma^2c^2\omega^2 \tag{391}$$

where M is the characteristic mass of the rotating system, a an internal separation, and ω a characteristic frequency, an orbital frequency for a binary say. Then

$$\bar{h}^{ij} \sim \frac{2GMa^2\omega^2}{c^4 r} \simeq 7\times 10^{-22}(M/M_\odot)(a_{11}^2\omega_7^2/r_{100}) \tag{392}$$

where $M/M_\odot$ is the mass in solar masses, a_{11} the separation in units of 10^{11} cm (about a separation of one solar radius), ω_7 the frequency associated with a 7 hour orbital period (similar to PSR193+16) and r_{100} the distance in units of 100 parsecs, some 3×10^{20} cm. A typical rather large h one might expect at earth from a local astronomical source is then of order 10^{-21}.

What about the LIGO source, GW150914? How does our formula work in this case? The distance in this case is cosmological, not local, with $r = 1.2\times 10^{22}$ km, or in astronomical parlance, about 400 megaparsecs (Mpc). In this case, we write (392) as

$$\bar{h}^{ij} \sim \frac{2GMa^2\omega^2}{c^4 r} = \left(\frac{2.9532}{r_{km}}\right)\left(\frac{M}{M_\odot}\right)\left(\frac{a\omega}{c}\right)^2 \simeq 1\times 10^{-22}\frac{M/M_\odot}{r_{Gpc}}\left(\frac{a\omega}{c}\right)^2, \tag{393}$$

since $2GM_\odot/c^2$ is just the Sun's Schwarzschild radius. (One Gpc$=10^3$Mpc $= 3.0856\times 10^{22}$km.) The point is that $(a\omega/c)^2$ is a number not very different from 1 for a relativistic source, perhaps 0.1 or so. Plugging in numbers with $M/M_\odot = 60$ and $(a\omega/c)^2 = 0.1$, we find $\bar{h}_{ij} = 1.5\times 10^{-21}$, just about as observed at peak amplitude.

Exercise. Prove that $\bar{h}^{ij}$ given by (389) is an *exact* solution of $\Box\bar{h}^{ij} = 0$, for any r, even if r is not large.

7.4 Radiated Energy

7.4.1 A useful toy problem

We have yet to make the link between $h_{\mu\nu}$ and the actual energy flux that is carried off by these time varying metric coefficients. Relating metric coefficients to energy is not trivial. To see how to do this, start with a simpler toy problem. Imagine that the wave equation for general relativity looked like this:

$$-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} + \nabla^2\Phi = 4\pi G\rho \tag{394}$$

This is what a relativisitic theory would look like if the source ρ were just a simple scalar quantity, instead of a component of a stress tensor. Then, if we multiply by $(1/4\pi G)\partial\Phi/\partial t$, integrate $(\partial\Phi/\partial t)\nabla^2\Phi$ by parts and regroup, this leads to

$$-\frac{1}{8\pi G}\frac{\partial}{\partial t}\left[\left(\frac{\partial\Phi}{\partial t}\right)^2 + |\boldsymbol{\nabla}\Phi|^2\right] + \boldsymbol{\nabla\cdot}\left(\frac{1}{4\pi G}\frac{\partial\Phi}{\partial t}\boldsymbol{\nabla}\Phi\right) = \rho\frac{\partial\Phi}{\partial t}. \tag{395}$$

But

$$\rho\frac{\partial\Phi}{\partial t} = \frac{\partial(\rho\Phi)}{\partial t} - \Phi\frac{\partial\rho}{\partial t} = \frac{\partial(\rho\Phi)}{\partial t} + \Phi\boldsymbol{\nabla\cdot}(\rho\boldsymbol{v}) = \frac{\partial(\rho\Phi)}{\partial t} + \boldsymbol{\nabla\cdot}(\rho\boldsymbol{v}\Phi) - \rho\boldsymbol{v\cdot\nabla}\Phi \tag{396}$$

where $\boldsymbol{v}$ is the velocity and the mass conservation equation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla\cdot}(\rho\boldsymbol{v}) = 0$$

has been used in the second "$=$ " sign from the left. Combining (395) and (396), and then rearranging the terms a bit leads to

$$\frac{\partial}{\partial t}\left[\rho\Phi + \frac{1}{8\pi G}\left(\left(\frac{\partial\Phi}{\partial t}\right)^2 + |\boldsymbol{\nabla}\Phi|^2\right)\right] + \boldsymbol{\nabla\cdot}\left(\rho\boldsymbol{v}\Phi - \frac{1}{4\pi G}\frac{\partial\Phi}{\partial t}\boldsymbol{\nabla}\Phi\right) = \rho\boldsymbol{v\cdot\nabla}\Phi \tag{397}$$

The right side is just minus the rate at which work is being done on the sources per unit volume. (The force per unit volume, you recall, is $-\rho\boldsymbol{\nabla}\Phi$.) For the usual case of interest when the source ρ vanishes outside a certain radius, the left side may then be readily interpreted as a far-field wave energy density of $[(\partial_t\Phi)^2 + |\boldsymbol{\nabla}\Phi|^2]/8\pi G$ and a wave energy flux of $-(\partial_t\Phi)\boldsymbol{\nabla}\Phi/4\pi G$. (Is the sign of the flux sensible for outgoing waves?) The question we raise here is whether an analogous method might work on the more involved linear wave equation of tensorial general relativity. The answer is YES, but we have to set things up properly. We can't be casual. And, needless to say, it is a bit more messy index-wise!

7.5 A conserved energy flux for linearised gravity

Start with equation (347):

$$\Box\bar{h}_{\mu\nu} - \frac{\partial^2\bar{h}^\lambda_\mu}{\partial x^\nu\partial x^\lambda} - \frac{\partial^2\bar{h}^\lambda_\nu}{\partial x^\mu\partial x^\lambda} + \eta_{\mu\nu}\frac{\partial^2\bar{h}^{\lambda\rho}}{\partial x^\lambda\partial x^\rho} = -\frac{16\pi G T_{\mu\nu}}{c^4}. \tag{398}$$

Contract on $\mu\nu$: the first term on the left becomes $\Box\bar{h}$, the second and third each become $-\partial^2\bar{h}^{\lambda\rho}/\partial x^\lambda\partial x^\rho$, while the final contraction turns $\eta_{\mu\nu}$ into a factor of 4. (Why?) This leads us to

$$\Box\bar{h} + 2\frac{\partial^2\bar{h}^{\lambda\rho}}{\partial x^\lambda\partial x^\rho} = -\kappa T \tag{399}$$

where we have written $\kappa = 16\pi G/c^4$. We then recast our original equation as

$$\Box\bar{h}_{\mu\nu} - \frac{\partial^2\bar{h}^\lambda_\mu}{\partial x^\nu\partial x^\lambda} - \frac{\partial^2\bar{h}^\lambda_\nu}{\partial x^\mu\partial x^\lambda} + \eta_{\mu\nu}\Box\bar{h} = -\kappa S_{\mu\nu} \tag{400}$$

where we have introduced the source function

$$S_{\mu\nu} = T_{\mu\nu} - \frac{\eta_{\mu\nu}T}{2} \tag{401}$$

Now multiply (400) by $\partial\bar{h}^{\mu\nu}/\partial x^\sigma$, summing over μ and ν as usual but keeping σ free. The first term on the left becomes

$$\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}\Box\bar{h}_{\mu\nu} = \frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}\frac{\partial^2\bar{h}_{\mu\nu}}{\partial x_\rho\partial x^\rho} = \frac{\partial}{\partial x_\rho}\left(\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}\right) - \frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial^2\bar{h}^{\mu\nu}}{\partial x_\rho\partial x^\sigma}$$

$$= \frac{\partial}{\partial x_\rho}\left(\frac{\partial \bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial \bar{h}^{\mu\nu}}{\partial x^\sigma}\right) - \frac{\partial \bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial}{\partial x^\sigma}\frac{\partial \bar{h}^{\mu\nu}}{\partial x_\rho} = \frac{\partial}{\partial x_\rho}\left(\frac{\partial \bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial \bar{h}^{\mu\nu}}{\partial x^\sigma}\right) - \frac{\partial}{\partial x^\sigma}\left(\frac{1}{2}\frac{\partial \bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial \bar{h}^{\mu\nu}}{\partial x_\rho}\right) \quad (402)$$

Do you see why the final equality is valid for the $\partial/\partial x^\sigma$ exact derivative? It doesn't matter which group of $\mu\nu$ on the $\bar{h}$'s is the up group and which is the down group.

Now that you've seen the tricks of the trade, you should be able to juggle the indices with me and recast all the terms as exact derivatives: we are aiming to get a pure divergence on the left side. The second term is

$$-\frac{\partial^2 \bar{h}^\lambda_\mu}{\partial x^\nu \partial x^\lambda}\frac{\partial \bar{h}^{\mu\nu}}{\partial x^\sigma} = -\frac{\partial^2 \bar{h}^{\lambda\mu}}{\partial x_\nu \partial x^\lambda}\frac{\partial \bar{h}_{\mu\nu}}{\partial x^\sigma} = -\frac{\partial}{\partial x_\nu}\left(\frac{\partial \bar{h}^{\lambda\mu}}{\partial x^\lambda}\frac{\partial \bar{h}_{\mu\nu}}{\partial x^\sigma}\right) + \frac{\partial \bar{h}^{\lambda\mu}}{\partial x^\lambda}\frac{\partial^2 \bar{h}_{\mu\nu}}{\partial x^\sigma \partial x_\nu}$$

or, replacing ν with ρ in the first group on the right,

$$-\frac{\partial^2 \bar{h}^\lambda_\mu}{\partial x^\nu \partial x^\lambda}\frac{\partial \bar{h}^{\mu\nu}}{\partial x^\sigma} = -\frac{\partial}{\partial x_\rho}\left(\frac{\partial \bar{h}^{\lambda\mu}}{\partial x^\lambda}\frac{\partial \bar{h}_{\mu\rho}}{\partial x^\sigma}\right) + \frac{1}{2}\frac{\partial}{\partial x^\sigma}\left(\frac{\partial \bar{h}^{\lambda\mu}}{\partial x^\lambda}\frac{\partial \bar{h}_{\mu\nu}}{\partial x_\nu}\right) \quad (403)$$

The third term is

$$-\frac{\partial^2 \bar{h}^\lambda_\nu}{\partial x^\mu \partial x^\lambda}\frac{\partial \bar{h}^{\mu\nu}}{\partial x^\sigma}$$

But this is exactly the same as the term we've just done: just interchange the dummy indices μ and ν and remember that $\bar{h}^{\mu\nu}$ is symmetric in $\mu\nu$. So there is no need to do any more here. The fourth and final term of the left side of equation is

$$-\frac{1}{2}\frac{\partial \bar{h}}{\partial x^\sigma}\frac{\partial^2 \bar{h}}{\partial x^\rho \partial x_\rho} = -\frac{1}{2}\frac{\partial}{\partial x_\rho}\left(\frac{\partial \bar{h}}{\partial x^\rho}\frac{\partial \bar{h}}{\partial x^\sigma}\right) + \frac{1}{4}\frac{\partial}{\partial x^\sigma}\left(\frac{\partial \bar{h}}{\partial x^\rho}\frac{\partial \bar{h}}{\partial x_\rho}\right). \quad (404)$$

Thus, after dividing our fundamental equation by 2κ, the left side of equation (400) takes on a nice compact form, and we find

$$\frac{\partial \mathcal{U}_{\rho\sigma}}{\partial x_\rho} = -\frac{1}{2}S_{\mu\nu}\frac{\partial \bar{h}^{\mu\nu}}{\partial x^\sigma}, \quad (405)$$

where

$$\mathcal{U}_{\rho\sigma} = \mathcal{T}_{\rho\sigma} + \eta_{\rho\sigma}\mathcal{S}. \quad (406)$$

$\mathcal{S}$ is the scalar density:

$$\mathcal{S} = -\left(\frac{1}{4\kappa}\frac{\partial \bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial \bar{h}^{\mu\nu}}{\partial x_\rho}\right) + \frac{1}{2\kappa}\left(\frac{\partial \bar{h}^{\lambda\mu}}{\partial x^\lambda}\frac{\partial \bar{h}_{\mu\nu}}{\partial x_\nu}\right) + \frac{1}{8\kappa}\left(\frac{\partial \bar{h}}{\partial x^\rho}\frac{\partial \bar{h}}{\partial x_\rho}\right), \quad (407)$$

and $\mathcal{T}_{\rho\sigma}$ is a flux tensor:

$$\mathcal{T}_{\rho\sigma} = \frac{1}{2\kappa}\left(\frac{\partial \bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial \bar{h}^{\mu\nu}}{\partial x^\sigma}\right) - \frac{1}{\kappa}\left(\frac{\partial \bar{h}^{\lambda\mu}}{\partial x^\lambda}\frac{\partial \bar{h}_{\mu\rho}}{\partial x^\sigma}\right) - \frac{1}{4\kappa}\left(\frac{\partial \bar{h}}{\partial x^\rho}\frac{\partial \bar{h}}{\partial x^\sigma}\right). \quad (408)$$

By working with plane waves in standard harmonic coordinates, $\partial \bar{h}^{\lambda\mu}/\partial x^\lambda = 0$, and $\mathcal{U}_{\rho\sigma}$ becomes symmetric in $\rho\sigma$. Remembering $k_\rho k^\rho = 0$ for the TT gauge, we find the simple result

$$\mathcal{U}_{\rho\sigma} = \frac{c^4}{32\pi G}\left(\frac{\partial \bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial \bar{h}^{\mu\nu}}{\partial x^\sigma}\right) \quad \text{(TT gauge)}. \quad (409)$$

Why did we choose to divide by 2κ for our overall constant? Why not just κ, or for that matter, 4κ? It is the right side of our energy equation that tells this story. This is

$$-\frac{1}{2}S_{\mu\nu}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma} = -\frac{1}{2}\left(T_{\mu\nu} - \frac{\eta_{\mu\nu}}{2}T\right)\left(\frac{\partial h^{\mu\nu}}{\partial x^\sigma} - \frac{\eta^{\mu\nu}}{2}\frac{\partial h}{\partial x^\sigma}\right) = -\frac{1}{2}T_{\mu\nu}\frac{\partial h^{\mu\nu}}{\partial x^\sigma}. \tag{410}$$

Choose $\sigma = 0$, the time component. We work in the Newtonian limit $h^{00} \simeq -2\Phi/c^2$, where Φ is a Newtonian gravitational potential. In the $\mu\nu$ summation on the right side of the equation, we are then dominated by the 00 components of both $h^{\mu\nu}$ and $T_{\mu\nu}$. Now, we are about to do a number of integration by parts. But we will always ignore the exact derivative! Why? Because the exact derivative of a perioidic function (and everything here *is* periodic) must oscillate away to zero on average. But in general the products of the periodic functions don't oscillate to zero; for example the average of $\cos^2(\omega t)^2 = 1/2$. Thus we keep these product terms, but only if they are not an exact derivative. Using the right arrow $\rightarrow$ to mean "integrate by parts and ignore the pure derivatives" (as inconsequential for wave losses), we perform the following manipulations on the right side of equation (410):

$$-\frac{1}{2}T_{00}\frac{\partial h^{00}}{\partial x^0} \rightarrow \frac{1}{2}\frac{\partial T_{00}}{\partial x^0}h^{00} = -\frac{1}{2}\frac{\partial T^{0i}}{\partial x^i}h^{00} \rightarrow \frac{1}{2}T^{0i}\frac{\partial h^{00}}{\partial x^i} \simeq -\rho\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{\nabla}\Phi, \tag{411}$$

where the first equality follows from $\partial_\nu T^{0\nu} = 0$. We have arrived on the right at an expression for the rate at which the effective Newtonian potential does net work on the matter. Why is that $1/c$ there? Don't worry, it cancels out with the same factor on the left (flux) side of the original equation. What about the sign of this? This expression is *negative* if the force $-\rho\boldsymbol{\nabla}\Phi$ is oppositely directed to the velocity, so that the source is losing energy by generating outgoing waves. Our harmonic gauge expression (409) for $\mathcal{T}_{0i}$ is also negative for an *outward* flowing wave that is a function of the argument $(r - ct)$, r being spherical radius and t time. By contrast, $\mathcal{T}^{0i}$ would be positive, as befits an outward moving wave energy.

The fact that division by 2κ produces a source corresponding to the rate at which work is done on the Newtonian sources (when $\sigma = 0$) means that our overall normalisation is indeed correct. The $\sigma = 0$ energy flux of (409) is the true energy flux of gravitational radiation in the weak field limit:

$$\mathcal{F}^i = \mathcal{F}_i = c\mathcal{T}^{i0} = -c\mathcal{T}_{i0} = -\frac{c^4}{32\pi G}\left(\frac{\partial\bar{h}_{\mu\nu}}{\partial x^i}\frac{\partial\bar{h}^{\mu\nu}}{\partial t}\right) \quad \text{(TT gauge)}. \tag{412}$$

7.6 The energy loss formula for gravitational waves

Our next step is to evaluate the transverse and traceless components of h_{ij}, denoted h_{ij}^{TT}, in terms of the transverse and traceless components of I_{ij}. Begin with the traceless component, denoted J_{ij}:

$$J_{ij} = I_{ij} - \frac{\delta_{ij}}{3}I \tag{413}$$

where I is the trace of I_{ij}. Next, we address the transverse property. The projection of a vector $\boldsymbol{v}$ onto a plane perpendicular to a unit direction vector $\boldsymbol{n}$ is accomplished simply by removing the component of $\boldsymbol{v}$ along $\boldsymbol{n}$. Denoting the resulting projected vector as $\boldsymbol{w}$,

$$\boldsymbol{w} = \boldsymbol{v} - (\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} \tag{414}$$

or

$$w_j = (\delta_{ij} - n_i n_j)v_i \equiv P_{ij}v_i \tag{415}$$

where we have introduced the projection tensor

$$P_{ij} = \delta_{ij} - n_i n_j,$$

with the easily shown properties

$$n_i P_{ij} = n_j P_{ij} = 0, \qquad P_{ij} P_{jk} = P_{ik}, \qquad P_{ii} = 2. \tag{416}$$

Projecting tensor components presents no difficulties,

$$w_{ij} = P_{ik} P_{jl} v_{kl} \rightarrow n_i w_{ij} = n_j w_{ij} = 0, \tag{417}$$

nor does the extraction of a projected tensor that is both traceless and transverse:

$$w_{ij}^{TT} = \left(P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \right) v_{kl} \quad \rightarrow w_{ii}^{TT} = (P_{ik} P_{il} - P_{kl}) v_{kl} = (P_{kl} - P_{kl}) v_{kl} = 0. \tag{418}$$

Let us define

$$J_{ij}^{TT} = \left(P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \right) J_{kl}. \tag{419}$$

Notice now that $(J_{ij} - J_{ij}^{TT}) J_{ij}^{TT}$ is the contraction of the nontranverse part of J_{ij} with its fully transverse part. It ought to vanish, if there is any justice. Happily, it does:

$$(J_{ij} - J_{ij}^{TT}) J_{ij}^{TT} = J_{ij} J_{ij}^{TT} - J_{kl} (P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl})(P_{im} P_{jn} - \frac{1}{2} P_{ij} P_{mn}) J_{mn} \tag{420}$$

Following the rules carefully in (416) and remembering $P_{ij} = P_{ji}$, this is

$$J_{ij} J_{ij}^{TT} - J_{kl} (P_{mk} P_{nl} - \frac{1}{2} P_{kl} P_{mn} - \cancel{\frac{1}{2} P_{mn} P_{kl}} + \cancel{\frac{1}{2} P_{mn} P_{kl}}) J_{mn} = J_{ij} J_{ij}^{TT} - J_{kl} J_{kl}^{TT} = 0 \tag{421}$$

This will come in very handy in a moment.

Next, we write down the traceless-transverse part of the quadrupole formula:

$$h_{ij}^{TT} = \frac{2G}{c^6 r} \frac{d^2 J_{ij}^{TT}}{dt'^2}. \tag{422}$$

Recalling that $t' = t - r/c$ and the J^{TT}'s are functions of t' (not t!),

$$\frac{\partial h_{ij}^{TT}}{\partial t} = \frac{2G}{c^6 r} \frac{d^3 J_{ij}^{TT}}{dt'^3}, \qquad \frac{\partial h_{ij}^{TT}}{\partial r} = -\frac{2G}{c^7 r} \frac{d^3 J_{ij}^{TT}}{dt'^3} \tag{423}$$

where, in the second expression we retain only the dominant term in $1/r$. The radial flux of gravitational waves is then given by (412):

$$\mathcal{F}_r = \frac{G}{8\pi r^2 c^9} \frac{d^3 J_{ij}^{TT}}{dt'^3} \frac{d^3 J_{ij}^{TT}}{dt'^3} \tag{424}$$

The $1/c^9$ dependence ultimately translates into a $1/c^5$ dependence for Newtonian sources, since each of the J's carries a c^2 factor.

The final step is to write out J_{ij}^{TT} in terms of the J_{ij} via the projection operator. It is here that the fact that J_{ij} is traceless is a computational help.

$$\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right) = (\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l) - \frac{1}{2}(\delta_{ij} - n_i n_j)(\delta_{kl} - n_k n_l) \tag{425}$$

Thus, with J_{kl} traceless, we find

$$J_{ij}^{TT} = \left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right) J_{kl} = J_{ij} + \frac{1}{2}(\delta_{ij} + n_i n_j) n_k n_l J_{kl} - n_i n_k J_{jk} - n_j n_k J_{ik} \tag{426}$$

If we now write

$$\dddot{J}_{ij}^{TT}\dddot{J}_{ij}^{TT} = [\dddot{J}_{ij} + (\dddot{J}_{ij}^{TT} - \dddot{J}_{ij})]\dddot{J}_{ij}^{TT}, \tag{427}$$

then we've seen in (420) and (421) that

$$(\dddot{J}_{ij}^{TT} - \dddot{J}_{ij})\dddot{J}_{ij}^{TT} = 0,$$

and we are left with

$$\dddot{J}_{ij}\dddot{J}_{ij}^{TT} \equiv \dddot{J}_{ij}(\dddot{J}_{ij} + \frac{1}{2}(\delta_{ij} + n_i n_j) n_k n_l \dddot{J}_{kl} - n_i n_k \dddot{J}_{jk} - n_j n_k \dddot{J}_{ik}) =$$

$$\dddot{J}_{ij}\dddot{J}_{ij} - 2 n_j n_k \dddot{J}_{ij}\dddot{J}_{ik} + \frac{1}{2} n_i n_j n_k n_l \dddot{J}_{ij}\dddot{J}_{kl} \tag{428}$$

We conclude:

$$\dddot{J}_{ij}^{TT}\dddot{J}_{ij}^{TT} = \dddot{J}_{ij}\dddot{J}_{ij}^{TT} = \dddot{J}_{ij}\dddot{J}_{ij} - 2\dddot{J}_{ij}\dddot{J}_{ik} n_j n_k + \frac{1}{2}\dddot{J}_{ij}\dddot{J}_{kl} n_i n_j n_k n_l \tag{429}$$

The gravitational wave luminosity is an integration of this distribution over all solid angles,

$$L_{GW} = \int r^2 \mathcal{F}_r \, d\Omega \tag{430}$$

To evaluate this, you will need

$$\int n_i n_j \, d\Omega = \frac{4\pi}{3}\delta_{ij}. \tag{431}$$

This is pretty simple: if the two vector components of n are not the same, the integral vanishes by symmetry (e.g. the average of xy over a sphere is zero). That means it is proportional to a delta function, say $C\delta_{ij}$. To get the constant of proportionality C, take the trace of both sides: $\int d\Omega = 4\pi = 3C$. More scary looking is the other identity you'll need:

$$\int n_i n_j n_k n_l \, d\Omega = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}), \tag{432}$$

but keep calm and think. The only way the integral cannot vanish is if two of the indices agree with one another and the remaining two indices also agree with one another. (Maybe the second pair is just the same pair as the first, maybe not.) This pairwise index agreement requirement is precisely what the symmetric combination of delta functions ensures, summed over the three different ways the agreement can occur. To get the $4\pi/15$ factor, set $i = j$ and sum, and the same thing with $l = k$. The integral on the left is then trivially $\int n_i n_i n_l n_l \, d\Omega =$

$\int d\Omega = 4\pi$. The combination of delta functions is $9 + 3 + 3 = 15$. Hence the normalisation factor $4\pi/15$. Putting this all together via (424), (429), (431) and (432), remembering $J_{ii} = 0$, and carrying out the angular integral, the total gravitational luminosity is given by a beautifully simple formula, first derived by Albert Einstein[7] in 1918:

$$L_{GW} = \frac{G}{8\pi c^9} \times \left[4\pi - 2 \times \frac{4\pi}{3} + \frac{1}{2} \times \frac{4\pi}{15} \times (0 + 1 + 1)\right] \dddot{J}_{ij} \dddot{J}_{ij}$$

which amounts to:

$$\boxed{L_{GW} = \frac{G}{5c^9} \dddot{J}_{ij} \dddot{J}_{ij} = \frac{G}{5c^9} \left(\dddot{I}_{ij} \dddot{I}_{ij} - \frac{1}{3} \dddot{I}_{ii} \dddot{I}_{jj} \right)} \qquad (433)$$

7.7 Gravitational radiation from binary stars

In W72, the detection of gravitational radiation looms as a very distant possibility, and rightly so. The section covering this topic devotes its attention to the possibility that rapidly rotating neutron stars might, just might, be a good source. Alas, for this to occur the neutron star would have to possess a sizeable and rapidly varying quadrupole moment, and this neutron stars do not seem to possess. Neutron stars are nearly exact spheres, even when rotating rapidly as pulsars. They are in essence perfectly axisymmetric; were they to have any quadrupole moment, it would hardly change with time.

The possibility that *Keplerian orbits* might be interesting from the point-of-view of measuring gravitational radiation is never mentioned in W72. Certainly ordinary orbits involving ordinary stars are not a promising source. But compact objects (white dwarfs, neutron stars or black holes) in very close binaries, with orbital periods measured in hours, were discovered within two years of the book's publication, and these turn out to be *extremely* interesting. They are the central focus of modern day gravitational wave research. As we have noted earlier, the first confirmation of the existence of gravitational radiation came from the binary pulsar system 1913+16, in which the change in the orbital period from the loss of wave energy was inferred via the changing interval of the arrival times of the pulsar signal. The radiation level of the gravitational waves itself was well below the threshold of direct detection at the time (and still today at the frequencies of interest). Over long enough time scales, a tight binary of compact objects, black holes in the most spectacular manifestation, may lose enough energy through gravitational radiation that the resulting inspiral goes all the way to completion and the system either coalesces or explodes. Predictions suggest that there are enough merging binaries in the universe to produce a rather high detection rate: several per year at a minimum. LIGO has already published its first detection, and given how quickly it was found when the threshold detector upgrade was made, there are grounds for optimism for more to come[8]. The final frenzied seconds of black holes coalescence will emit detectable gravitational wave signatures rich in physical content at frequencies that LIGO is tuned for. Such waveforms can now also be determined numerically to high precision (F. Pretorius 2005, Phys. Rev. Lett. 95, 121101). In the near future, they will very likely be detected on a regular basis.

Let us apply equation (433) to the case of two point masses in a classical Keplerian orbit. There is of course no contradiction between assuming a classical orbit and calculating its

[7] Actually, Einstein found a coefficient of 1/10, not 1/5. Eddington put matters right a few years later. Tricky business, this gravitational radiation.

[8] Update: yes indeed! There is now a second confirmed black hole merger, GW151226, and a third likely merger, though at a formal statistical level short of full GW status: LVT151012. LVT stands for "LIGO VIRGO Transient."

gravitational energy loss. We are working here in the regime in which the losses themselves exert only a tiny change on the orbit over one period, and the objects themselves, while close by ordinary astronomical standards, are separated by a distance well beyond their respective Schwarzschild radii. (Pretorius [2005] does not make this restriction, of course!)

The orbital elements are defined on page 71. The separation r of the two bodies is given as a function of azimuth ϕ as

$$r = \frac{L}{1 + \epsilon \cos\phi} \tag{434}$$

where L is the semilatus rectum and ϵ is the orbital eccentricity. With M being the total mass of the individual objects, $M = m_1 + m_2$, l the constant specific angular momentum (we forego J for angular momentum to avoid confusion with J_{ij}), and a is the semi-major axis, we have

$$r^2 \frac{d\phi}{dt} = l, \qquad L = \frac{l^2}{GM} = a(1 - \epsilon^2) \tag{435}$$

and thus

$$\frac{d\phi}{dt} = \left(\frac{GM}{a^3(1-\epsilon^2)^3}\right)^{1/2} (1 + \epsilon\cos\phi)^2 \qquad \frac{dr}{dt} = \left(\frac{GM}{a(1-\epsilon^2)}\right)^{1/2} \epsilon \sin\phi \tag{436}$$

The distance from the center-of-mass of each body is denoted r_1 and r_2. Writing these as vector quantities,

$$\boldsymbol{r_1} = \frac{m_2 \boldsymbol{r}}{M}, \qquad \boldsymbol{r_2} = -\frac{m_1 \boldsymbol{r}}{M} \tag{437}$$

Thus the coordinates in the xy orbital plane are

$$\boldsymbol{r_1} = \frac{m_2 r}{M}(\cos\phi, \sin\phi), \qquad \boldsymbol{r_2} = \frac{m_1 r}{M}(-\cos\phi, -\sin\phi) \tag{438}$$

The nonvanishing moment tensors I_{ij} are then

$$I_{xx} = \frac{m_1 m_2^2 + m_1^2 m_2}{M^2} r^2 \cos^2\phi = \mu r^2 \cos^2\phi \tag{439}$$

$$I_{yy} = \mu r^2 \sin^2\phi \tag{440}$$

$$I_{xy} = Iyx = \mu r^2 \sin\phi\cos\phi \tag{441}$$

$$I_{ii} = I_{xx} + I_{yy} = \mu r^2 \tag{442}$$

where μ is the reduced mass $m_1 m_2/M$. It is a now lengthy, but entirely straightforward task to differentiate each of these moments three times. You should begin with the relatively easy $\epsilon = 0$ case when reproducing the formulae below, though I present the results for finite ϵ here:

$$\frac{d^3 I_{xx}}{dt^3} = \alpha(1 + \epsilon\cos\phi)^2 (2\sin 2\phi + 3\epsilon \sin\phi\cos^2\phi), \tag{443}$$

$$\frac{d^3 I_{yy}}{dt^3} = -\alpha(1 + \epsilon\cos\phi)^2 [2\sin 2\phi + \epsilon\sin\phi(1 + 3\cos^2\phi)], \tag{444}$$

$$\frac{d^3 I_{xy}}{dt^3} = \frac{d^3 I_{yx}}{dt^3} = -\alpha(1 + \epsilon\cos\phi)^2 [2\cos 2\phi - \epsilon\cos\phi(1 - 3\cos^2\phi)], \tag{445}$$

where

$$\alpha^2 \equiv \frac{4G^3 m_1^2 m_2^2 M}{a^5(1-\epsilon^2)^5} \tag{446}$$

Equation (433) yields, after some assembling:

$$L_{GW} = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2 M}{a^5(1-\epsilon^2)^5}(1+\epsilon\cos\phi)^4\left[(1+\epsilon\cos\phi)^2 + \frac{\epsilon^2}{12}\sin^2\phi\right] \tag{447}$$

Our final step is to average L_{GW} over an orbit. This is not simply an integral over $d\phi/2\pi$. We must integrate over time, i.e., over $d\phi/\dot{\phi}$, and then divide by the orbital period to do a *time* average. The answer is

$$\langle L_{GW}\rangle = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2 M}{a^5}f(\epsilon) = 1.00\times 10^{25}\ m_{\odot 1}^2 m_{\odot 2}^2 M_\odot (a_\odot)^{-5}\ f(\epsilon)\ \text{Watts}, \tag{448}$$

where

$$f(\epsilon) = \frac{1 + (73/24)\epsilon^2 + (37/96)\epsilon^4}{(1-\epsilon^2)^{7/2}} \tag{449}$$

and $\odot$ indicates solar units of mass (1.99×10^{30} kg) and length (one solar radius is 6.955×10^8 m). (Peters and Mathews 1963).

Exercise. Show that following the procedure described above, the time-averaged luminosity $\langle L_{GW}\rangle_{\text{time}}$ is given by the expression

$$\langle L_{GW}\rangle_{\text{time}} = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2 M}{a^5(1-\epsilon^2)^{7/2}}\left\langle(1+\epsilon\cos\phi)^2\left[(1+\epsilon\cos\phi)^2 + \frac{\epsilon^2}{12}\sin^2\phi\right]\right\rangle_{\text{angle}},$$

where the average on the right is over 2π angles in ϕ. Use the fact that the *angular* average of $\cos^2\phi$ is 1/2 and the average of $\cos^4\phi$ is 3/8 to derive equation (448).

Equations (448) and (449) give the famous gravitational wave energy loss formula for a classical Keplerian orbit. Notice the dramatic effect of finite eccentricity via the $f(\epsilon)$ function. The first binary pulsar to be discovered, PSR1913+16, has an eccentricity of about 0.62, and thus an enhancement of its gravitational wave energy loss that is boosted by more than an order of magnitude relative to a circular orbit.

This whole problem must have seemed like an utter flight of fancy in 1963: the concept of a neutron star was barely credible and not taken seriously; the notion of pulsar timing was simply beyond conceptualisation. A lesson, perhaps, that no good calculation of an interesting physical problem ever goes to waste!

Exercise. When we studied Schwarzschild orbits, there was an exercise to show that the total Newtonian orbital energy of a bound two body system is $-Gm_1m_2/2a$ and that the system period is proportional to $a^{3/2}$, independent of the eccentricity. Use these results to show that the orbital period change due to the loss of gravitational radiation is given by

$$\dot{P} = -\frac{192\pi}{5}\left(\frac{m_1 m_2}{M^2}\right)\left(\frac{GM}{ac^2}\right)^{5/2} f(\epsilon)$$

with $M = m_1 + m_2$ as before. This $\dot{P}$ is a measurable quantity! Stay tuned.

Exercise. Now that you're an expert in the the two-body gravitational radiation problem, let's move on to three! Show that three equal masses revolving around their common centre-of-mass emit no quadrupole gravitational radiation.

7.8 Detection of gravitational radiation

7.8.1 Preliminary comments

The history of gravitational radiation has been somewhat checkered. Albert Einstein himself stumbled several times, both conceptually and computationally. Arguments of fundamental principle persisted through the early 1960's; technical arguments still go on.

At the core of the early controversy was the question of whether gravitational radiation existed at all! The now classic Peters and Mathews paper of 1963 begins with a disclaimer that they are assuming that the "standard interpretation" of the theory is correct. The confusion concerned whether the behaviour of $h_{\mu\nu}$ potentials were just some sort of mathematical coordinate effect, devoid of any actual physical consequences. For example, if we calculate the affine connection $\Gamma^{\mu}_{\nu\lambda}$ and apply the geodesic equation,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^{\mu}_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \tag{450}$$

and ask what happens to a particle initially at rest with $dx^\nu/d\tau = (-c, \mathbf{0})$. The subsequent evolution of the spatial velocity components is then

$$\frac{d^2x^i}{d\tau^2} + \Gamma^{i}_{00}c^2 = 0 \tag{451}$$

But equation (334) clearly shows that $\Gamma^{i}_{00} = 0$ since any h with a zero index vanishes for our TT plane waves. The particle evidently remains at rest. Is there really no effect of gravitational radiation on ordinary matter?!

Coordinates, coordinates, coordinates. The point, once again, is that coordinates by themselves mean nothing, any more than does the statement "My house is located at the vector (2, 1.3)." By now we should have learned this lesson. We picked our gauge to make life simple, and we have simply found a coordinate system that is frozen to the individual particles. There is nothing more to it than that. The *proper* spatial separation between two particles with coordinate separation dx^i is $ds^2 = (\eta_{ij} - h_{ij})dx^i\,dx^j$, and that separation surely is not constant because h_{11}, h_{22}, and $h_{12} = h_{21}$ are wiggling even while the dx^i are fixed. Indeed, to first order in h_{ij}, we may write

$$ds^2 = \eta_{ij}(dx^i - h_{ik}dx^k/2)(dx^j - h_{jm}dx^m/2).$$

This makes the physical interpretation easy: the passing wave increments the initially undisturbed spatial interval dx^i by an amount $-h_{ik}dx^k/2$. It was Richard Feynman who in 1955 seems to have given the simplest and most convincing argument for the existence of gravitational waves. If the separation is between two beads on a rigid stick and the beads are free to slide, they will oscillate with the tidal force of the wave. If there is now a tiny bit of stickiness, the beads will heat the stick. Where did that energy come from? It could only be the wave. The "sticky bead argument" became iconic in the relativity community.

The two independent states of linear polarisation of a gravitational wave are sometimes referred to as $+$ and $\times$, "plus" and "cross." The behave similarly, but rotated by 45°. The $+$ wave as it passes initially causes a prolate distortion along the vertical part of the plus sign, squeezes from prolate to oblate distorting along the vertical axis, then squeezes inward from oblate to prolate once again. The $\times$ wave shows the same oscillation pattern along a rotation pattern rotated by 45°. (An excellent animation is shown in the Wikipedia article "Gravitational Waves.") These are true physical distortions caused by the tidal force of the gravitational wave.

In the midst of what had been intensively theoretical investigations and debate surrounding the nature of gravitational radiation, in 1968 a physicist named Joseph Weber calmly announced that he had detected gravitational radiation experimentally in his basement lab, coming in prodigious amounts from the centre of the Milk Way Galaxy, thank you very much. His technique was to use what are now called "Weber bars", giant cylinders of aluminum fitted with special piezoelectric devices that can convert tiny mechanical oscillations into electrical signals. The gravitational waves distorted these great big bars by a tiny, tiny amount, and the signals were picked up. Or at least that was the idea. The dimensionless relative strain $\delta l/l$ of a bar of length l due to passing wave would be of order h_{ij}, or 10^{-21} by our optimistic estimate. To make a long, rather sad story very short, Weber was in error in several different ways, and ultimately his experiment was completely discredited. Yet his legacy was not wholly negative: the possibility of actually detecting gravitational waves hadn't been taken very seriously up to this point. Post Weber, the idea gradually took hold in the physics establishment. People asked themselves how we might actually go about detecting these signals. It became part of the mainstream, with leading figures in relativity getting directly involved. The detection of gravitational radiation is not a task for a clever lone researcher working in the basement of university building, any more than was, say, finding the Higgs boson. Substantial resources of the National Science Foundation in the US and a research team numbering in the thousands were needed for the construction and testing of viable gravitational wave receptors. Almost fifty years after Weber, the LIGO facility has at last cleanly detected the exquisitely gentle tensorial strains of gravitational waves at the level of $h \sim 10^{-21}$. The LIGO mirrors did not crack from side-to-side, but they did flutter a bit in the gravitational breeze. This truly borders on magic: if the effective length of LIGO's interferomter arm is taken is taken to be $l = 10$ km, then δl is 10^{-15} cm, one percent of the radius of a proton!

The next exercise is strongly recommended.

Exercise. **Weaker than weak interactions.** Imagine a gravitational detector of two identical masses m separated by a distance l symmetrically about the origin along the x-axis. Along comes a plane wave gravitational wave front, propagating along the z-axis, with $h_{xx} = -h_{yy} = A_{xx}\cos(kz - \omega t)$ and no other components. The masses vibrate in response. Show that, to linear order in A_{xx},

$$\dddot{I}_{xx} = \frac{1}{2} m c^4 \omega^3 l^2 A_{xx} \sin\omega t,$$

that there are no other $\dddot{I}_{ij}$, and that the masses radiate an average gravitational wave luminosity of

$$\langle L_{GW}\rangle = \frac{G}{60c^5} m^2 \omega^6 l^4 A_{xx}^2$$

Next, show that the average energy flux for our incoming plane wave radiation is, from equation (412),

$$\mathcal{F} = \frac{c^3\omega^2 A_{xx}^2}{64\pi G}.$$

The cross section for gravitational interaction (dimensions of area) is defined to be the ratio of the average luminosity to the average incoming flux. Why is this a good definition for the cross section? Show that this ratio is

$$\sigma = \frac{16\pi G^2 m^2 \omega^4 l^4}{15c^8} = \frac{4}{15}\pi R_S^2 \left(\frac{\omega l}{c}\right)^4$$

where $R_S = 2Gm/c^2$ is the Schwarzschild radius of each mass. Evaluate this numerically for $m = 10$kg, $l = 10$m, $\omega = 20$ rad s^{-1} (motivated by GW150914). Compare this with a typical weak interaction cross section of 10^{-48}m^2. Just how weak is gravitational scattering?

7.8.2 Indirect methods: orbital energy loss in binary pulsars

In 1974, a remarkable binary system was discovered by Hulse and Taylor (1975, *ApJ Letters,* 195, L51). One of the stars was a pulsar with a pulse period of 59 milliseconds, i.e., a neutron star that rotates about 17 times a second. The orbital period was 7.75 hours, a very tight binary with a separation of about the radius of the Sun. The other star was not seen, only inferred, but the very small separation between the two stars together with the absence of any eclipse of the pulsar suggested that the companion was also a compact star. (If the binary orbital plane were close to being in the plane of the sky to avoid observed eclipses, then the pulsar pulses would show no Doppler shifts, in sharp contradiction to observations.)

What made this yet more extraordinary is that pulsars are among the most accurate clocks in the universe, until recently more accurate than any earthbound atomic clock. The most accurately measured pulsar has a pulse period known to 17 significant figures! Indeed, pulsars can be calibrated only by ensemble averages of large numbers of atomic clocks. Pulsars are now directly used as clocks. [9] Nature has placed its most accurate clock in the middle of binary system in which fantastically precise timing is required. This is the ultimate general relativity laboratory.

Classic nonrelativistic binary observation techniques allow one to determine five parameters from observations of the pulsar: the semimajor axis projected against the plane of the sky ($a\sin i$), the eccentricity e, the orbital period P, and two parameters related to the periastron (the point of closest separation): its angular position within the orbit and a time reference point for when it occurs.

Relativistic effects, something new and beyond standard analysis, give two more parameters. The first is the advance of the perihelion (exactly analogous to Mercury) which in the case of PSR 1913+16 is 4.2° per year. (Recall that Mercury's is only 43 arc seconds per century!) The second is the *second order* ($\sim v^2/c^2$) Doppler shift of the pulse period from both the gravitational redshift of the combined system and the rotational kinematics. These seven parameters allow a complete determination of the masses and orbital components of the system, a neat achievement in itself. The masses of the neutron stars are $1.4414\,M_\odot$ and $1.3867\,M_\odot$, remarkably similar to one another and remarkably similar to the Chandrasekhar mass $1.42\,M_\odot$[10]. (The digits in the neutron stars' masses are all significant!) More importantly, there is a third relativistic effect also present, and therefore the problem is over-constrained. That is to say, it is possible to make a *prediction.* The orbital period changes slowly with time, shortening in duration due to the gradual approach of the two bodies. This "inspiral" is caused by the loss of orbital energy that has been carried off by gravitational radiation, equation (448). Thus, by monitoring the precise arrival times of the pulsar signals emanating from this slowly decaying orbit, the *existence* of gravitational radiation could be quantitatively confirmed and Einstein's quadrupole formula verified—even though the radiation itself was not directly observable.

Figure [5] shows the results of many years of observations. The dots are the cumulative change in the time of periastron due to the more progressively more rapid orbital period as the neutron stars inspiral from gravitational radiation losses. Without the radiation losses, there would still be a perihelion advance of course, but the time *between* perihelia would not change–it would just be a bit longer than an orbital period. It is the cumulative change between perihelia that is an indication of actual energy loss. The solid line is *not* a fit to the data. It is the prediction of general relativity of what the cumulative change in the "epoch of perihelion" (as it is called) should be, according to the energy loss formula of Peters and

[9] Since 2011, a bank of six pulsars, observed from Gdansk Poland, has been monitored continuously as a timekeeping device.

[10] This is the upper limit to the mass of a white dwarf star. If the mass exceeds this value, it collapses to either a neutron star or black hole, but cannot remain a white dwarf.

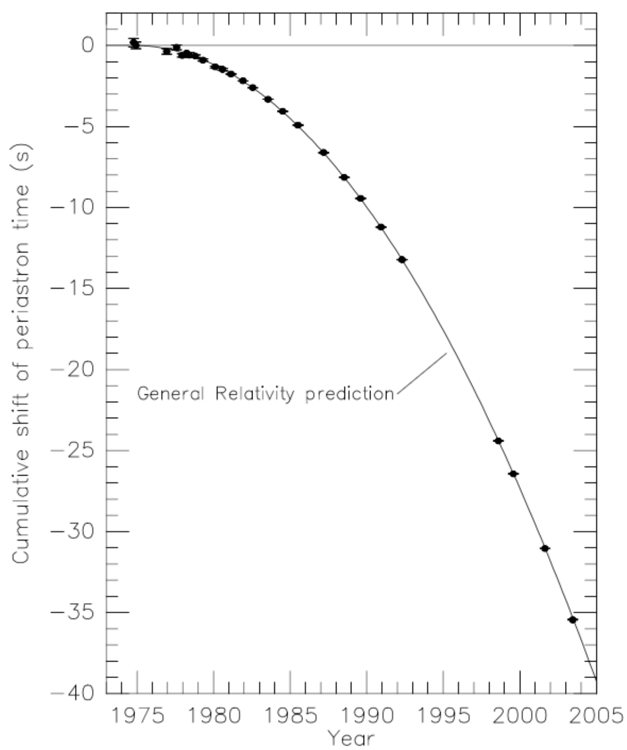

Figure 5: The cumulative change in the periastron event ("epoch") caused by the inspiral of the pulsar PSR1913+16. The dots are the data, the curve is the *prediction,* not the best fit! This prediction is confirmed to better than a fraction of a percent.

Mathews, (448). This beautiful precision fit leaves no doubt whatsover that the quadrupole radiation formula of Einstein is correct. For this achievement, Hulse and Taylor won a well-deserved Nobel Prize in 1993. (It must be just a coincidence that this is about the time that the data points seem to become more sparse.)

Direct detection of gravitational waves is a very recent phenomenon. There are two types of gravitational wave detectors currently in operation. The first is based on a classic 19th century laboratory apparatus: a Michelson interferometer. The second makes use of pulsar emission pulses—specifically their arrival times—as a probe of the $h_{\mu\nu}$ caused by gravitational waves as they propagate across our line of site to the pulsar. The interferometer detectors are designed for wave frequencies from ~ 10 Hz to 1000's of Hz. This is now up and running. By contrast, the pulsar measurements are sensitive to frequencies of tens to hundreds of *micro* Hz. A very different range, measuring physical processes on very different scales. This technique has yet to be demonstrated. The high frequency interferometers measure the gravitational radiation from stellar-mass black holes or neutron star binaries merging together. The low frequency pulsar timing will measure black holes merging, but with masses of order 10^9 solar masses. These are the masses of galactic core black holes in active galaxies.

7.8.3 Direct methods: LIGO

LIGO, or Laser Interferometer Gravitational-Wave Observatory, detects gravitational waves as described in figure (6). In the absence of a wave, the arms are set to destructively interfere, so that no light reaches the detector. The idea is that a gravitational wave passes through the apparatus from above or below, each period of oscillation slightly squeezing one arm, slightly extending the other. With coherent laser light traversing each arm, when it re-superposes at the centre, the phase will become *ever* so slightly out of precise cancellation, and photons will appear in the detector. In practise, the light makes many passages back and forth along a 4 km arm before analysis. The development of increased sensitivity comes from engineering greater and greater numbers of reflections, and thus a greater effective path length. There are two such interferometers, one in Livingston, Louisiana, the other in Hanford, Washington, a separation of 3000 km. Both must show a simultaneous wave passage (actually, with an offset of 10 milliseconds for speed of light travel time) for the signal to be verified.

This is a highly simplified description, of course. All kinds of ingenious amplification and noise suppression techniques go into this project, which is designed to measure induced strains at the incredible level of 10^{-21}. This detection is only possible because we measure not the flux of radiation, which would have a $1/r^2$ dependence with distance to the source, but the h_{ij} amplitude, which has a $1/r$ dependence.

Figure (7) shows a match of an accurate numerical simulation to the processed LIGO event GW150914. I have overlaid three measured wave periods P_1, P_2, and P_3, with each of their respective lengths given in seconds. (These were measured with a plastic ruler directly from the diagram!) The total duration of these three periods is 0.086 s. Throughout this time the black holes are separated by a distance in excess of of 4 R_S, so we are barely at the limit for which we can trust Newtonian orbit theory. Let's give it a try for a circular orbit. (Circularity is not unexpected for the final throes of coalescence.)

Using the zero eccentricity orbital period decrease formula from the previous exercise, but remembering that the orbital period P is twice the gravitational wave period P_{GW},

$$\dot{P}_{GW} = -\frac{96\pi}{5}\left(\frac{m_1 m_2}{M^2}\right)\left(\frac{GM}{ac^2}\right)^{5/2}$$

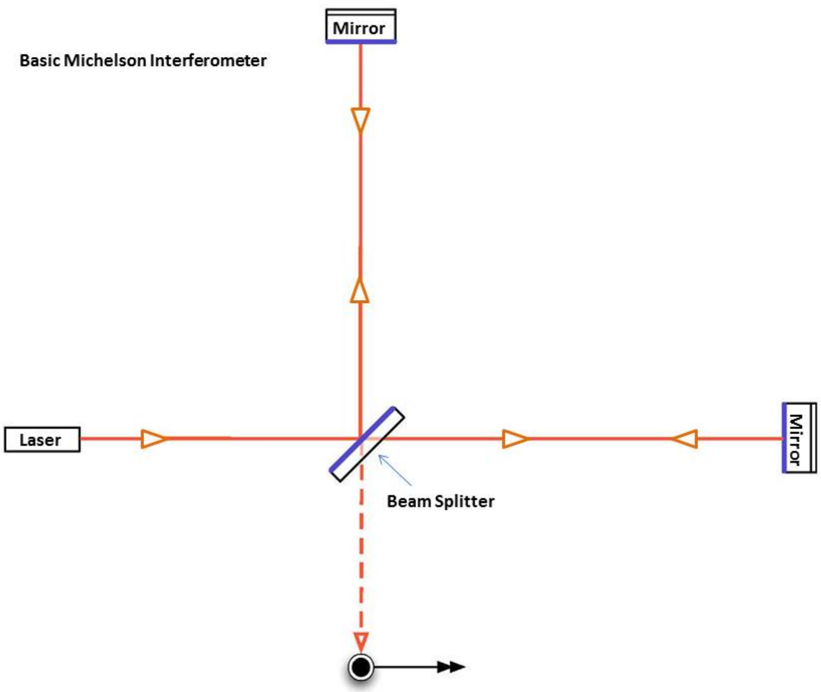

Figure 6: A schematic interferometer. Coherent light enters from the laser at the left. Half is deflected 45° upward by the beam splitter, half continues on. The two halves reflect from the mirrors. The beams re-superpose at the splitter, interfere, and are passed to a detector at the bottom. If the path lengths are identical or differ by an integral number of wavelengths they interfere constructively; if they differ by an odd number of half-wavelengths they cancel one another. In “null” mode, the two arms are set to destructively interfere so that no light whatsoever reaches the detector. A passing gravity wave just barely offsets this precise destructive interference and causes laser photons to appear in the detector.

We eliminate the semi-major axis a in favour of the measured period P_{GW},

$$P^2 = \frac{4\pi^2 a^3}{GM}, \quad \text{whence } P_{GW}^2 = \frac{\pi^2 a^3}{GM}$$

This gives

$$\dot{P}_{GW} = -\frac{96\pi^{8/3}}{5c^5}\left(\frac{GM_c}{P_{GW}}\right)^{5/3} \tag{452}$$

where we have introduced what is known as the "chirp mass" M_c,

$$M_c = \frac{(m_1 m_2)^{3/5}}{M^{1/5}} \tag{453}$$

The chirp mass (so-named because if the gravitational wave were audible at the same frequencies, it would indeed sound like a chirp!) is the above combination of m_1 and m_2, which is directly measurable from P_{GW} and its derivative. It can be shown (try it!) that $M = m_1 + m_2$ is a minimum when $m_1 = m_2$, in which case

$$m_1 = m_2 \simeq 1.15 M_c.$$

Now, putting numbers in (452), we find

$$M_{c\odot} = -5.522 \times 10^3 P_{GW} \dot{P}_{GW}^{3/5} \tag{454}$$

where $M_{c\odot}$ is the chirp mass in solar masses and P_{GW} is measured in seconds. From the GW150914 data, we estimate

$$\dot{P}_{GW} \simeq \frac{P_3 - P_1}{P_1 + P_2 + P_3} = \frac{-0.0057}{0.086} = -0.0663,$$

and for P_{GW} we use the midvalue $P_2 = 0.0283$. This yields

$$M_{c\odot} \simeq 30.7 \tag{455}$$

compared with "$M_c \simeq 30M_\odot$" in Abbot et al (2016)! I'm sure this remarkable level of agreement is somewhat (but not entirely!) fortuitous. Even in this, its simplest presentation, the wave form presents a wealth of information. The "equal mass" coalescing black hole system comprises two $35M_\odot$ black holes, and certainly at that mass a compact object can only be a black hole!

The two masses need not be equal of course, so is it possible that this is something other than a coalescing black hole binary? We can quickly rule out any other possibility, without a sophisticated analysis. It cannot be any combination of white dwarfs or neutron stars, because the chirp mass is too big. Could it be, say, a black hole plus a neutron star? With a fixed observed $M_c = 30M_\odot$, and a neutron star of at most $\sim 2M_\odot$, the black hole would have to be some $1700M_\odot$. So? Well, then the Schwarzschild radius would have to be very large, and coalescence would have occured at a separation distance too large for any of the observed high frequencies to be generated! There are frequencies present toward the end of the wave form event in excess of 75 Hz. This is completely incompatible with a black hole mass of this magnitude.

A sophisticated analysis using accurate first principle numerical simulations of gravitational wave from coalescing black holes tells an interesting history, though one rather

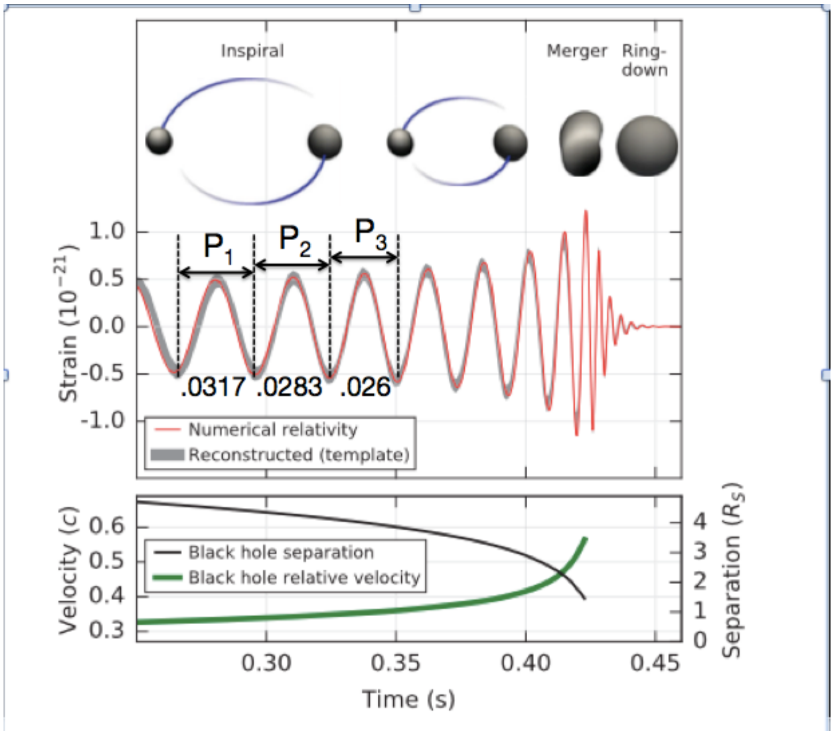

Figure 7: From Abbot et al. (2016). The upper diagram is a schematic rendering of the black hole inspiral process, from slowly evolution in a quasi-Newtonian regime, to a strongly interacting regime, followed by a coalescence and "ring-down," as the emergent single black hole settles down to its final, nonradiating geometry. The middle figure is the gravitational wave strain, overlaid with three identified periods discussed in the the text. The final bottom plot shows the separation of the system and the relative velocity as a function of time, from insprial just up to the moment of coalescence.

well-captured by our naive efforts. Using a detailed match to the waveform, the following can be deduced. The system lies at a distance of some 400 Mpc, with significant uncertainties here of order 40%. At these distances, the wave form needs to be corrected for cosmological expansion effects, and the masses in the source rest frame are $36M_\odot$ and $29M_\odot$, with $\pm15\%$ uncertainties. The final mass, $62M_\odot$ is less than the sum of the two, $65M_\odot$: some $3M_\odot c^2$ worth of energy has disappeared in gravitational waves! A release of 5×10^{47}J is, I believe, the largest explosion of any kind every recorded. A billion years later, some of that energy, in the form of ripples in space itself, tickles the interferometer arms in Louisiana and Washington. It is, I believe, at 10^{-15} cm, the smallest amplitude mechanical motion ever recorded.

What a story.

7.8.4 Direct methods: Pulsar timing array

Pulsars are, as we have noted, fantastically precise clocks. Within the pulsar cohort, those with millisecond periods are the most accurate of all. The period of PSR1937+21 is known to

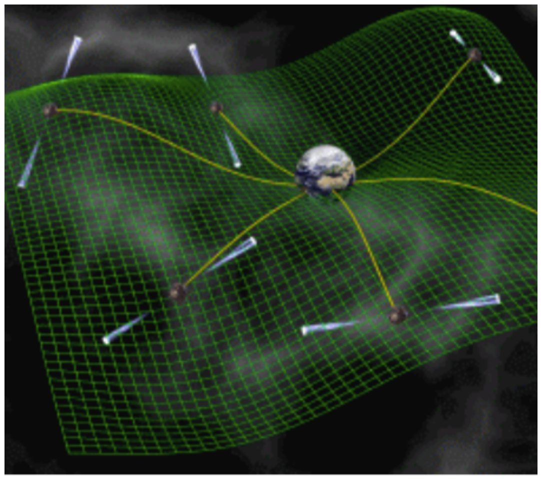

Figure 8: A schematic view of a gravitational wave passing through an array of pulsar probes.

be 1.5578064688197945 milliseconds, an accuracy of one part in 10^{17}. One can then predict the arrival time of a pulse to this level of accuracy as well. By constraining variations in pulse arrival times from a single pulsar, we can set an upper limit to amount of gravitational radiation that the signal has traversed. But we don't just have one pulsar. So why settle for one pulsar and mere constraints? We know of many pulsars, distributed more or less uniformly through the galaxy. If the arrival times from this "pulsar timing array" (PTA) were correlated with one another in a mathematically calculable manner, this would be a direct indication of the the deformation of space caused by the passage of a gravitational wave. This technique is sensitive to very long wavelength gravitational radiation, light-years in extent. This is very difficult to do because all other sources introducing a spurious correlation must be scrupulously eliminated. LIGO too has noise issues, but unlike pulsar blips propagating through the interstellar medium, LIGO's signal is very clean and all hardware is accessible. Thus, PTA has its share of skeptics. At the time of this writing, there are only upper limits from the PTA measurements.

Despite its name, the big bang theory is not really a theory of a bang at all. It is really only a theory of the aftermath of a bang.

— Alan Guth

8 Cosmology

8.1 Introduction

8.1.1 Newtonian cosmology

The subject of the origin of the Universe is irresistible to the scientist and layperson alike. What went bang? Where did the Universe come from? What happened along the way? Where are we headed? The theory of general relativity, with its rigorous mathematical formulation of the large-scale geometry of spacetime, provides both the conceptual and technical apparatus to understand the structure and evolution of the Universe. We are fortunate to live in an era in which many precise answers to these great questions are at hand. Moreover, while we need general relativity to put ourselves on a truly firm footing, we can get quite far using very simple ideas and hardly any relativity at all! Not only *can*, we absolutely *should* begin this way. Let us start with some very Newtonian dynamics and see what there is to see. Then, knowing a bit of what to expect and where we are headed, we will be in a much better position to revisit "the problem of the Universe" on a fully relativisitic basis.

A plausible but naive model of the Universe might be one in which space is ordinary static Euclidian space, and the galaxies fill up this space uniformly (on average) everywhere. Putting aside the question of the origin of such a structure (let's say it has existed for all time) and the problem that the cumulative light received at any location would be infinite ("Olber's paradox"–that's tougher to get around: let's say maybe we turned on the galaxies at some finite time in the past[11]), the static Euclidian model is not even mathematically self-consistent.

Consider the analysis from Figure [9]. There are two observers, one at the centre of the sphere labelled of radius r_1, the other at the centre of sphere r_2. Each calculates the expected acceleration at the location of the big black dot, which is a point on the surface of each of the spheres. Our model universe is spherically symmetric about r_1, but it is also spherically symmetric about r_2. Hence the following conundrum:

The observer at the centre of the r_1 sphere ignores the effect of the spherically symmetric mass exterior to the black point and concludes that the acceleration at the dot's location is

$$a_1 = \frac{GM(\text{within r}_1)}{r_1^2} = \frac{4\pi G\rho r_1}{3} \tag{456}$$

directed toward the centre of the r_1 sphere. (Here ρ is meant to be the average uniform mass density of the Universe.) But the observer at the origin of the r_2 sphere claims, by identical reasoning, that the acceleration must be $a_2 = 4\pi G\rho r_2/3$ directed toward the centre of r_2! Both cannot be correct.

[11]See W72, pp. 611-13.

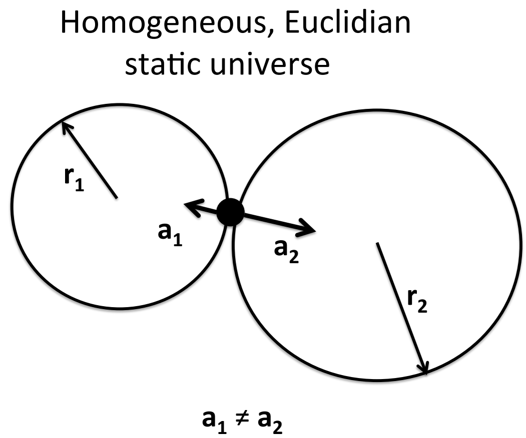

Figure 9: In a static homogeneous Euclidian universe, an observer at the center of the r_1 circle would calculate a different gravitational acceleration for the dot than an observer at the centre of the r_2 circle, i.e. $\boldsymbol{a_1} \neq \boldsymbol{a_2}$. But if we take into account the relative acceleration of the two observers, each considers the other to be in a *noninertial* frame, and the calculation is self-consistent with the included fictitious force. (See text.)

What if the Universe is dynamically active? Then we must put in the gravitational acceleration, in the form of a noninertial reference frame, from the very start of the calculation. If the observers at the centres of r_1 and r_2 are actually accelerating relative to one another, there is no reason to expect that their separate calculations for the black dot acceleration to agree, because the observers are not part of the same inertial frame! Can we make this picture self-consistent somehow for any two r_1 and r_2 observers? Yes. If the Universe exhibits a relative acceleration between two observers that is proportional to the vector difference $\boldsymbol{r_2} - \boldsymbol{r_1}$ between the two observers' positions, all is well.

Here is how it works. The observer at the centre of circle 1 measures the acceleration of the black dot to be $-4\pi G\rho \boldsymbol{r_1}/3$ as above, with $\boldsymbol{r_1}$ indicating a vector pointing from the centre of circle 1 to the surface dot. (In figure [9], $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are shown pointing to arbitrary boundary points for clarity; but think of them now as both pointing directly to the common, big dot boundary point.) The same circle 1 observer finds that the acceleration of the centre of circle 2 is $-4\pi G\rho(\boldsymbol{r_1} - \boldsymbol{r_2})/3$, where $\boldsymbol{r_2}$ is the position vector oriented from the circle 2 centre toward the big dot. (Defined this way, these particular $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ vectors are colinear.) Thus, the person at the centre of circle 1 would say that the acceleration of the big dot, *as measured by an observer moving in the (noninertial) centre of circle 2 frame* is the circle 1 acceleration of $-4\pi G\boldsymbol{r_1}/3$, minus the acceleration of the circle 2 centred observer:

$$-\frac{4\pi G\rho \boldsymbol{r_1}}{3} - \frac{-4\pi G\rho(\boldsymbol{r_1} - \boldsymbol{r_2})}{3} = -\frac{4\pi G\rho \boldsymbol{r_2}}{3} \tag{457}$$

Lo and behold, this is the result that the observer at the centre of circle 2 finds self-consistently in the privacy of his own study, without worrying about what anyone else thinks might be going on. A Euclidian, "linearly accelerating" universe is therefore perfectly self-consistent, at least at this level of dynamics. A dynamically active, expanding universe is essential. The expansion itself is essentially Newtonian, not, as originally thought at the time of its discovery, a mysterious effect of general relativity. Now, the rate of expansion naively ought to be slowing, since this is what gravity does: an object thrown from the surface of the earth slows down as its distance from the surface increases. As we shall soon see however, our Universe is a bit more devious than that. There is still some mystery here beyond the realm of the purely Newtonian.

8.1.2 The dynamical equation of motion

A simple way to describe the internal acceleration of the Universe is to begin with the spatially homogeneous but time-dependent relative expansion between two locations. The separation between two arbitrary points separated by a distance $r(t)$ may be written

$$r(t) = R(t)l \tag{458}$$

where l is a comoving coordinate that labels a fixed radial distance from us in the space—fixed in the sense of being fixed to the expanding space, like latitude and longitude would be on the surface of an inflating globe. If we take l to have dimensions of length, then $R(t)$ is a dimensionless function of time alone. It is a scale factor that embodies the dynamical behaviour of the Universe. The velocity $v = dr/dt \equiv \dot{r}$ of a "fixed" point expanding with space is then

$$v(t) = \dot{R}l = (\dot{R}/R)r. \tag{459}$$

We should emphasise the vector character of this relationship:

$$\boldsymbol{v}(t) = (\dot{R}/R)\boldsymbol{r} \tag{460}$$

where $\boldsymbol{r}$ is a vector pointing outward from our arbitrarily chosen origin. Then, the acceleration is

$$\boldsymbol{a}(t) = \frac{d\boldsymbol{v}}{dt} = (\ddot{R}/R)\boldsymbol{r} \tag{461}$$

(Why didn't we differentiate $\boldsymbol{r}(t)/R(t)$?) But we already *know* the relative acceleration between two points, because we know Newtonian physics. We've just worked it out a moment ago! You can easily see that the above discussions (especially equation [456]) imply that we must have :

$$-\frac{4\pi G\rho}{3} = \frac{\ddot{R}}{R} \tag{462}$$

Notice how l disappears: this is an equation for the scale factor R, and that does not depend on where you are. Next, multiply this by $\dot{R}R$ and integrate, assuming that mass is conserved in the usual way, i.e. ρR^3 is constant. (Why is this "the usual way?") We then obtain

$$\dot{R}^2 - \frac{8\pi G\rho R^2}{3} = 2E \tag{463}$$

where E is an energy-like integration constant. This, in a simple, apparently naive Euclidian-Newtonian approach, would be our fundamental dynamical cosmological equation for the evolution of the Universe. Amazingly, providing that we are prepared to allow the mass density ρ is to be upgraded to an energy density divided by c^2 that includes *all* contributions (in particular radiation and vacuum energy), this innocent little equation turns out to be far more general: it is *exactly* correct in full relativity theory! More on this anon.

8.1.3 Cosmological redshift

The expansion of the Universe leads to a very important kinematic effect known as the cosmological redshift. Since the Universe is expanding, a travelling photon is constantly overtaking sources that are moving away from it. If a photon has a wavelength λ at some location r, when the photon passes an observer a distance $dr = cdt$ away, moving at a relative velocity $\dot{R}\,dr/R$, the observer measures a Doppler change in wavelength $d\lambda$ determined by equation (460):

$$\frac{d\lambda}{\lambda} = \frac{v}{c} = \frac{\dot{R}}{cR}dr = \frac{\dot{R}}{R}dt, \tag{464}$$

or in other words

$$\frac{1}{\lambda}\frac{d\lambda}{dt} \equiv \frac{\dot{\lambda}}{\lambda} = \frac{\dot{R}}{R}. \tag{465}$$

Solving for λ, we find that it is linearly proportional to R. It is as though the wavelength stretches with the rest of the Universe! This is a very general kinematic result, a property of any model that is symmetrically expanding. (Sneak preview: this of course means that the frequency goes *down* as $1/R$. But the frequency of a photon is, in essence, its energy. The entire Universe must be radiatively cooling. Energy is not conserved in the expansion; the entropy in a voulme R^3, which is proportional to the third power of the temperature times R^3, is.)

We are free, and it is customary, to choose our coordinates in such a way that the current value of R is 1, with R becoming smaller and smaller as we go back in time. If a photon is emitted with a wavelength λ_e at some time t in the past, the wavelength we would now measure (λ_0) is formally expressed as

$$\lambda_0 = \lambda_e(1+z) \tag{466}$$

where z is defined by this equation and known as the redshift parameter. Therefore,

$$\frac{\lambda_0}{\lambda_e} = 1 + z = \frac{1}{R(t)} \tag{467}$$

The advantage to using z, as opposed to the more geometrical quantity R, is that z is directly observed by astronomers. But the two are mathematically equivalent via this completely general equation, $R = 1/(1+z)$. If you measure a redshift of 2, it has come from a time when the Universe had one-third of its current size.

We have been, you will notice, pretty informal, just organising our common sense. It is customary for cosmology courses to begin with a heavy dose of historical material related to the discovery of the expansion of the Universe. We will get to this in short order, but I have taken a somewhat different tack here, in part because much of the historical material is rather well-known these days, but mainly because it is not often appreciated how far direct Newtonian reasoning can take one in establishing a viable cosmological model of the Universe. Let us examine one very simple model, and then see how formal general theory gets us to the same place in the end.

8.2 Cosmology models for the impatient

8.2.1 The large-scale spacetime metric

Euclid, Newton and pure thought can take us very far, even farther than we have ventured up to now. Because the Newtonian approach on which we are about to embark is going to work remarkably well, here is a brief reminder as to why we actually do need relativity in our study of the large scale structure of the Universe. Let us understand what it is at stake.

First, we require a Riemannian metric structure to ensure that the speed of light is a universal constant c, especially when traversing a dynamically evolving spacetime background. It is rather easy to see what the form this metric must take in the simplest model of an expanding Euclidian Universe. Symmetry demands that time must flow the same for all observers comoving with the universal expansion, and we can always choose time to be a linear function of the time coordinate. Space is uniformly expanding at the same rate everywhere. So if space itself is Euclidian, the spacetime metric practically leaps out of the page,

$$-c^2 d\tau^2 = -c^2 dt^2 + R^2(dx^2 + dy^2 + dz^2) \tag{468}$$

where we have used the usual (x, y, z) Cartesian coordinates, and R satisfies equation (463). Here x, y, z are all comoving with the expansion, in essence the l coordinate of the previous section. In particular, for a photon heading directly toward us along our line of sight from a distant source,

$$R\frac{dl}{dt} = -c \tag{469}$$

where dl is interpreted as the change in radial comoving coordinate induced by the photon's passage. This equation describes an ant crawling along the surface of an expanding sphere from, say, the pole to the equator, moving at a constant velocity c. In this case, think of dl as the change in latitude.

Equation (468) really does appear to be the true form of the spacetime metric for our Universe. Space is in fact very nearly, or perhaps even precisely, Euclidian. As a mathematical point, this need not be the case even if we demand perfect symmetry, any more than a perfectly symmetric two-dimensional surface must be a plane. We could preserve our global maximal spatial symmetry and have a curved space, just like the surface of sphere. This, in common with a flat plane, is symmetric about every point, but is obviously distorted relative to a plane. The case of a two-dimensional spherical surface is readily grasped because we can easily embed it in three dimensions and form a mental image. It is finite in area and said to be positively curved. There is also a perfectly viable flaring, *negatively* curved two-dimensional surface. A saddle begins to capture its essence, but not quite, because the curvature is not uniform in a saddle. The case of a uniformly negatively curved surface cannot be embedded in three dimensions, so it is hard to picture in your mind's eye! There are perfectly good positively and negatively *three-dimensional spaces* as well, which are logically possible alternative symmetric structures for the space of our Universe. They just happen not to fit the data. It is fortunate for us that the real Universe seems also to be mathematically the simplest. We will study these other symmetric spaces later; for now we confine our attention to expanding, good old, "flat" Euclidian space.

Second, we need relativity in the form of the Birkhoff theorem to justify properly the argument neglecting exterior contributions from outside the arbitraily chosen spheres we used in section 8.1. The Newtonian description strictly can't be applied to an infinite system, whereas nothing prevents us from using Birkhoff's theorem applied to an unbounded symmetric spacetime.

Third, we need a relativistically valid argument to arrive at equation (463). Nothing in the Newtonian derivation hints at this level of generality. We shall return to this carefully in section 8.3.

Fourth, we need relativity theory to relate the constant E to the geometry of our space. For now, we restrict ourselves to the case $E = 0$, which will turn out to be the only solution consistent with the adoption of a flat Euclidian spatial geometry, the sort of universe we do seem to live in.

8.2.2 The Einstein-de Sitter universe: a useful toy model

Consider equation (463) for the case $E = 0$ in the presence of ordinary matter, for which ρR^3 is a constant. Remember that we are free to choose coordinates in which $R = 1$ at the present time $t = t_0$. We may then choose the constant ρR^3 to be equal to its present day value, ρ_{M0}. Equation (463) becomes

$$R^{1/2}\dot{R} = \left(\frac{8\pi G\rho_{M0}}{3}\right)^{1/2}. \tag{470}$$

Then,

$$\frac{2}{3}R^{3/2} = \left(\frac{8\pi G\rho_{M0}}{3}\right)^{1/2} t \tag{471}$$

where the integration constant has been set to zero under the assumption that R was very small at early times. We finally obtain

$$R = \left(\frac{3H_0 t}{2}\right)^{2/3} \tag{472}$$

where H_0, the value of $\dot{R}/R$ and the current time t_0, is known as the Hubble constant,

$$H_0 = \dot{R}(t_0) = \left(\frac{8\pi G\rho_{M0}}{3}\right)^{1/2}. \tag{473}$$

More generally, the Hubble parameter is defined as

$$H(t) = \frac{\dot{R}}{R} \tag{474}$$

for any time t. The solution (472) is known, for historical reasons, as the Einstein-de Sitter model.

Exercise. Show that $H(t) = H_0(1+z)^{3/2}$ for our simple model $R = (t/t_0)^{2/3}$.

The Hubble constant is in principle something that we may observe directly, "simply" by measuring the distances to nearby galaxies as well as their redshift, and then using equation (460). In practise this is hardly simple. On the contrary, it is a very difficult task for reasons we will discuss a bit later, but the bottom line is that *the measured value of H_0 and the measured value of the density of ordinary matter ρ_{M0} do not satisfy (473) in our Universe.* There is not enough ordinary matter ρ_{M0} to account for the measured H_0. Yet, equation (463) does seem to be precisely valid, with $E = 0$. As the energy density of radiation in the contemporary Universe is much less than $3H_0^2/8\pi G$, how is all this possible?

The answer is stunning. While the energy density of *ordinary* matter does indeed dominate over radiation, there is strong evidence now of an energy density associated with the vacuum of spacetime itself! This energy density, ρ_V, is the dominant energy density of the real Universe on cosmolgical scales, though not at present overwhelmingly so: ρ_V is about 73% of the energy budget whereas matter (ordinary baryons and "dark matter") comes in at about 27%. However, ρ_V remains constant as the Universe expands, so that at later times vacuum energy dominates the expansion: ρ_M drops off as $1/R^3$, and ρ_V completely dominates. Moreover, with an effective vacuum Hubble parameter

$$H_V \equiv \left(\frac{8\pi G\rho_V}{3}\right)^{1/2}, \tag{475}$$

equation (463) at later times takes the form

$$\dot{R} = H_V R \tag{476}$$

or

$$R \propto \exp(H_V t) \tag{477}$$

the Universe will expand exponentially! In other words, rather than gravity slowly decelerating the expansion by the mutual Newtonian attractive force, the vacuum energy density will actively drive an ever more vigorous repulsive force. The Universe was expanding more slowly in the past than in the present. It is this particular discovery which led to our current understanding of the remarkable expansion dynamics. The Nobel Prize in Physics was awarded to Perlmutter, Schmidt and Riess in 2011 for the use of distant supernovae as a tool for unravelling the dynamics of the Universe from early to later times. We are currently in the epoch where exponential expansion is taking over.

Equations (467), (469) and (473) may be combined to answer the following question. If we measure a photon of redshift z, from what value of l did it originate? This is an important question because it provides the link between observations and geometry. With $R(t) = (t/t_0)^{2/3} = (3H_0 t/2)^{2/3}$, equation (469) may be be integrated over the path of the photon from its emission at l at time t:

$$\int_0^l dl' = -c\int_{t_0}^{t} \frac{dt'}{R} = ct_0^{2/3}\int_t^{t_0} \frac{dt'}{t'^{2/3}} = 3ct_0\left(1-\sqrt{R}\right) = \frac{2c}{H_0}\left(1-\frac{1}{\sqrt{1+z}}\right) \equiv l(z) \tag{478}$$

This is a very interesting equation for many reasons. First, note that as $z \to \infty$, $l \to 2c/H_0$, a constant. The most distant photons—and therefore the most distant regions that may causally influence us—come from $l = 2c/H_0$. This quantity is known as the *horizon,* denoted l_H. Beyond the horizon, we can see—and be influenced by—nothing. This particular value of $l_H = 2c/H_0$ is associated with the Einstein–de Sitter model, but the existence of a horizon is a general feature of many cosmologies and will generally be a multiple of c/H_0. Do not confuse this cosmological horizon, sometimes called a "particle horizon" with the *event horizon* of a black hole. The particle horizon is notionally outward, the length over which a a causal effect may be exerted. The event horizon is notionally inward, the radius within which no causal contact with the outside world is possible.

Physical distances are given not by l, but by Rl. Since $R = 1$ currently, $2c/H_0$ is the current physical scale of the horizon as well. There is nothing special about *now,* however. We could be doing this analysis at any time t, and the scale of the horizon at time t in the E-de S model would then be $2c/H(t) = 2c/[H_0(1+z)^{3/2}]$. So here is another interesting question. The farthest back in time that we can see is to a redshift of about $z = 1500$. At higher redsifts the Universe was completely opaque. Just as we can only see to the opaque surface of the Sun, but not its interior, we can only see back in time to when the Universe became opaque to photons. Question: what would be the subtended angular size of the horizon Θ_H at $z = 1500$, as we it measure today? This is an important question because we would not expect the Universe to be very smooth or regular or correlated in any way on angular scales bigger than this. (The E-dS model doesn't actually hold during the radiation dominated phase, but it will serve to make our point.) The angular size of the horizon is given by the following expression:

$$\Theta_H = \frac{2c}{H_0(1+z)^{3/2}R(z)l(z)} = \frac{2c}{H_0(1+z)^{1/2}l(z)} = \frac{1}{\sqrt{1+z}-1} \tag{479}$$

The first equality sets the horizon angle equal to the actual physical size of the horizon at redshift z, divided by $d = R(z)l(z)$, the distance to redshift z *at the earlier time corresponding* *to redshift z* (not now!). This d is the relevant distance to the photon sources at the moment of the radiation emission. The Universe was a smaller place then, and we cannot, we must not, use the *current* proper distance, which would be $l(z)$ times $R_0 = 1$. If you plug in $z = 1500$ into (479) and convert Θ_H to degrees, you'll find $\Theta_H = 1.5°$, about 3 times the diameter of the full moon. But the Universe looks very regular on much, much larger angular scales, indeed over the entire sky! Even if our model is only crude, it highlights an important problem. We will see later in this course how modern cosmology addresses this puzzle. Simply put, how does the Universe know about itself in a global sense, given that it takes signals, even signals travelling at the speed of light, so long to cross it? What we have here, ladies and gentleman, is a failure to communicate.

By the way, the fact that the Universe was "on top of us" at early times has another surprising consequence. Assuming that the average physical size of a galaxy isn't changing very much with time, if we calculate the average *angular* size of a galaxy, we find that at low redshift, all is normal: the more distant galaxies appear to be smaller. But then, at higher and higher redshifts, the galaxies appear to be growing larger on the sky! Why? Because at large z, the Universe was, well, on top of us. The "distance-to-redshift-z" formula is

$$l(z)R(z) = \frac{2c}{H_0(1+z)^{3/2}}\left(\sqrt{1+z} - 1\right) \tag{480}$$

At low z, this is increasing linearly with z, which is intuitive: bigger redshift, more distant. But at large z this declines as $1/z$, since the Universe was a smaller place.

Exercise. At what redshift would the average galaxy appear to be smallest in this model? (Answer: $z = 5/4$.)

8.3 The Friedman-Robertson-Walker Metric

Let us start very simply. The ordinary metric for a planar two-dimensional space ("2-space") without curvature may be written in cylindrical coordinates as

$$ds^2 = d\varpi^2 + \varpi^2 d\phi^2 \tag{481}$$

where the radial ϖ and angular ϕ polar coordinates are related to ordinary Cartesian x and y coordinates by the familiar formulae:

$$x = \varpi\cos\phi, \qquad y = \varpi\sin\phi \tag{482}$$

As we have noted, this flat 2-space is not the most general globally symmetric 2-space possible. The space could, for example, be distorted like the 2-surface of a sphere, yet retain the symmetry of every point being equivalent. The metric for a spherical surface is well-known:

$$ds^2 = a^2 d\theta^2 + a^2\sin^2\theta d\phi^2 \tag{483}$$

where a is the radius of the sphere. Now, we know how to go between cylindrical polar and spherical coordinates: set $\varpi = a\sin\theta$. This may be viewed as a purely formal transformation of coordinates, but in our mind's eye we picture θ in physical terms as the colatitude measured down from the z axis. But don't expect to recover (481) from (483) via this simple coordinate change $\sin\theta = \varpi/a$. Instead, we find that the spherical surface metric becomes:

$$ds^2 = \frac{d\varpi^2}{1 - \varpi^2/a^2} + \varpi^2 d\phi^2 \tag{484}$$

which is a different space from the planar surface (481) altogether. It is of course the same spherical surface space we started with by any other name, agreeing with (481) only in the limit $a \to \infty$. Changing coordinates does not change the geometry, i.e., it does not change the curvature scalars. A plane may be regarded as the surface of a sphere only in the limit $a \to \infty$.

Let's stretch ourselves a bit and consider, at a formal level, the closely related metric

$$ds^2 = \frac{d\varpi^2}{1 + \varpi^2/a^2} + \varpi^2 d\phi^2, \tag{485}$$

which upon substituing $\sinh\chi = \varpi/a$ reverts to

$$ds^2 = a^2 d\chi^2 + a^2 \sinh^2\chi\, d\phi^2 \tag{486}$$

the fundamental symmetry properties of the metric are unaffected by the $\pm\varpi^2$ sign flip. The flip in sign simply changes the sign of the constant curvature surface from positive (convex, think sphere) to negative (flaring, think saddle). The characteristic form of the metric tensor component $g_{rr} = 1/(1 \pm \varpi^2/a^2)$ will reappear when we go from curved 2-space, to curved 3-space.

8.3.1 Maximally symmetric 3-spaces

It is perhaps best to begin with the conclusion, which ought not to suprise you. The most general form of the three dimensional metric tensor that is maximally symmetric—homogeneous and isotropic about every point—takes the form

$$-c^2 d\tau^2 = -c^2 dt^2 + R^2(t)\left(\frac{dr^2}{1 - r^2/a^2} + r^2 d\theta^2 + r^2 \sin^2\theta\right) \tag{487}$$

where we allow ourselves the liberty of taking a^2 to be positive or negative.

The derivation for a 3-space hypersurface for a sphere in four dimensions is not difficult. The surface of 4-sphere (Cartesian coordinates w, x, y, z) is given by

$$w^2 + x^2 + y^2 + z^2 = a^2 = \text{constant}. \tag{488}$$

Thus, on this surface, a small change in w^2 is restricted to satisfy:

$$d(w^2) = -d(x^2 + y^2 + z^2) \equiv -d(r^2) \to dw = -r dr/w \tag{489}$$

with $r^2 \equiv x^2 + y^2 + z^2$. Hence,

$$(dw)^2 = \frac{r^2(dr)^2}{w^2} = \frac{r^2(dr)^2}{a^2 - r^2} \tag{490}$$

The line element in 4-space is

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 + dw^2 \tag{491}$$

and with $(dw)^2$ given by (490), there follows immediately the line element of the 3-surface of a 4-sphere, analogous to our expression for the ordinary spherical surface metric (483):

$$ds^2 = \frac{dr^2}{1 - r^2/a^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{492}$$

just as we see in equation (487). An alternative form for (492) is sometimes useful without the singular denominator. Set $r = a\sin\chi$. Then

$$ds^2 = a^2 d\chi^2 + a^2 \sin^2\chi d\theta^2 + a^2 \sin^2\chi \sin^2\theta d\phi^2 \tag{493}$$

Exercise. Would you care to hazard a guess as to what the line element of the 4-surface of a 5-sphere looks like either in the form of (492) or (493)?

The corresponding negatively curved 3-surface has line elements

$$ds^2 = \frac{dr^2}{1 + r^2/a^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{494}$$

and, with $r = \sinh\chi$,

$$ds^2 = a^2 d\chi^2 + a^2 \sinh^2\chi d\theta^2 + a^2 \sinh^2\chi \sin^2\theta d\phi^2 \tag{495}$$

Not convinced that these are the unique metrics for the most general, maximally symmetric forms of these curved spaces? You say you want proof? I'll give you proof! W72, Chapter 13. Have fun. (Lite version: HEL06, Chapter 14.) We won't pursue the uniqueness question further in these notes, as it is too much of a mathematical diversion (and certainly not on the syllabus). We will keep physics front and centre.

It is customary in some textbooks to use $r' = r/a$ as the radial variable, and absorb the factor of a into the definition of what you mean by the scale factor $R(t)$. If you do that, then you give up on the convenience of setting the current value of R equal to 1. Dropping now the $'$ on r', the general FRW metric is written,

$$-c^2 d\tau^2 = -c^2 dt^2 + R^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\right) \tag{496}$$

where the constant k is $+1$ for a positively curved space, -1 for a negatively curved space, and 0 for a flat space. This will be our underlying fundamental geometrical model for the Universe:

$$g_{00} = -1, \quad g_{rr} = R^2(t)/(1 - kr^2), \quad g_{\theta\theta} = R^2(t)r^2, \quad g_{\phi\phi} = R^2(t)r^2 \sin^2\theta. \tag{497}$$

Be careful! In (496), R has dimensions of length, and r is *dimensionless.* In (487), R is dimensionless and r has dimensions of length.

In section 8.2.2, we used the notation $l(z)$ to denote the integral of cdt/R from some distant time t to present time t_0 in an Einstein-de Sitter model. We will continue to use this notation for FRW metrics with spatial curvature. Physical distances are then $Rl(z)$, where R is taken at some time t that depends upon the application of interest. The $r - l$ difference will appear only when we look at l as a function of the r coordinate in a curved spacetime,

$$dl = d(\sin^{-1} r) \quad (k = 1), \qquad dl = d(\sinh^{-1} r) \quad (k = -1).$$

But $dl = dr$ for $k = 0$, which fortunately seems to be our beloved Universe. In dealing with *our* Universe, therefore, we will use r or $r(z)$ for our comoving coordinate.

As an example of how to use our formalism, consider the question of how large a volume of space is being sampled out to a redshift of z_m, some maximum value. Photons arriving at the current epoch t_0 with redshift z come from a comoving coordinate

$$r = c\int_t^{t_0} \frac{dt'}{R(t')} = c\int_R^1 \frac{dR'}{\dot{R}'R'} = c\int_0^{z_m} \frac{dz}{(1+z)\dot{R}(z)} \equiv r(z)$$

where $\dot{R}$ is given by equation (463) quite generally, expressed in terms of $R = 1/(1+z)$. The volume of photons within redshift z_m is formally given by

$$V = 4\pi \int_0^r R^3 \frac{r'^2 dr'}{\sqrt{1-kr'^2}}, \tag{498}$$

a function of r related to redshift by using $r = r(z_m)$. But what do we use for R inside this integral? That depends on the question. If we are interested in the *current* net volume of these sources, then $R = R_0$, a constant, and life is simple. If we are interested in the net volume of all the sources occupied *at the time of their emission,* then we would use R as a function of r or z from equation (478) (whichever is simplest) to do the integral. For example, the current volume V in an E-dS ($k = 0$, $R_0 = 1$) universe for sources out to a maximum redshift z_m is simply

$$V = \frac{4\pi}{3}\left(\frac{2c}{H_0}\right)^3 \left(1 - \frac{1}{\sqrt{1+z_m}}\right)^3 \tag{499}$$

8.4 Large scale dynamics

8.4.1 The effect of a cosmological constant

Begin first with the Field Equations *including the cosmological constant,* equation (245):

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R = -\frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{500}$$

Recal the stress energy tensor for a perfect fluid:

$$T_{\mu\nu} = P g_{\mu\nu} + (\rho + P/c^2)U_\mu U_\nu. \tag{501}$$

We may arrange the right side source term of (500) as follows:

$$-\frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}\left[\tilde{P} g_{\mu\nu} + \left(\tilde{\rho} + \frac{\tilde{P}}{c^2}\right)U_\mu U_\nu\right], \tag{502}$$

where

$$\tilde{P} = P - \frac{c^4\Lambda}{8\pi G}, \quad \tilde{\rho} = \rho + \frac{c^2\Lambda}{8\pi G} \tag{503}$$

In other words, the effect of a cosmological constant is to leave the left side of the Field Equations untouched and to leave the right side of the Field Equations in the form of a stress tensor for a perfect fluid, but with the density acquiring a constant additive term $c^2\Lambda/8\pi G$ and the pressure acquiring a constant term of the opposite sign, $-c^4\Lambda/8\pi G$!

This is simple, almost trivial, mathematics, but profound physics. The effect of a cosmological constant is as if the vacuum itself had an energy density $\mathcal{E}_V = \rho_V c^2 = c^4\Lambda/8\pi G$ and a pressure $P_V = -c^4\Lambda/8\pi G$. Does it make sense that the vacuum has a negative pressure, equal to its energy density but opposite in sign? Yes! By the first law of thermodynamics, if the vacuum volume expands by dV, the first law states that the change in energy per unit volume of expansion, which is just $dE/dV \equiv \rho_V c^2$, must be $-P_V$, which indeed it is.

More revealingly, if we recall the form of the stress energy tensor of the vacuum, *but without assuming* $P_V = -\rho_V c^2$, then

$$T_{\mu\nu\ (V)} = P_V g_{\mu\nu} + (\rho_V + P_V/c^2)U_\mu U_\nu, \tag{504}$$

and the last group of terms would change the form of the vacuum stress energy going from one constant velocity observer to another. In other words, you could tell if you were moving relative to the vacuum! That is strictly not allowed. The vacuum stress must always be proportional to $g_{\mu\nu}$, and to $g_{\mu\nu}$ alone. The only way this can occur is if $P_V = -\rho_V c^2$.

An early general relativity advocate, Sir Arthur Eddington was particularly partial to a cosmological constant, and was fond of commenting that to set $\Lambda = 0$ would be to "knock the bottom out of space." At the time this was probably viewed as Eddington in his customary curmudgeon mode; today the insight seems downright prescient. Nowadays, physicists like to think less in terms of a cosmological constant and place more conceptual emphasis on the notion of a vacuum energy density. What is the reason for its existence? Why does it have the value that it does? If ρ_V is not strictly constant, general relativity would be wrong. Are the actual observational data supportive of a truly constant value for ρ_V? The value of ρ_V probably emerged from the same type of "renormalisation process" (for those of you familiar with this concept) that has produced finite values for the masses for the fundamental particles. How do we calculate this? These are some of the most difficult questions in all of physics.

For present purposes, we put these fascinating issues to the side, and continue our development of large scale models of the Universe without the formal appearance of a cosmological constant, but with the understanding that we may add the appropriate contributions to the density ρ and pressure P to account precisely for the effects of Λ.

8.4.2 Formal analysis

We shall use the Field Equations in terms of the source function $S_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu}T/2$,

$$R_{\mu\nu} = -\frac{8\pi G}{c^4} S_{\mu\nu}. \tag{505}$$

In our comoving coordinates, the only component of the 4-velocity U_μ that does not vanish is $U_0 = -c$ (from the relation $g^{00}(U^0)^2 = -c^2$ and $U_0 = g_{00}U^0$). The nonvanishing components of $T_{\mu\nu}$ are then

$$T_{00} = \rho c^2, \quad T_{ij} = P g_{ij}. \tag{506}$$

Which means

$$T^0_0 = -\rho c^2, \quad T^i_j = \delta^i_j P, \quad T = T^\mu_\mu = -\rho c^2 + 3P \tag{507}$$

We shall need

$$S_{00} = T_{00} - \frac{g_{00}}{2}T = \frac{1}{2}(\rho c^2 + 3P), \quad S_{ij} = \frac{1}{2}g_{ij}(\rho c^2 - P) \tag{508}$$

To calculate R_{00}, begin with our expression for the Ricci tensor, (253):

$$R_{\mu\kappa} = \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial x^\kappa \partial x^\mu} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \frac{\Gamma^\eta_{\mu\kappa}}{2}\frac{\partial \ln|g|}{\partial x^\eta}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$ given by (497). Defining

$$f(r) = 1/(1 - kr^2),$$

we have

$$|g| = R^6(t)\, r^4\, f(r) \sin^2\theta \tag{509}$$

For diagonal metrics, $\Gamma^a_{ab} = \partial_b(\ln g_{aa})/2$ and $\Gamma^b_{aa} = -(\partial_b g_{aa})/2g_{bb}$ (no sum on a). Therefore $\Gamma^\lambda_{00} = 0$, and our expression for R_{00} simplifies to

$$R_{00} = \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial x^0 \partial x^0} + \Gamma^\eta_{0\lambda}\Gamma^\lambda_{0\eta} = \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial x^0 \partial x^0} + \Gamma^r_{0r}\Gamma^r_{0r} + \Gamma^\theta_{0\theta}\Gamma^\theta_{0\theta} + \Gamma^\phi_{0\phi}\Gamma^\phi_{0\phi} \tag{510}$$

With

$$\frac{1}{2}\frac{\partial^2 \ln|g|}{\partial x^0 \partial x^0} = \frac{3\ddot{R}}{Rc^2} - \frac{3\dot{R}^2}{R^2c^2}, \qquad \Gamma^r_{0r} = \Gamma^\theta_{0\theta} = \Gamma^\phi_{0\phi} = \frac{\dot{R}}{Rc}, \tag{511}$$

the 00 component of (500) becomes

$$\ddot{R} = -\frac{4\pi G R}{3}\left(\rho + \frac{3P}{c^2}\right) \tag{512}$$

which differs from our Newtonian equation (462) only by an additional, apparently very small, term of $3P/c^2$ as an effective source of gravitation. However, during the time when the Universe was dominated by radiation, this term was important, and even now it turns out to be not only important, but *negative* as well! During the so-called inflationary phase, $3P/c^2$ was hugely important. We will have much more to say about all of this later.

The rr component of the Field Equations is a bit more involved. Ready? With $f' \equiv df/dr$, we prepare a working table in advance of all the results we will need:

$$R_{rr} = \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial r^2} - \frac{\partial \Gamma^\lambda_{rr}}{\partial x^\lambda} + \Gamma^\eta_{r\lambda}\Gamma^\lambda_{r\eta} - \frac{1}{2}\Gamma^\eta_{rr}\frac{\partial \ln|g|}{\partial x^\eta} \tag{513}$$

$$\Gamma^r_{rr} = \frac{1}{2g_{rr}}\frac{\partial g_{rr}}{\partial r} = \frac{f'}{2f}, \quad \Gamma^0_{rr} = -\frac{1}{2cg_{00}}\frac{\partial g_{rr}}{\partial t} = \frac{fR\dot{R}}{c}, \quad \Gamma^r_{0r} = \frac{1}{2cg_{rr}}\frac{\partial g_{rr}}{\partial t} = \frac{\dot{R}}{Rc} \tag{514}$$

$$\Gamma^\phi_{\phi r} = \frac{1}{2g_{\phi\phi}}\frac{\partial g_{\phi\phi}}{\partial r} = \Gamma^\theta_{\theta r} = \frac{1}{2g_{\theta\theta}}\frac{\partial g_{\theta\theta}}{\partial r} = \frac{1}{r} \tag{515}$$

$$\frac{\partial \ln|g|}{\partial r} = \frac{f'}{f} + \frac{4}{r}, \quad \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial r^2} = \frac{1}{2}\frac{f''}{f} - \frac{1}{2}\frac{(f')^2}{f^2} - \frac{2}{r^2}, \quad \frac{\partial \Gamma^0_{rr}}{\partial x^0} = \frac{1}{c^2}\left(f\dot{R}^2 + fR\ddot{R}\right) \tag{516}$$

$$\frac{\partial \ln|g|}{\partial t} = \frac{6\dot{R}}{R}, \quad \frac{\partial \Gamma^r_{rr}}{\partial r} = \frac{f''}{2f} - \frac{(f')^2}{2f^2} \tag{517}$$

Putting it all together:

$$R_{rr} = \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial r^2} - \frac{\partial \Gamma^r_{rr}}{\partial r} - \frac{\partial \Gamma^0_{rr}}{\partial x^0} + (\Gamma^r_{rr})^2 + \left(\Gamma^\theta_{\theta r}\right)^2 + \left(\Gamma^\phi_{\phi r}\right)^2 + 2\Gamma^0_{rr}\Gamma^r_{r0} - \frac{\Gamma^r_{rr}}{2}\frac{\partial \ln|g|}{\partial r} - \frac{\Gamma^0_{rr}}{2c}\frac{\partial \ln|g|}{\partial t}$$

$$= \cancel{\frac{f''}{2f}} - \cancel{\frac{(f')^2}{2f^2}} - \cancel{\frac{2}{r^2}} - \left[\cancel{\frac{f''}{2f}} - \cancel{\frac{(f')^2}{2f^2}}\right] - \frac{1}{c^2}\left(f\dot{R}^2 + fR\ddot{R}\right) + \cancel{\frac{(f')^2}{4f^2}} + \cancel{\frac{2}{r^2}} + \frac{2f\dot{R}^2}{c^2} - \frac{f'}{4f}\left(\cancel{\frac{f'}{f}} + \frac{4}{r}\right) - \frac{3f\dot{R}^2}{c^2}$$

Thus, with $f'/f = 2krf$,

$$R_{rr} = -\frac{2f\dot{R}^2}{c^2} - \frac{fR\ddot{R}}{c^2} - 2kf = -\frac{8\pi G}{c^4}S_{rr} = -\frac{4\pi G}{c^4}g_{rr}(\rho c^2 - P) = -\frac{4\pi R^2 G f}{c^4}(\rho c^2 - P)$$

or

$$2\dot{R}^2 + R\ddot{R} + 2kc^2 = 4\pi G R^2(\rho - P/c^2) \tag{518}$$

Notice that r has disappeared, as it must! (Why must it?) Eliminating $\ddot{R}$ from (518) via equation (512) and simplifying the result leads to

$$\dot{R}^2 - \frac{8\pi G\rho R^2}{3} = -kc^2 \tag{519}$$

This is exactly the Newtonian equation (463) with the constant $2E$ "identified with" $-kc^2$. *But be careful.* The Newtonian version (463) was formulated with R dimensionless. In equation (519), R has been rescaled to have dimensions of length. To compare like-with-like we should repeat the calculation with the radial line element of the metric (487). Don't panic: this just amounts to replacing k with $1/a^2$. We then have more properly, $2E = -c^2/a^2$, carrying dimensions of $\dot{R}^2$ or $1/t^2$. But the major point is that equation (463) is valid in full general relativity! And, as we have just seen, the dynamical Newtonian energy constant may be identified with the geometrical curvature of the space.

One final item in our analysis. We have not yet made use of the equation for the conservation of energy-momentum, based on (175) and the Bianchi Identities:

$$T^{\mu\nu}_{;\mu} = \frac{1}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}T^{\mu\nu})}{\partial x^\mu} + \Gamma^\nu_{\mu\lambda}T^{\mu\lambda} = 0. \tag{520}$$

This constraint is of course built into the Field Equations themselves, and so adds no new information to our problem. But we may ask whether use of this equation from the start might have saved us some labour in getting to (519): it was a long derivation after all. The answer is an interesting "yes" and "no."

The $\nu = 0$ component of (520) reads

$$T^{\mu 0}_{;\mu} = \frac{1}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}T^{00})}{\partial x^0} + \Gamma^0_{\mu\lambda}T^{\mu\lambda} = 0. \tag{521}$$

Only affine connections of the form Γ^0_{ii} (spatial index i, no sum) are present, and with

$$\Gamma^0_{ii} = -\frac{1}{2g_{00}}\frac{\partial g_{ii}}{\partial x^0} = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^0}, \quad T^{00} = \rho c^2, \quad T^{ii} = Pg^{ii} = \frac{P}{g_{ii}}, \quad \text{(NO } i \text{ SUMS)}$$

equation (521) becomes:

$$\frac{1}{R^3}\frac{\partial(R^3\rho c^2)}{\partial t} + 3P\frac{\dot{R}}{R} = 0 \to \dot{\rho} + 3\left(\rho + \frac{P}{c^2}\right)\frac{\dot{R}}{R} = 0. \tag{522}$$

Notice how this embodies at once the law of conservation of mass (in the large c limit), and the first law of thermodynamics, $dE = -PdV$.

Exercise. Justify the last statement, and show that a pure vacuum energy density universe satisfies (522).

If we now use (522) to substitute for P in (512), after a little rearranging we easily arrive at the result

$$\frac{d}{dt}\left(\dot{R}^2\right) = \frac{8\pi G}{3}\frac{d}{dt}\left(\rho R^2\right) \tag{523}$$

which in turn immediately integrates to (519)! This is surely a much more efficient route to (519), except...except that we cannot relate the integration constant that emerges from (523) to the spatial curvature constant k. True, we have a faster route to our final equation, but with only dynamical information. Equation (519) is after all just a statement of energy conservation, as seen clearly from our Newtonian derivation. Without explicitly considering the Ricci R_{rr} component, we lose the geometrical connection between Newtonian E (just an integration constant) and $-kc^2$. We therefore have *consistency* between the energy conservation and Ricci approaches, but not true *equivalence.* A subtle and interesting distinction.

We may, however, work with the dynamical cosmological equation in its Newtonian form with absolutely no loss of generality

$$\dot{R}^2 - \frac{8\pi G\rho R^2}{3} = 2E, \tag{524}$$

where E is an energy integration constant, which can now be set by the convenient convention that $R = 1$ at the current time t_0. We write $2E$ in terms of observable quantities, H_0, the Hubble constant $\dot{R}_0$ and $\rho_0 c^2$, the current average value of the energy density of the Universe:

$$H_0^2\left(1 - \frac{8\pi G\rho_0}{3H_0^2}\right) \equiv H_0^2(1-\Omega_0) = 2E, \tag{525}$$

where Ω_0 parameter is

$$\Omega_0 = \frac{8\pi G\rho_0}{3H_0^2} \tag{526}$$

The Universe is positively curved (closed) or negatively curved (open) according to whether the measured value of Ω_0 is larger or smaller than unity. Defining the critical mass density ρ_c by

$$\rho_c = \frac{3H_0^2}{8\pi G} \tag{527}$$

the critical *energy* density in the Universe is $\rho_c c^2$. We have not yet assumed anything about the sources of ρ; they could involve a vacuum energy. The currently best measured value of H_0 (in standard astronomical units) is 67.6 km s^{-1} Mpc^{-1},[12] or $H_0 \simeq 2.2 \times 10^{-18}$ s¹. This number implies a critical density of 8.6×10^{-27} kg (about 5 hydrogen atoms) per cubic meter. The best so-called "concordance models" all point to an $\Omega_0 = 1$, $E = 0$, universe, but only if 70% of $\rho_0 c^2$ comes from the vacuum! About 25% comes from dark matter, which is matter that is not luminous but whose presence is inferred from its gravitational effects, and just 5% comes from ordinary baryonic matter in the form of gas and stars. We will denote the current vacuum contribution to Ω as Ω_{V0} and the matter contribution as Ω_{M0}. We have in addition a contribution from radiation, $\Omega_{\gamma 0}$, and while it is quite negligible now, in the early universe ($z \geq 1500$) it was completely dominant, even over the vacuum component. Indeed, all the Ω's are time-dependent quantities. The Universe went through a radiation-dominated phase, followed by a mass-dominated phase, and at about a redshift of 2, it started to switch to a vacuum-dominated phase, a transition we are currently still experiencing.

We live in interesting times.

[12] One "megaparsec," or Mpc is one million parsecs. One parsec ("pc") is the distance from the solar system at which the Earth–Sun semi-major axis would subtend an angle of one arcsecond: 1 pc $= 3.8056\times10^{13}$ km, 1 Mpc $= 3.8056\times10^{18}$ km. Stars are typically separated by 1 pc in galaxies, galaxies from one another by about 1Mpc.

8.5 The classic, matter-dominated universes

Let us return to the good old days, when the idea that the Universe was driven by "the energy density of the vacuum" was the stuff of *Star Trek* conventions, not something that serious-minded physicists spent their time with. The expansion dynamics of the Universe was thought to be dominated by matter, pure and simple, after a relatively brief phase when radiation dominated. Matter obeys the constraint that $\rho(t)R^3(t)$ remains constant with time. The dynamical equation of the Universe may then be written

$$\dot{R}^2 = H_0^2\left(1 - \Omega_{M0} + \frac{\Omega_{M0}}{R}\right) \tag{528}$$

or

$$\int_0^R \frac{dR'}{\sqrt{1 - \Omega_{M0} + (\Omega_{M0}/R')}} = \int_0^R \frac{R'^{1/2}dR'}{\sqrt{(1 - \Omega_{M0})R' + \Omega_{M0}}} = H_0 t \tag{529}$$

The nature of the integral depends upon whether Ω_{M0} is less than, equal to, or greater than, 1. The case $\Omega_{M0} = 1$ is trivial and leads immediately to

$$R = \left(\frac{3H_0 t}{2}\right)^{2/3} \tag{530}$$

our Einstein-de Sitter solution (472). According to (524) with $E = 0$, the average density is then given explicitly by

$$\rho = \frac{\dot{R}^2}{R^2}\frac{3}{8\pi G} = \frac{1}{6\pi G t^2} \tag{531}$$

Consider next the case $\Omega_{M0} > 1$, that of a Closed Universe. Then (529) may be written:

$$\int_0^R \frac{R'^{1/2}dR'}{\sqrt{1 - (1 - \Omega_{M0}^{-1})R'}} = \Omega_{M0}^{1/2} H_0 t \tag{532}$$

Set

$$(1 - \Omega_{M0}^{-1})R' = \sin^2\theta' \tag{533}$$

Then (532) becomes

$$\frac{2}{(1 - \Omega_{M0}^{-1})^{3/2}}\int_0^\theta \sin^2\theta'\, d\theta' = \frac{1}{(1 - \Omega_{M0}^{-1})^{3/2}}\int_0^\theta (1 - \cos 2\theta')\, d\theta' = \Omega_{M0}^{1/2} H_0 t \tag{534}$$

or

$$\frac{2\theta - \sin 2\theta}{2(1 - \Omega_{M0}^{-1})^{3/2}} = \Omega_{M0}^{1/2} H_0 t \tag{535}$$

With $\eta = 2\theta$, our final solution is in the form of a parameterisation:

$$R = \frac{1 - \cos\eta}{2(1 - \Omega_{M0}^{-1})}, \quad H_0 t = \frac{\eta - \sin\eta}{2\sqrt{\Omega_{M0}(1 - \Omega_{M0}^{-1})^3}} \tag{536}$$

Some readers may recognise these equations for R and $H_0 t$ as describing a cycloid, which is the path taken by a fixed point on the circumference of a wheel as the wheel rolls forward.

Precisely analogous expressions emerge for the case $\Omega_{M0} < 1$, which are easily verified from (529):

$$R = \frac{\cosh\eta - 1}{2(\Omega_{M0}^{-1} - 1)}, \quad H_0 t = \frac{\sinh\eta - \eta}{2\sqrt{\Omega_{M0}(\Omega_{M0}^{-1} - 1)^3}} \tag{537}$$

This is the solution for negative curvature, an Open Universe.

Exercise. Show that the equations analogous to (530), (536) and (537) for the FRW metric (496) are respectively

$$R = \omega^{1/3} t^{2/3} \quad \text{(Einstein − deSitter)}$$

$$R = \frac{\omega}{c^2}\left(\frac{1-\cos\eta}{2}\right), \quad ct = \frac{\omega}{c^2}\left(\frac{\eta - \sin\eta}{2}\right) \quad \text{(closed)}$$

$$R = \frac{\omega}{c^2}\left(\frac{\cosh\eta - 1}{2}\right), \quad ct = \frac{\omega}{c^2}\left(\frac{\sinh\eta - \eta}{2}\right) \quad \text{(open)}$$

where $\omega = 8\pi G\rho_{M0}R_0^3/3$. Show that $cdt = Rd\eta$ for the last two cases.

Exercise. Show that as $\eta \to 0$, both (536) and (537) reduce to $R = \Omega_0^{1/3}[3H_0t/2]^{2/3}$ (the same of course as [530] for $\Omega_0 = 1$), and at late times (537) becomes the "coasting" solution, $R = \sqrt{1-\Omega_0}H_0 t$. This means that a plot of all possible solutions of $R(t)$ versus $\Omega_0^{1/2}H_0 t$ would converge to exactly same solution at early times, regardless of Ω_0. Figure (10) shows this behaviour quite clearly.

What if $\Omega_0 = 0$? Show then that $R = H_0 t$ for all times. Wait. Could a universe *really* be expanding if there is nothing in it? Expanding with respect to exactly what, please? See Problem Set.

8.6 Our Universe

8.6.1 Prologue

Throughout most the 20th century, the goal of cosmology was to figure out which of the three *standard* model scenarios actually holds: do we live in an open, closed, or critical universe? Solutions with a cosmological constant were relegated to the realm of disreputable speculation, perhaps the last small chapter of a textbook, under the rubric of "Alternative Cosmologies." If you decided to sneak a look at this, you would be careful to lock your office door. All of that changed in 1998-9, when the results of two cosmological surveys of supernovae (Perlmutter et al. 1998 ApJ, **517,** 565; Riess et al. 1999 Astron J., 116, 1009) produced compelling evidence that the rate of expansion of the Universe is increasing with time, and that the spatial geometry of the Universe was flat ($k = 0$), even with what had seemed to be an under density of matter.

Though something of a shock at the time, for years there had been evidence that something was amiss with standard models. Without something to increase the rate of the Universe's expansion, the measured rather large value of H_0 consistently gave an embarrassingly short lifetime for the Universe, less than the inferred ages of the oldest stars! (The current stellar record holder is HE 1523-0901 at a spry 13.2 billion years.) People were aware that a cosmological constant could fix this, but to get an observationally reasonable balance between the energy density of ordinary matter and a vacuum energy density seemed like a desperate appeal to "it-just-so-happens" fine-tuning. But the new millenium brought with it unambiguous evidence that this is the way things are: our Universe is about 30% non-relativistic matter, 70% vacuum energy, and boasts a Euclidian spatial geometry. So right now *it just so happens* that there is a bit more than twice as much energy in the vacuum as there is in ordinary matter. Nobody has the foggiest idea why.

8.6.2 A Universe of ordinary matter and vacuum energy

It is perhaps some consolation that we can give a simple mathematical function for the scale factor $R(t)$ of our Universe. With $E = 0$ for a Euclidian space, (524) is

$$\dot{R} = \left(\frac{8\pi G\rho}{3}\right)^{1/2} R \tag{538}$$

The energy density ρc^2 is a combination of nonrelativistic matter ρ_M, for which ρR^3 is a constant, and a vacuum energy density ρ_V which remains constant. With ρ_{M0} the current value of ρ_M and $R_0 = 1$, $\rho_M = \rho_{M0}/R^3$ and therefore

$$\rho R^2 = \rho_V R^2 + \frac{\rho_{M0}}{R} \tag{539}$$

Substituting (539) into (538) leads to

$$\frac{R^{1/2}\, dR}{\sqrt{\rho_{M0} + \rho_V R^3}} = \sqrt{\frac{8\pi G}{3}}\, dt. \tag{540}$$

Integration then yields

$$\int \frac{R^{1/2}\, dR}{\sqrt{\rho_{M0} + \rho_V R^3}} = \frac{2}{3}\int \frac{d(R)^{3/2}}{\sqrt{\rho_{M0} + \rho_V R^3}} = \frac{2}{3}\frac{1}{\sqrt{\rho_V}} \sinh^{-1}\left(\sqrt{\frac{\rho_V}{\rho_{M0}}} R^{3/2}\right) = \sqrt{\frac{8\pi G}{3}} t \tag{541}$$

Equation (541) then tells us:

$$R^{3/2} = \sqrt{\frac{\rho_{M0}}{\rho_V}} \sinh\left(\frac{3}{2}\sqrt{\frac{8\pi G\rho_V}{3}} t\right). \tag{542}$$

In terms of the Ω parameters, we have

$$\Omega_M = \frac{8\pi G\rho_{M0}}{3H_0^2}, \quad \Omega_V = \frac{8\pi G\rho_V}{3H_0^2}, \tag{543}$$

henceforth dropping the 0 subscript on the Ω's with the understanding that these parameters refer to current time. The dynamical equation of motion (538) at the present epoch tells us directly that

$$\Omega_M + \Omega_V = 1. \tag{544}$$

In terms of the observationally accessible quantity Ω_M, the scale factor$R$ becomes

$$R = \frac{1}{(\Omega_M^{-1} - 1)^{1/3}} \sinh^{2/3}\left(\frac{3}{2}\sqrt{1 - \Omega_M}\; H_0 t\right), \tag{545}$$

or, with $\Omega_M = 0.27$,

$$R = 0.7178 \sinh^{2/3}(1.282 H_0 t). \tag{546}$$

This gives a current age of the Universe t_0 of

$$H_0 t_0 = \frac{2\sinh^{-1}(\sqrt{\Omega_M^{-1} - 1})}{3\sqrt{1-\Omega_M}} = \frac{\sinh^{-1}[(0.7178)^{-3/2}]}{1.282} = 0.992, \tag{547}$$

i.e., t_0 is almost precisely $1/H_0 \simeq 13.7$ billion years. (This calculation ignores the brief period of the Universe's history when it was radiation dominated.)

Exercise. Show that the redshift z is related to the time t since the big bang by

$$z = 1.393 \sinh^{-2/3}\left(\frac{1.271t}{t_0}\right) - 1$$

If a civilisation develops 5 billion years after the big bang and we detect their signals(!), at what redshift would they be coming from?

8.6.3 The parameter q_0

As we look back in time the Hubble parameter $\dot{R}/R$ may be expanded in a Taylor series in time:

$$H(t) = \frac{\dot{R}}{R} = H_0 + (t - t_0)\left(\frac{\ddot{R}}{R} - H_0^2\right) + ... \tag{548}$$

since $\dot{R}^2/R^2$ at the present is H_0^2. This may be written

$$H(t) = H_0 + H_0^2(t_0 - t)(1 + q_0) \tag{549}$$

where

$$q_0 \equiv -\frac{\ddot{R}R}{\dot{R}^2} \to -\frac{\ddot{R}_0}{H_0^2} \quad \text{for } R_0 = 1 \tag{550}$$

is known as the deceleration parameter. In units where $R_0 = 1$, $q_0 = -\ddot{R}_0/H_0^2$. Then, since $(1+z)R(t) = 1$, to leading order in z, $z = H_0(t_0 - t)$ and

$$H(z) = H_0[1 + z(1 + q_0)] \tag{551}$$

Note that if $q_0 < 0$, the Universe is accelerating. To illustrate the usefulness of q_0, consider the often needed integral $\int c dt/R(t)$. With $R_0 = 1$, $t - t_0 = \delta t$,

$$\frac{1}{R(t_0 + \delta t)} = 1 - \delta t \dot{R}_0 + (\delta t)^2(\dot{R}_0^2 - \frac{\ddot{R}_0}{2}) + ... = 1 + z \tag{552}$$

hence $\delta t = -z/H_0$ to leading order and to next order (with $(\delta t)^2 = z^2/H_0^2$ in the correction quaratic term),

$$-\delta t = t_0 - t = \frac{1}{H_0}\left[z - \left(1 + \frac{q_0}{2}\right)z^2\right] + ... \tag{553}$$

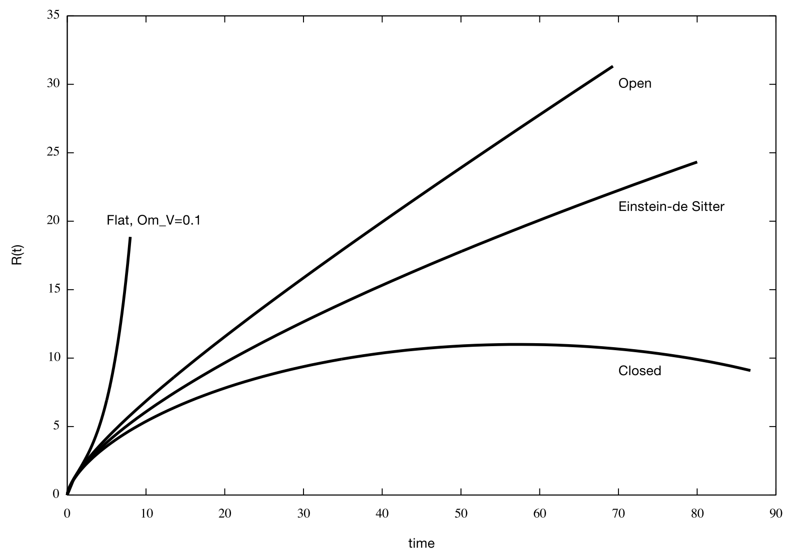

Figure 10: $R(t)$ versus time, units of $\Omega_M^{-1/2} H_0^{-1}$, for four model universes. 'Closed' is eq. (536); 'Einstein- de Sitter' eq. (530); 'Open' eq. (537). 'Closed' has $\Omega_M = 1.1$, for 'Einstein–de Sitter' $\Omega_M = 1$, 'Open' has $\Omega_M = 0.9$. The curve labelled 'Flat, Om_V= 0.1', eq. (545), includes a cosmological constant with a vacuum contribution of $\Omega_V = 0.1$, so that $\Omega_M + \Omega_V = 1$ and the spatial geometry is Euclidian.

Next recall equation (478):

$$l(z) = c\int_t^{t_0} \frac{dt'}{R} = c(t_0 - t) + \frac{cH_0}{2}(t - t_0)^2 + ... \tag{554}$$

and using (553):

$$l(z) = \frac{c}{H_0}\left[z - \frac{z^2}{2}(1 + q_0) + ...\right] \tag{555}$$

Equation (555) is general for any FRW model. It may be seen that q_0 embodies the leading order deviations from a simple expansion model in which $l(z) \propto z$, and that it is the only cosmological parameter that is available to observers.

8.7 Radiation-dominated universe

The thermal history of the early universe is discussed in §8.10; it is useful at this stage, however, to understand the dynamical expansion of a universe dominated by radiation.

In a universe whose energy density $\rho_\gamma c^2$ is dominated by radiation (or more generally by relativistic particles),

$$\rho_\gamma \propto 1/R^4.$$

This follows because the energy density of radiation is proportional to T_γ^4 where T_γ is the radiation temperature, and the temperature has the same evolutionary history of a photon with energy $h\nu$: the frequency ν and T_γ both decrease as $1/R$. The current Universe is of course not radiation-dominated, but for the first several hundred thousand years it was. (See

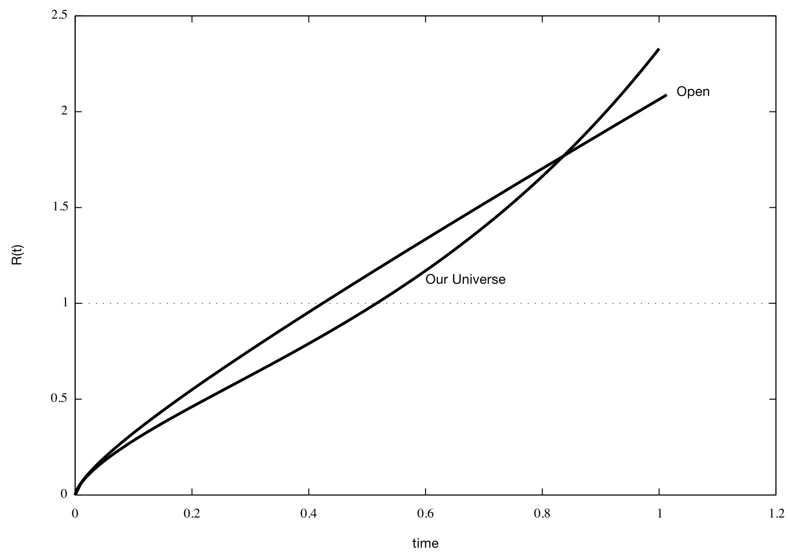

Figure 11: Comparison of $R(t)$ versus $\Omega_M^{1/2} H_0 t$ for our Universe eq. (545) and an open universe eq. (537), both with $\Omega_M = 0.27$. Crossing of the dotted line $R = 1$ (at the current age) occurs earlier for the positive energy open universe, which is about 20% younger than our Universe. This would be too young for the oldest stars. The positive E value kickstarts the open universe ahead, but it eventually lags behind when the vacuum energy starts to play a dominant role...when the expansion of space has produced more vacuum!

§below.) In these early times the dynamical equation of motion (524) is dominated by the left side of the equation, both of whose terms are very large. The dynamical equation of motion is then

$$R^2 \dot{R}^2 = \frac{8\pi G \rho_{0\gamma}}{3} = \text{(constant)}, \tag{556}$$

where $\rho_{0\gamma} c^2$ is the current energy density in relativistic particles. This implies that

$$R(t) \propto t^{1/2} \quad \text{(radiation dominated universe)}. \tag{557}$$

If we now go back to the dynamical equation of motion, we may solve for $\rho_\gamma c^2$:

$$\rho_{0\gamma} c^2 = \frac{3c^2}{8\pi G} \left(\frac{\dot{R}^2}{R^2} \right) = \frac{3c^2}{32\pi G t^2}. \tag{558}$$

We then have an *exact* expression for what the total energy density in *all* relativistic particles must be.

This is rather neat. At at time of one second, the Universe had an energy density of 4×10^{25} J m^{-3}, and that is that. Moreover, since the total energy density is now fixed, the greater the number of relativistic particle species there was at the time of the early Universe, the *smaller* the temperature at a fixed time. During the epoch when hydrogen was being fused into helium and only a few other low atomic weight nuclei, the abundances were very sensitive to temperature. This sensitivity, combined with observationally-determined abundances, has been used to limit the number of types of neutrinos that could have been present during the era of nucleosynthesis.

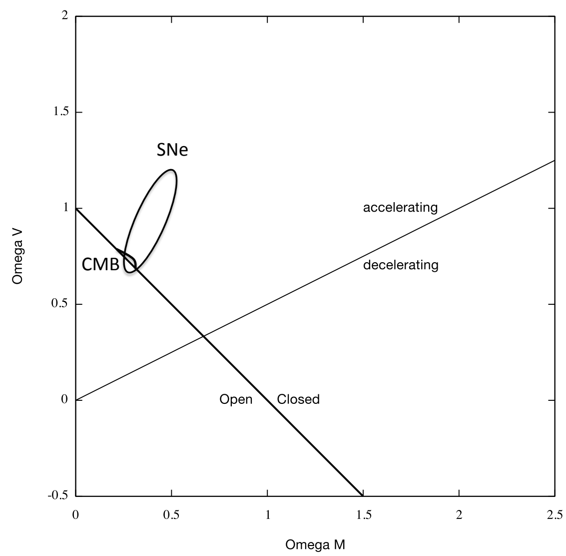

Figure 12: Parameter plane of Ω_V versus Ω_M assuming no radiation contribution. Regions of open/closed geometry and currently accelerating/decelerating dynamics are shown. Also shown are approximate zones of one standard deviation uncertainties for the distant supernova data (SNe) and —you have to squint—for fluctuations in the cosmic microwave background radiation (CMB), which came a decade later. Note the powerful constraint imposed by the latter: we no longer depend on the SNe data. That the Universe is accelerating is beyond reasonable doubt.

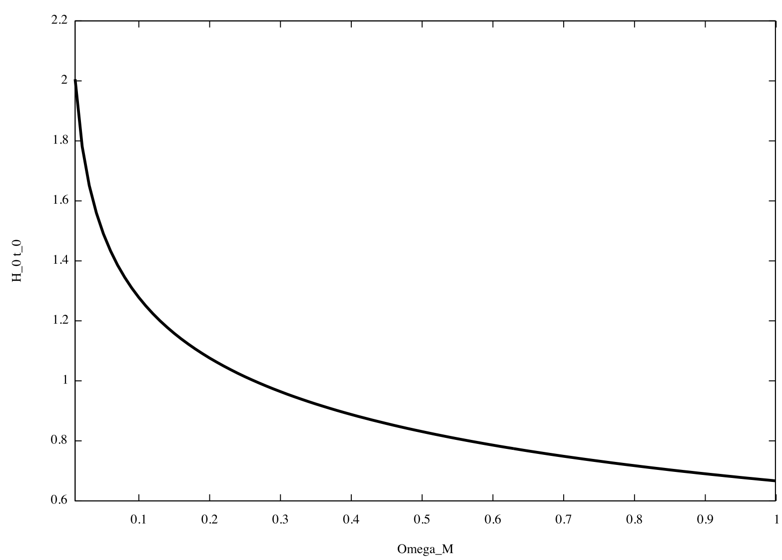

Figure 13: The current age of the Universe H_0t_0 as a function of Ω_M for flat, matter plus vacuum energy models with negligible radiation. As Ω_M approaches unity, the model recovers the Einstein–de Sitter value $H_0t_0 = 2/3$; as $\Omega_M \to 0$, H_0t_0 becomes proportional to $-\ln \Omega_M$ (Show!), and we recover a "logarithmic eternity," first highlighted by Sir Arthur Eddington.

8.8 Observational foundations of cosmology

8.8.1 Detection of cosmlogical redshifts

To me, the name of Vesto Melvin Slipher has always conjured up images of some 1930's J. Edgar Hoover FBI G-man in a fedora who went after the bad guys. But Vesto was a mild-mannered and very careful astronomer. If we allow that Edwin Hubble was the father of modern cosmology, then Slipher deserves the title of grandfather.

In 1912, using a 24-inch reflecting telescope, Slipher was the first person to measure the redshifts of external galaxies. He didn't quite know that that was what he had done, because the notion of galaxies external to our own was not one that was well-formed at the time. Nebular spectroscopy was hard, tedious work, spreading out the light from the already very faint, low surface brightness smudges of spiral nebulae through highly dispersive prisms. Slipher worked at Lowell Observatory, a small, isolated, private outpost in Flagstaff, Arizona. Percival Lowell, the proprietor, was obsessed at the time with mapping what he thought were the Martian canals. Slipher was an assiduous worker, but one who was far from the centres of great astronomical activity. By 1922, he had accumulated 41 spectra of spiral nebulae, of which 36 showed a shift toward the red end of the spectrum. But he had no way of organising these data to bring out the linear scaling of the redshift with distance for the simple reason that he hadn't any idea what the distances to the nebulae were. Other observations by Wirtz and Lundmark were at this time showing an apparent trend of greater redshift with fainter nebulae, but the decisive step was taken by Edwin Hubble. With the aid of Milton Humason, Hubble found a linear relationship between galactic distance and redshift (E. Hubble 1929, PNAS, **15,** 168.) How did Hubble determine the distances to the nebulae? Using the new 100-inch telescope on Mt Wilson, Hubble had earlier resolved individual Cepheid variables, a class of variable star, in the outer arms of the Andromeda spiral *galaxy*, as we shall henceforth refer to it. For, in obtaining the distances to the nebulae, Hubble also showed that they must be galaxies in their own right.

Cepheid variable stars were well-studied in our own galaxy, where it was found that they have a well-defined relationship between the time period over which the star's brightness oscillates, and the absolute mean luminosity of the star. To obtain a distance, one proceeds as follows. Find a Cepheid variable. Measure its oscillation period. Determine thereby its true luminosity. Measure the star's flux (which is all that can actually be measured). The flux is the true luminosity L divided by $4\pi r^2$, where r is the distance to the star. By measuring the flux and inferring L, deduce r. Simple—if you just happen to have a superb quality, 100-inch telescope handy. To such an instrument, only Hubble and a small handful of other astronomers had access[13].

8.8.2 The cosmic distance ladder

As observations improved through the 1930's the linear relation between velocity and distance, which became known as the *Hubble Law,* $v = H_0 r$, became more firmly established. There are two major problems with collecting data in support of the Hubble expansion.

First, galaxies need not be moving with the Hubble expansion (or "Hubble flow"): their motions are affected by neighbouring masses. The best known example of this is the An-

[13]Much later, in 1952, Hubble's quantitative results were found to be inaccurate, because Cepheids come in two quite separate populations, with very different Period-Luminosity relations. This was discovered by W. Baade, introducing the concept of distinct stellar populations of very different ages into astronomy. It also completely revised the whole extragalactic distance scale, though it kept intact the linear redshift-distance relation.

dromeda galaxy, whose is redshift is in fact a blueshift! It is approaching us (the Milky Way Galaxy) at about 300 km per second. This problem is greatest for nearby galaxies whose "peculiar velocity" (deviation from Hubble flow) is a large fraction of its recession velocity. Second, it is very difficult to establish distances to cosmological objects. We can establish distances to relatively nearby objects realtively easily, but these galaxies are precisely the ones affected by large peculiar velocities. Those galaxies unaffected by large peculiar velocities are the ones whose distances are difficult to establish!

But observational astronomers are resourceful, and they have come up with a number of ingenious techniques which have served them well. The idea is to create a cosmic "distance ladder" (perhaps better described as a linked chain) in which you start with direct measurements on certain objects, and use those measurments to calibrate other objects more distant. Then repeat. Here is how it works.

Start with our solar system. These days, we can bounce radar singals off planets and measure the time of flight (even testing general relativity in the process, as we have seen) to get extremely accurate distances. Next, we make use of our knowledge of the astronomical unit (AU) thus obtained to use the classic technique of trigonometric parallax. This makes use of the fact that the earth's motion around the sun creates a baseline of about 2 AU from which we have a different perspective on nearby stars. We see nearby stars shift in angular position on the sky relative to their much more distant counterparts. The angular shift is inversely proportional to the distance from the solar system. We define the unit of distance known as a parsec (pc) as the distance that 1 AU would subtend 1 second of arc. We can turn this perspective around and say that the parallax angle, perversely denoted as π in the astronomical literature and deduced from the shift in a star's apparent position, corresponds to a distance to the star of $1/\pi$ parsecs when π is in arcseconds. (One parsec is $\simeq 3.085678 \times 10^{16}$ m. Because 1 AU is a defined exact quantity, so is 1 pc.)

The next rung in the ladder is known as spectroscopic parallax. There is no actual parallax, just an analogue. The idea is that you use the method of trigonometric parallax to obtain distances stars of a given spectral class. For this, you need to know what the stellar spectrum looks like in detail. Detailed agreement between spectra mean the same type of star, with the same mass and luminosity. Knowing the distance to a star of a given spectral type and measuring its flux, you know the intrinsic luminosity L, since $F = L/4\pi r^2$ where F is the measured flux and r the measured distance to the star. Then, when you see a similar spectrum in a much more distant star, you know its intrinsic luminosity. You then measure its F, and deduce *its* distance!

Keep going. Distances to Cepheid variables can be calibrated by spectroscopic parallax, and these bright stars have, as we have seen, a well-defined period-luminosity relationship. They are bright enough that they can be seen indivdually in external galaxies. Measure, thereby, the distances to these external galaxies.

Keep going. In studying the properties of external galaxies, Tully and Fisher showed that the rotation velocities of spiral galaxies (measured by the Doppler shifts in the moving stellar spectra) was tightly correlated with the intrinsic luminosity of the galaxy. This is not terribly surprising in itself: the larger the stellar mass of a galaxy the larger the luminosity and the larger the mass the larger the rotational velocity. Tully and Fisher crafted this notion into a widely used tool for establishling the distances to very distant galaxies. Elliptical galaxies, which are supported by the dispersion of stellar velocities rather than their systematic rotation were also turned into useful distance indicators. Here the correlation between luminosity and velocity dispersion is known as the Faber-Jackson relation.

The final step in the cosmic distance ladder involves Type Ia supernovae. Type Ia supernovae are thought to occur when a white dwarf in a tight binary system accretes just enough matter from its companion to tip itself over the "Chandrasekhar mass." (This mass is the maximum possible mass a white dwarf can sustain by electron degeneracy pressure, about

1.4 times the mass of the sun.) When this mass is exceeded, the white dwarf implodes, overwhelmed by its now unsupportable self-gravitational attraction. In the process, carbon and oxygen nuclei are converted to ^{56}Ni, triggering a thermonuclear explosion that can be seen quite literally across the Universe. What is nice about type Ia supernovae, from an astronomical perspective, is that they always occur in a white dwarf of the same mass. Therefore there is little variation in the absolute intrinsic luminosity of the supernova explosion. To the extent that there is some variation, it is reflected not just in the luminosity, but in the rise and decay times of the emission, the "light curve." The slower the decline, the larger the luminosity. So you can correct for this. This relation has been well calibrated in many galaxies with well-determined distances.

The Type Ia supernova data were the first to provide compelling evidence that the Universe was expanding. To understand how this was obtained we need to return to our concept of flux, $F = L/4\pi r^2$, and understand how this changes in an expanding, possibly curved, spatial geometry.

8.8.3 The redshift–magnitude relation

The flux that is measured from a source at cosmological distances differs from its simple $L/4\pi r^2$ form for several important reasons. It is best to write down the answer, and then explain the appearance of each modification. The flux from an object at redshift z is given by

$$\mathcal{F}(z) = \frac{LR^2(t)}{4\pi R_0^4 l^2(z)} = \frac{L}{4\pi(1+z)^2 R_0^2 l^2(z)} \tag{559}$$

where $l(z)$ is computed, as always, by $c\int_{t_0}^{t} dt'/R(t')$, with this definite integral written as a function of z for the cosmological model at hand. (Recall $R(t)/R_0 = 1/(1+z)$, relating t to z.) The luminosity distance is then defined by $\mathcal{F}(z) = L/4\pi d_L^2$ or

$$d_L(z) = (1+z)R_0 l(z), \tag{560}$$

which reduces to the conventional Euclidian distance at small z.

Explanation: The two factors of $1+z$ in the denominator of (559) arise from the change in the luminosity L. First, the photons are emitted with Doppler-shifted energies. But even if you were measuring only the rate at which the photons were being emitted like bullets, there is an additional second $1+z$ factor, quite separate from the first, due to the emission interval time dilation. The proper radius of the sphere over which the photons from the distant source at z are now distributed is $R_0 l(z)$, where R_0 is as usual the *current* value of $R(t)$. One is free to use a metric where $R_0 = 1$, but we present the form (559) for complete generality. For the Einstein-de Sitter universe with $R_0 = 1$,

$$\mathcal{F}(z) = \frac{LH_0^2}{16\pi c^2(1+z)(\sqrt{1+z}-1)^2} \qquad \text{(Einstein} - \text{de Sitter)} \tag{561}$$

The simple Euclidian value of $L/4\pi r^2$ is recovered at small z by rembering $cz = v = H_0 r$, whereas at high z, all the photons come from the current horizon distance $2c/H_0$, and the $1/z^2$ behaviour in $\mathcal{F}$ is due entirely to $1+z$ Doppler shifts.

A "magnitude" is an astronomical conventional unit used for, well, rather arcane reasons. It is a logarithmic measure of the flux. Explicitly:

$$\mathcal{F} = \mathcal{F}_0 10^{-0.4m} \tag{562}$$

where $\mathcal{F}$ is the measured flux, $\mathcal{F}_0$ is a constant that changes depending upon what wavelength range you're measuring. m is then defined as the apparent magnitude. (Note: a larger magnitude is *fainter*. Potentially confusing.) The "bolometric magnitude" covers a wide wavelength range and is a measure of the total flux; in that case $F_0 = 2.52 \times 10^{-8}$ J m^{-2}. Astronomers plot m versus z for many objects that ideally have the same intrinsic luminosity, like type Ia supernovae. Then they see whether the curve is well fit by a formula like (559) for an FRW model. It was just this kind of exercise that led to the discovery in 1998-9 by Perlumtter , Riess and Schmidt that our Universe must have a large value of Ω_V: we live in an accelerating Universe!

Exercise. Show that

$$\mathcal{F}(z) = \frac{LH_0^2}{4\pi c^2 z^2}\left[1 + (q_0 - 1)z + ...\right]$$

for any FRW model. Thus, with knowledge of L, observers can read off the value of H_0 from the dominant $1/z^2$ leading order behaviour of the redshift-magnitude data, but that knowledge of q_0 comes only once the leading order behaviour is subtracted off.

With the determination of Ω_V, the classical problem of the large scale structure of the Universe has been solved. There were 6 quantities to be determined:

- The Hubble constant, $H_0 = \dot{R}/R$ at the present epoch.
- The age of the Universe, t_0.
- The curvature of the Universe, in essence the integration constant E.
- The ratio Ω_0 of the current mass density ρ_0 to the critical mass density $3H_0^2/8\pi G$.
- The value of the cosmological constant or equivalently, the ratio Ω_V of the vacuum energy density $\rho_V c^2$ to the critical energy density.
- The value of the q_0 parameter.

Within the context of FRW models, these parameters are not completely independent, but are related by the dynamical equations for $\ddot{R}$ (512) and $\dot{R}^2$ (524). A quick summary:

- $H_0 \simeq 67$ km s^{-1} Mpc^{-1}.
- $t_0 \simeq 13.7$ billion years
- $E \simeq 0$
- $\Omega_0 \simeq 0.27$
- $\Omega_V \simeq 0.73$.
- $q_0 = \Omega_0/2 - \Omega_V \simeq -0.6$

Exercise. Derive the last result on this list.

This brief list of values hardly does justice to the century-long effort to describe our Universe with precision. Because astronomers were forced to use galaxies as "standard candles" (the colloquial term for calibrated luminosity sources), their measuring tools were fraught with uncertainties that never could be fully compensated for. It was only the combination

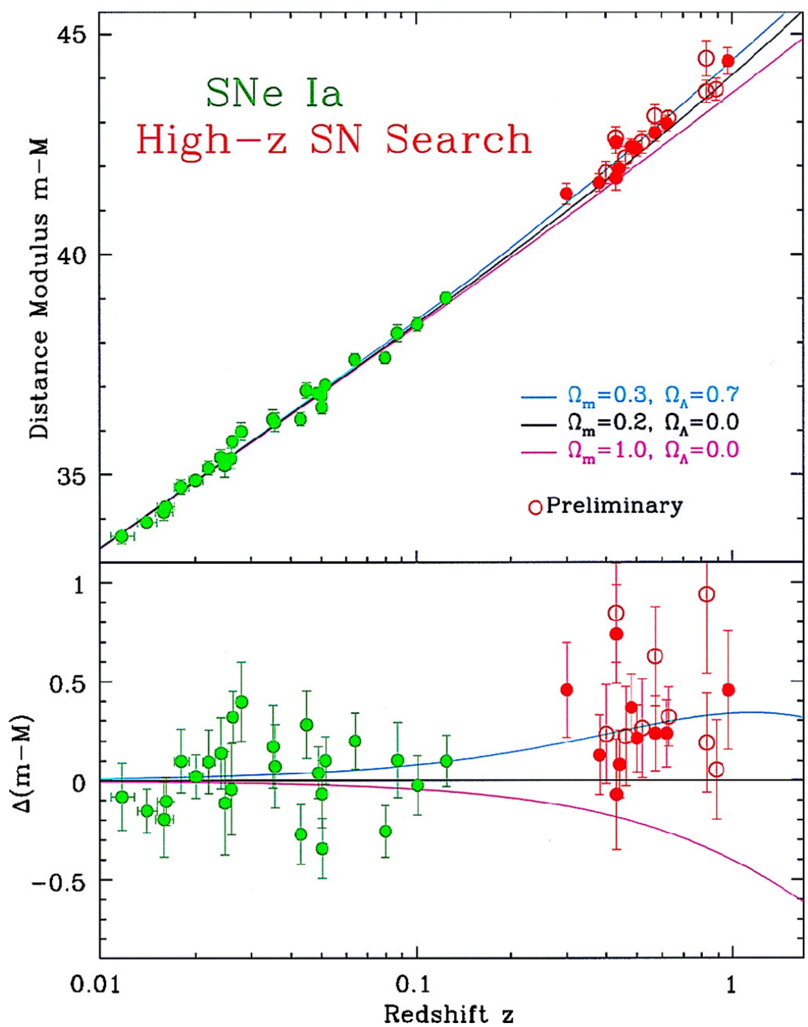

Figure 14: Evidence for an accelerating Universe from type Ia supernovae. The top figure shows a redshift-magnitude plot for three different FRW models, $\Omega_0 = 1$, $\Omega_V = 0$, an Einstein-de Sitter model in mangenta; $\Omega_0 = 0.2$, $\Omega_V = 0$, an open model, in black, and $\Omega_0 = 0.3$, $\Omega_V = 0.7$ an accelerating model, in blue. The bottom panel shows the same with the inverse square slope removed. The data are much better fit by the accelerating model.

of establishing a truly standard candle via the type Ia supernovae, together the tehcnical capability of high receiver sensitivity and automated searches that allowed the programme ("The Supervova Cosmology Project") to succeed.

Since the 1998/9 breakthrough, cosmologists have not been idle. The development of extremely sensitive receivers and sophisticated modelling and simulation techniques have turned the remnant radiation from the big bang itself into a vast treasure trove of information. In particular, the nature of the tiny fluctuations that are present in the radiation intensity—more specifically the radiation temperature—allow one to set very tight constraints on the the large scale parameters of our Universe. Not only are these measurements completely consistent with the supernova data, the results of the missions known as WMAP and Planck render them all but obsolete. Figure (12) speaks for itself.

8.9 The growth of density perturbations in an expanding universe

The Universe is expanding and the density of nonrelativistic matter decreasing as $1/R^3$. In such a background, as the raw material to form condensed objects is diminishing do quickly, does gravity even allow the sort of runaway collapse we think of when we envision a star or a galaxy forming? Determining the fate of a small overdensity or underdensity of nonrelativistic matter in an FRW universe is a problem that can be approached via (a not too difficult) analysis.

We require two equations. The first expresses the conservation of ordinary matter. The mass within a volume V, $\int_V \rho\, dV$ is changed only if matter flows in or out from the boundaries of V. The flux of mass (mass per kg per square meter) is $\rho\boldsymbol{v}$. Hence

$$\frac{d}{dt}\int_V \rho\, dV = \int_V \frac{\partial\rho}{\partial t}\, dV = -\int_{\partial V} \rho\boldsymbol{v}\cdot\boldsymbol{dA} = -\int_V \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\, dV$$

where ∂V is the volume's boundary and we have used the divergence. The volume V is arbitrary, so we much have

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0 \tag{563}$$

which is the equation of mass conservation. Newton's equation of motion states that if a mass element of fluid ρdV is accelerating, then it is acted on by a gravitational force $-\rho dV\boldsymbol{\nabla}\Phi$, where Φ is the associated potential function. In other words, the force equation reads after cancellation of ρdV,

$$\left[\frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\right]\boldsymbol{v} = -\boldsymbol{\nabla}\Phi \tag{564}$$

Note that the acceleration measured relative to a space-time coordinate background means that the "total time derivative" must be used, $\partial_t + v_i\partial_i$ in index notation.

The local behaviour of the density is entirely Newtonian. The expansion of the universe is described by

$$\boldsymbol{v} = \frac{\dot{R}}{R}\boldsymbol{r} \tag{565}$$

a familiar Hubble law. The mass equation then becomes

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\rho\frac{\dot{R}}{R}\right) = 0 \tag{566}$$

With the background ρ independent of position,

$$\frac{d\ln\rho}{dt} + \frac{3\dot{R}}{R} = 0 \tag{567}$$

whence ρR^3 is a constant, as we know. As for the force equation, a straightforward calculation yields

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \frac{\ddot{R}}{R}\boldsymbol{r} \tag{568}$$

and, as the discussion of §8.1.1 shows that $\ddot{R}/R = -4\pi G\rho/3$, our solution for the background expansion is correct. The Poisson equation

$$\nabla^2\Phi = 4\pi G\rho \tag{569}$$

is likewise solved by our solution. (Try $\Phi = 2\pi G\rho r^2/3$).

We are interested in the behaviour of small disturbances $\delta\rho$, $\delta\boldsymbol{v}$ and $\delta\Phi$ on top of this background:

$$\rho \to \rho + \delta\rho, \quad \boldsymbol{v} \to \boldsymbol{v} + \delta\boldsymbol{v}, \quad \Phi \to \Phi + \delta\Phi \tag{570}$$

We replace our dynamical variables as shown, and because the δ-quantities are small, we retain them only through linear order, ignoring quadratic and higher order tems. The mass equation is

$$\frac{\partial\delta\rho}{\partial t} + \boldsymbol{\nabla\cdot}(\rho\boldsymbol{\delta v}) + \boldsymbol{\nabla\cdot}(\boldsymbol{v}\delta\rho) = 0 \tag{571}$$

Noting that the gradient of ρ vanishes and that the equilibrium $\boldsymbol{v}$ satisfies $\boldsymbol{\nabla\cdot v} = -\partial_t \ln\rho$, it is straightforward to show that the perturbed linearised mass conservation equation simplifies to

$$\left[\frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\right]\frac{\delta\rho}{\rho} + \boldsymbol{\nabla\cdot\delta v} = 0 \tag{572}$$

The linearised equation of motion

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right]\boldsymbol{\delta v} + (\boldsymbol{\delta v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}\delta\Phi \tag{573}$$

becomes

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right]\boldsymbol{\delta v} + \frac{\dot{R}}{R}\boldsymbol{\delta v} = -\boldsymbol{\nabla}\delta\Phi \tag{574}$$

Now change to comoving coordinates. This is a simple task. Let $\boldsymbol{r} = R(t')\boldsymbol{r'}$ and $t = t'$. Then $\boldsymbol{r'}$ (or x'_i in index notation) is a comoving spatial coordinate. The partial derivative transformation is (sum over repeated i):

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \frac{\partial x'_i}{\partial t}\frac{\partial}{\partial x'_i}, \qquad \boldsymbol{\nabla} = \frac{1}{R}\boldsymbol{\nabla}', \qquad \frac{\partial x'_i}{\partial t} = -\frac{x_i}{R^2}\dot{R} = -\frac{v_i}{R}, \tag{575}$$

so that

$$\frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla}) = \frac{\partial}{\partial t'}, \tag{576}$$

which is a time derivative following a fluid element of the unperturbed expansion. Then, our two equations for mass conservation and dynamics are

$$\frac{\partial}{\partial t'}\frac{\delta\rho}{\rho} + \frac{1}{R}\boldsymbol{\nabla}'\boldsymbol{\cdot}\boldsymbol{\delta v} = 0 \tag{577}$$

$$\frac{\partial\boldsymbol{\delta v}}{\partial t'} + \frac{\dot{R}}{R}\boldsymbol{\delta v} = -\frac{1}{R}\boldsymbol{\nabla}'\delta\Phi \tag{578}$$

Taking $\boldsymbol{\nabla}'\boldsymbol{\cdot}$ of (578) and using (577) leads to

$$\frac{\partial^2}{\partial t'^2}\frac{\delta\rho}{\rho} + \frac{2\dot{R}}{R}\frac{\partial}{\partial t'}\frac{\delta\rho}{\rho} = \frac{1}{R^2}\nabla'^2\delta\Phi = 4\pi G\rho\left(\frac{\delta\rho}{\rho}\right) \tag{579}$$

where the last equality is the linearised Poisson equation $\nabla^2\delta\phi = 4\pi G\delta\rho$. For an Einstein-de Sitter univese, recall that $\rho = 1/6\pi Gt^2$ and $\dot{R}/R = 2/3t$, where we have dropped the primes. Using the notation $\delta = \delta\rho/\rho$ and a dot $\dot{\delta}$ for a time derivative, our differential equation for the growth of small perturbations in an Einstein de-Sitter universe takes on a very elegant form:

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0 \tag{580}$$

This differential equation has two very simple linearly independent solutions, one where δ decays as $1/t$, the other where it grows as $t^{2/3}$. (Show!) The important point is that it displays none of the explosive exponential growth typical of fluid instabilities in a static background. The growing solution of the small perturbation is $t^{2/3}$, growing no faster than the universe expands. This is a pretty torpid tempo. For the musically inclined, think of something between *larghissimo* and *adagio.*

How our Universe grew both its large and small scale structures has long been a great mystery, one that is remains far from understood. To make things grow in the barren soil of the Universe, you need to start out with very healthy-sized seeds. The questions of where those might come from and whether we can see their imprints in the cosmic background radiation are the topics of the next section.

8.10 The Cosmic Microwave Background Radiation

8.10.1 Prologue

The Universe is expanding, and expanding systems cools adiabatically. That means if we follow history backwards, the Universe was once extremely hot and dense. The frequency ν of a photon scales as $1/R$ and the energy density as $1/R^4$ because the radiation temperature T_γ is an average of ν. The energy density of matter (dominated by rest mass) scales as $1/R^3$, so going back in time, the Universe must at some point have become radiation dominated. We will see that this occurred at a redshift of about 1000.

At redshifts larger than 1000, not only was radiation dominant, it was also well-coupled to matter and thus in a state of complete thermodynamic equilibrium: the radiation set the matter temperature. In thermodynamic equilibrium, the number density of photons is given by the Planck function associated with a blackbody spectrum:

$$n(\nu, T)\,d\nu = \frac{(8\pi\nu^2/c^3)d\nu}{\exp(h\nu/kT_\gamma) - 1} \tag{581}$$

At about the same epoch that the radiation became subdominant, it also lost coupling with the matter and therefore was no longer maintained in themal equilibrium via collisions. Imagine an extreme, but realistic, situation in which the photons evolve with the expansion of the Universe and but are otherwise untouched. You might expect the shape of the spectrum to be maintained, but that the number density would perhaps be diluted below that of a true blackbody (e.g. sunlight on earth). In fact, the number density and effective temperature of the photons vary in exactly the right way to maintain a true blackbody distribution. The decreased number density is exactly correct for the cooling temperature.

To see this, start with the spectrum (581) at the time of "last scattering" at time t, which is the last moment of maintained thermal equilbrium, and let the group of photons with frequency ν and disperson $d\nu$ evolve with the Universe till time t'. The first thing to note is that is the t' number density $n(\nu', t', T')$ is diluted by an overall normalisaton factor of $R^3(t)/R^3(t')$ from the original time t blackbody. In addition, each t' photon that we observe at frequency ν' must have come from a t-frequency of $\nu = R'\nu'/R$ ($R' \equiv R(t')$), in the original blackbody distribution. Thus, to get the more recent time t' spectrum, simply take the original time t blackbody distribution, dilute it by R^3/R'^3, and replace ν everywhere in the formula by $R'\nu'/R$:

$$n(\nu', t', T')d\nu' = \frac{R^3}{R'^3}\frac{(8\pi/c^3)(R'\nu'/R)^2 d(\nu' R'/R)}{\exp(hR'\nu'/RkT_\gamma) - 1} = \frac{8\pi\nu'^2\, d\nu'/c^3}{\exp(h\nu'/kT'_\gamma) - 1} \tag{582}$$

where $T'_\gamma = T_\gamma R/R'$, a *cooler* temperature. The point is that the distribution (582) is *still* a blackbody at time t', but with a new temperature T'_γ that has cooled in proportion to $1/R$ with the expansion. This is remarkable because there is nothing maintaining thermal equilibrium. This scaling result follows mathematically for any number spectrum of the form $\nu^2 F(\nu^p/T)\, d\nu$, where F is an arbitrary function and p is a real number. There is no reason why, as a matter of principle, the orginal photon spectrum has to be maintained as a blackbody, if other interaction physics takes place. For years there were arguments whether the spectrum was a true blackbody. These were based on observational results from rocket-borne instrumentation that were later retracted. (It turns out that rocket exhaust is not helpful when hyper-accurate infrared sky measurements are required.) Moreover, as a matter of sensible physics, to change the radiation spectrum of the Universe takes a vast amount of energy! All doubts were erased in 1992 when the COBE (COsmic Background Explorer) satellite as launched. This satellite had sub mm detectors of unprecedented accuracy, showing that the backgound radiation was an almost perfect blackbody at $T_\gamma = 2.728K$. Figure (15) summarises the data. The error bars are shown at 400 times their actual value, just in order to be visible!

The photon distribution is not, however, absolutely *exactly* a blackbody, for two very interesting reasons. The first is that the earth is not at rest relative to the CMB frame. Why should it be? In fact the local group of galaxies seems to be moving at about 630 km s^{-1} relative to the CMG, a surprisingly large value. This corresponds to a measured radiation temperature that is about 2×10^{-3}K warmer in one direction and the same amount cooler in the oppostion direction.

The second reason is that matter has collapsed out of the expanding background, forming galaxies, stars, and great clusters. The seed for these structures could not have formed recently; there isn't enough time for them to have grown. They must have been present during the era when the Universe was radiation-dominated, and matter and radiation interacted strongly. This interaction would have left its imprint at the time of last scattering in the form of fluctuations of the microwave background temperature. The relative fluctuation $\Delta T/T$ would have had to have be, it would seem, at least $\sim 10^{-3}$ in order to be able to form nonlinear structures by the current epoch. Why? Because $\Delta T_\gamma/T_\gamma$ and $\Delta\rho/\rho$ in matter are comparable and $\Delta\rho/\rho$ grows about in proportion to $t^{2/3}$, the Einstein-de Sitter value, and this means it is proportional to the scale factor R. R has grown be about a factor of 1500 since the time of last scattering, whence $\Delta T_\gamma / T_\gamma \sim 10^{-3}$.

The search for these fluctuations proved frustrating. There were no temperature fluctuations found at this level for many years. When COBE and later satellites did find them, starting in the early 1990's, they were an order of magnitude smaller. So how does structure form in the Universe? The answer is that there is more to matter than the usual baryons we know and love. The Universe is pervaded by what is known as "dark matter," and this leaves no imprint on the microwave background. More on the importance of dark matter later.

8.10.2 An observable cosmic radiation signature: the Gamow argument

The idea that the Universe had a residual radiation field left over from its formation, and that this radiation is potentially observable, seems to have originated with George Gamow in the 1940's. Gamow was theoretical physicist with a brilliant common sense instinct that allowed him to make contributions to a wide variety of problems, from the theory of radioactivity to DNA coding. He spent many years developing what became known as the Big Bang theory.

Gamow pointed out that Helium is about 25% of the Universe by mass. That is neither overwhelming nor negligible. It implies that at the time of nucelosynthesis—and Gamow was convinced that most of the Helium was made this way in the early Universe—the expansion rate and the nuclear reaction rate were comparable. Too rapid an expansion, no Helium. Too slow an expansion rate, all the protons get fused into Helium, and there is nothing left over to make physcists. This turns out to be a remarkably constraining observation, leading to a prediction that there should be a residual radiation field of about 10 K. Let's see how it works.

At some point, the Universe passes downward through the temperature range of $\sim 10^9$ K. When that happens, neutrons n and protons p can combine to form Deuterium nuclei. The reaction is

$$n + p \to D + \gamma$$

where γ is a gamma ray. Helium synthesis then follows rapidly. Gamow took equal numbers of protons, neutrons and electrons. In order to get a 25% yield of Helium, the reaction rate and the age of the Universe (of order the expansion time) should be comparable. If the cross section for Deuterium formation is σ (units of area), then the reaction rate per proton due to an incoming flux nv of neutrons is $nv\sigma$ (number per unit area per unit time). At time t, on average about one reaction per proton should have occured, because of the order unity fraction of Helium—that is the heart of the Gamow argument:

$$nv\sigma t \sim 1 \tag{583}$$

The product σv is nearly independent of v (because the cross section σ depends on v) and, as Gamow knew from nuclear experiments, about 4.6×10^{-26} m^3 s^{-1}. As for the time t, we are interested in the epoch when the temperature $T_\gamma \sim 10^9$K. Using equation (558) for the density in relativistic particles, and following Gamow by assuming (not quite correctly, but let it go) that this was all radiation, the time t is

$$t = \frac{c}{T_\gamma^2} \left(\frac{3}{32\pi G a} \right)^{1/2} \tag{584}$$

which amounts to 230 s for $T_\gamma = 10^9$K. This give n very close to 10^{23} m^{-3}. Gamow estimated a present day average particle density of about 0.1 per cubic meter based on astronomical

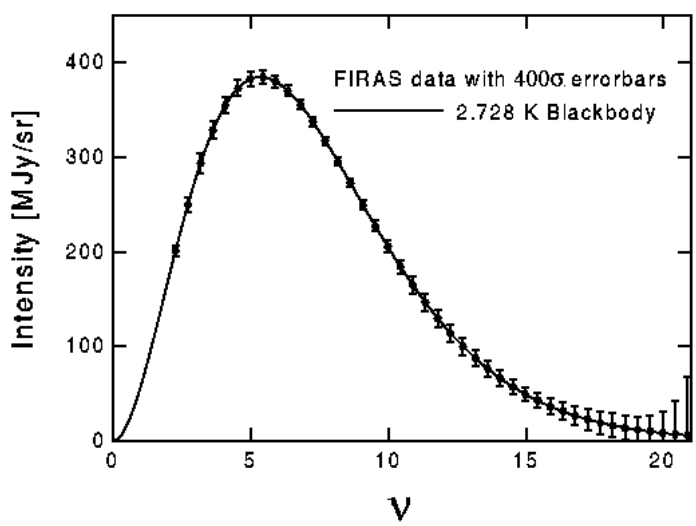

Figure 15: COBE satellite data showing a perfect fit to a blackbody at 2.728 K. To be visible, the error bars are shown at 400 times their actual value! When shown at an American Astronomical Society Meeting, this plot triggered a spontaneous standing ovation. Units of ν are cm^{-1}, i.e., the wavelength in cm is the reciprocal of the number on the x axis.

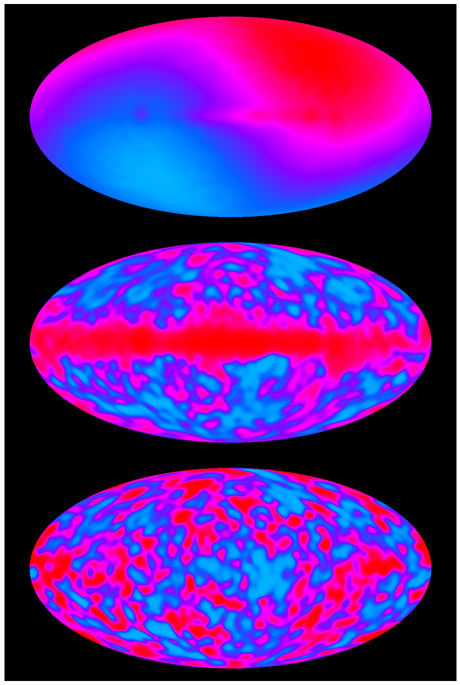

Figure 16: COBE temperature fluctuations. *Top:* Temperature dipole variation dominated by motion of the galaxy. *Middle:* Residual with kinematic dipole subtracted off. *Bottom:* Further residual with galactic foreground subtracted off. These represent primordial fluctuations in the CMB.

estimates, so that the Universe had expanded by a factor of 10^8 in the scale factor R, reducing n by 10^{24}. But an expansion of a factor of 10^8 in R means that the current T_γ should be 10 K! The millimeter detectors that would have been required for this observation were just at the leading edge of technology in the late 1940s, a by-product of the development of radar during the Second World War. But Gamow did not pursue this actively, and the prediction was gradually forgotten.

His argument would be put slightly differently now, but it is in essence correct, and a brilliant piece of intuitive reasoning.

Exercise. Repeat the Gamow argument with modern cosmological numbers. Keep $T_\gamma = 10^9$K and the σv value, but note that relativistic species present at the time include not only photons but three types of neutrinos and three types of antineutrions, each neutrino-anti neutrino combination contributing $(7/8)aT_\gamma^4$ to the background energy density. We can neglect e^+ and e^- pairs in our relativistic fermion population. Why? The 7/8 factor arises beacause the neutrinos obey fermi statistics:

$$\int_0^\infty \frac{x^3\,dx}{e^x+1} = \frac{7}{8}\int_0^\infty \frac{x^3\,dx}{e^x-1}$$

Can you prove this mathematically *without* doing either integral explicitly? (Hint: consider the difference

$$\frac{1}{e^x-1} - \frac{1}{e^x+1},$$

and integrate x^3 times the residual.) Note that for each neutrino specie the factor is actually 7/16, because only *one* spin helicity is present, as opposed to *two* spin states for photons.

The neutrinos get back up to 7/8 with their antiparticles, which photons lack.) Take the current density to be 5% of the critical density $3H_0^2/8\pi G$.

In 1965, the problem of determining $T_{\gamma 0}$ attracted the attention of an able team of physicists at Princeton University. They rediscovered for themselves the Gamow argument. The senior investigator, Robert Dicke, a hugely talented physicist (both theoretical and experimental), realised that there was likely to be a background radiation field that survived to the present day. Moreover, it could be detected by an instrument, the Dicke radiometer, that he himself had invented twenty years before! (A Dicke radiometer is a device that switches 100 times per second between looking at the sky source and a calibrated thermal heat bath of liquid helium. This imparts a 100Hz fourier component to the desired signal and eliminates extraneous variability occuring on longer time scales.)

The A-Team assembled: Dicke, the scientific leader; J. Peebles, the brilliant young theorist who would become the world's leading cosmologist in the decades ahead, and P. Roll and D. Wilkinson, superb craftsmen who designed and built the contemporary Dicke radiometer. The were all set up to do the observation from the roof of their Princeton office buliding, when a phone call came from nearby Bell Laboratories. Two radio engineers named Arno Penzias and Robert Wilson had found a nuisance extraneous signal in their detector, an instrument designed to receive signals relayed by some of the first commercial television satellites. (The *Telstar* series.) They were trying to chase down all sources of backgound confusion. The "effective radiation temperature" of the unwanted diffuse signal was about 3 K. Penzias and Wilson had no idea what to make of it, but were advised by a colleague to give the Dicke team at Princeton a call. Those guys are very clever you know, they might just be able to help.

The 1978 Nobel Prize in Physics went to Penzias and Wilson for the discovery of the cosmic microwave background radiation.

8.10.3 The cosmic microwave background (CMB): subsequent developments

The initial observations of the CMB were at one wavelength: 7 cm. Needless to say, a single point does not establish a blackbody spectrum. The task of establishing the broader spectrum was fraught with difficulties, with many disputed and ultimately withdrawn claims of large deviations from a blackbody. In 1992 matters were finally laid to rest when the COBE satellite returned its dramatically undramatic finding that the CMB is, very very nearly, but not *exactly*, a blackbody. There are in fact small fluctuations in the temperature, the largest of which amount to a few parts in 10^4. This value, it turns out, is just of the order needed to account for the nonlinear structure in the Universe that we see today, provided that there is a healthy component of dark matter that does not react with the radiation. This unseen, and unseeable, dark matter component had been invoked long before the COBE results to account for the large internal velocities measured in galaxies and within clusters of galaxies. These velocities are not maintainable without most of the mass of the galaxy (or cluster) being in the form of dark, non light-emitting or scattering, matter. Now the COBE results became another indication of the presence of dark matter in the Universe.

The next important result was the Wilkinson Microwave Anisotropy Probe, or WMAP. This is named for David Wilkinson, the Princeton Researcher who was instrumental in the earliest CMB studies. With the launch of WMAP in 2003, cosmology truly became a precision science. WMAP revealed the structure of the tmeperature fluctuations in such exquisite detail in angular resolution on the sky, it became possible to determine the key physical parameters of the Universe: H_0, Ω_0, Ω_V, t_0 and many others, to several significant figure accuracy. The Planck satellite was launched in 2009 and provided a further shrinking of the error bars, higher angular resolution coverage of the CMB on the sky, better frequency

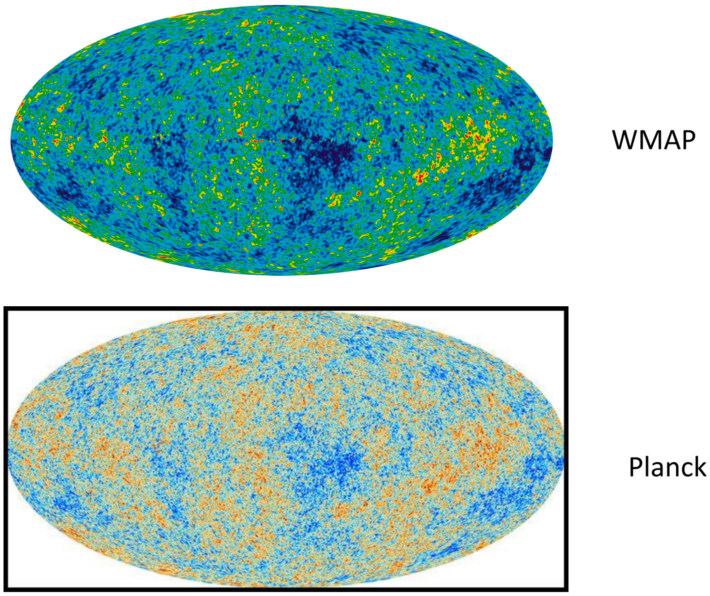

Figure 17: WMAP (top) and Planck (bottom) temperature fluctuations, with different colour scales. Think of these as the Universe's baby pictures. Note the much higher angular resolution compared with COBE; Planck is another factor of 3 better than WMAP. The angular information encoded on these smalle scales is valuable as a means of tightly constraining cosmological parameters.

coverage (very important for subtracting off the effects of the Milky Way Galaxy), and better constraints on null results (absence of polarisation, for example), but no qualitatively new physical findings. The story is essentially that revealed by WMAP.

Think of the evolution of the maps and globes of the Earth, from ancient to modern times. Gross inaccuracies in basic geometry gradually evolved to Googlemap standard over a period of thousands of years. Contrast this with the observation that, well within the professional careers of currently active researchers, serious models of the Universe went from wild misconceptions (e.g. the "steady-state model," in which the universe supposedly never changed in appearance and hydrogen atoms were spontaneously created out of the vacuum) to three-significant-figure accuracy in its structural parameters. By any measure, this is one of the great scientifc achievements of our time. As with any great scientific advance, its true meaning and implications will take years to elucidate.

8.11 Thermal history of the Universe

8.11.1 Prologue

The recorded history of civilisation began when Sumerian merchants inscribed grain inventories on clay tablets. (BTW, the concept of "inventory" would not have been possible without the invention of large scale agriculture, which would not have been possible without the invention of astronomy. Stick with this for a moment: despite what you're thinking, I'm on message...) Etched in the clay matrix, these markings were frozen in time because the background clay matrix, once dried, remained unaltered and intact. It turns out that the Universe has its own gigantic clay tablet, which allowed "inscriptions" first to be imprinted, then to be preserved, effectively unaltered for 13.7 billion years. We know the inscriptions as temperature fluctuations, and the clay tablet as the cosmic microwave background.

The temperature fluctuations are a tracer of the initial density fluctuations, which coupled to the radiation by electron scattering. (Also known as *Thompson scattering.*) While this coupling remained strong, the CMB dutifully recorded and re-recorded the evolving changes in the density. Then, when matter and radiation became decoupled, the pattern imprinted on the CMB abruptly stopped being recorded. The CMB instead retained only the pattern imprinted on it at the time of the "last scattering." It is *this* pattern that we receive, redshifted by the expansion of the Universe, in our detectors today. Think of this either as an ancient inscription passed on through the eons, or more congenially as the Universe's baby picture.

Exercise. Let σ_T be the Thompson cross section for a photon to collide with an electron, a constant number equal to 6.7×10^{-29} m^2. When an electron has moved relative to the photon gas a distance l such that the swept-out volume $l\sigma_T$ captures a single photon, l is said to be one *mean free path (mfp):* the average distance between scatterings.

Justify this definition, and show that the scattering rate per electron is $n_\gamma c\sigma = c/l$. (And what is n_γ here?) In a radiation-dominated Universe, show that the ratio of the photon scattering mfp to the horizon size grows like $\sqrt{t}$; and the mfp relative to the scale factor R grows like t. (t as usual is time.)

Exactly how to decipher the CMB inscriptions and turn them into a model of the Universe is a very complicated business, one that we will only be able to treat very superficially in this course. We will go about this in two steps. In the first, we will describe what I will call the "classical theory" of the thermal history of the Universe: how matter and radiation behaved in each other's presence from temperatures of 10^{12} K through 3000K, the time of recombination. In the second step, we will discuss the "modern theory" of the very early Universe. This puts a premium on the notion of inflation, a period in the history of the Universe in which it seems to have undergone a rapid exponential growth phase. First put forth in 1980, this idea has been the most important theoretical advance in modern cosmology in recent decades. There are very good theoretical reasons for invoking the process of inflation, even if the mechanism is not well-understood (not an unusual state of affairs in science), for there is by now very good observational evidence for it. To my mind, the best evidence there is for inflation is that we have in fact entered another inflationary stage of the Universe's history. Inflation is real, full stop.

In what follows, we will have reason to consider physics on what is known as the "Planck scale." These are fundamental values set by the three fundamental dimensional constants Newton's G, Planck's h, and the speed of light c. There are unique dimensional combinations to form a mass m_P, a length l_P and a time t_P from these constants:

$$m_P = \left(\frac{hc}{G}\right)^{1/2} = 5.456 \times 10^{-8} \text{ kg} \tag{585}$$

$$l_P = \left(\frac{hG}{c^3}\right)^{1/2} = 4.051 \times 10^{-35} \text{ m} \tag{586}$$

$$t_P = l_P/c = \left(\frac{hG}{c^5}\right)^{1/2} = 1.351 \times 10^{-43} \text{ s} \tag{587}$$

These are, in some sense, the limits of our knowledge. It is on these scales that we may expect quantum gravity effects to be important, and on which we remain quite ignorant. We cannot expect to have anything like a classical picture of the early Universe for time $t < t_P$, or horizon scales $c/H < l_P$. The Planck mass may not, at first glance, appear remarkable, but remember the comparison is with elementary particle masses. A proton is close to 1 GeV in rest energy; $m_P c^2 = 3.06 \times 10^{19}$ GeV.

8.11.2 Helium nucleosynthesis

Let us begin the story when the temperature of the Universe is just under 10^{12}K. This is very early on, about 10^{-4}s after the big bang. In the next section we will go even before this interval, but this is a good place to begin for now. At this stage, the Universe consists of a relativistic cocktail of photons, neutrinos and their antiparticles, muons and their antiparticles and electrons and their antiparticles (positrons). This cocktail is well-mixed (shaken, not stirred) with all particles, even the neutrinos, in complete thermal equilibrium, freely created and destroyed.[14] There is also a population of protons and neutrons, energetically unimportant at these temperatures. But keep track of them! They are going to make the Universe we know and love, including us.

Once the temperature slips below 10^{12} K, the muons and antimuons annihilate, but without being able to maintain their populations from production by the other relativistic populations present. The energy from the photons and e^+e^- pairs produced in the annihilation are equilibrated amongst these populations, and everyone is heated, including the neutrinos.

The Universe continues to expand, the temperature falls. Now, neutrons and protons do not have the same mass. More precisely, with m_p the proton mass and m_n the neutron mass,

$$\Delta mc^2 \equiv (m_n - m_p)c^2 = 1.293 \text{ MeV} = 2.072 \times 10^{-13} \text{ J}. \tag{588}$$

This is very small compared with either $m_n c^2$ or $m_p c^2$. Except when we are specifically concerned with the mass difference or doing very accurate calculations, there is no need to distinguish m_n for m_p. Textbooks on statistical mechanics tells us that the ratio of the probabilities for finding a system in state i or state j depends only on the energies E_i and E_j of the states, and nothing else. In particular, the ratio of the probability of finding the system in i to the probability of finding it in j is given by the *Boltzmann equation:*

$$\frac{P_i}{P_j} = \frac{g_i}{g_j} \exp\left[-\left(\frac{E_i - E_j}{kT}\right)\right] \tag{589}$$

where the g's are the *statistical weights,* basically how many distinct states there are with energy E_i or E_j. (If you've forgotten this and don't have a statistical mechanics textbook handy, see the Exercise below for a quick justification.) Once we fall below 10^{11}K, the so-called Boltzmann factor $\exp(-\Delta mc^2/kT)$ starts to differ noticeably from unity, and as we approach 10^{10} it becomes small. Since neutrons are more massive than nprotons, they become more scarce at cooler temperatures. The temperature corresponding to $kT = \Delta mc^2$ is 1.5×10^{10}K. (By way of comparison, for electrons and positrons, $m_e c^2/k = 5.9 \times 10^9$K.)

Exercise. Let the probability of finding a system in state i be $f(E_i)$, where E_i is the energy of the state. The ratio of the probability of finding the system in state i versus j is then $f(E_i)/f(E_j)$. But this ratio must be a function only of the difference of the energies, because a constant additive constant in energy can't affect the physics! The potential energy is always

[14] By using the word "freely," we ignore the electron and neutrino rest masses.

defined only up to such a constant. Hence

$$\frac{f(E_i)}{f(E_j)} = F(E_i - E_j)$$

If $E_i = E_j + \delta$, show that if this equation is to hold even to first order in δ, that

$$f(E) \propto \exp(-\beta E)$$

where β is an as yet undetermined constant. We can determine β by physics, e.g. by demanding that the ideal gas equation of state be satisfed, pressure P equals nkT, where n is the number density and k is the Boltzmann constant and T the temperature. Then it follows (show!) $\beta = 1/kT$.

The reactions that determine the neutron n proton p balance are

$$n + \nu \leftrightarrow p + e$$

$$n + e^+ \leftrightarrow p + \bar{\nu}$$

$$n \leftrightarrow p + e + \bar{\nu}$$

The reaction rates for the two body processes are about 0.1 per second per nucleon at $T = 10^{10}$K. However, the rate drops rapidly as the thermometer falls. Below 10^{10}K, the reactions can't keep up with the expansion rate of the Universe. When this occurs, whatever the ratio of n/p is, it remains "frozen" with time, as the Universe is too cold to keep the reactions cooking! At $T = 10^{10}$K, the Universe is about 1 s old. The n/p ratio at this temperature is $\exp(-\Delta mc^2/kT) \simeq 0.2$, and is frozen in.

Exercise. Show that the early Universe "temperature clock" is given conveniently by

$$t \simeq 1/T_{10}^2,$$

where t is the time in seconds and T_{10} the temperature in $10^{10}K$. Assume a Universe of photons, electrons and positrons, and three neutrinos and anti-neutrinos.

This figure of 20% is interesting, because it is neither close to unity nor tiny. Without further production of neutrons, they will decay in minutes by the third reaction channel. If all the neutrons had decayed into protons and electrons and antineutrinos, there would have been no cosmological nucleosynthesis. But just as the neutrons start to slip away as we approach 10^9K, the remaining neutrons are rescued from decay by being packed into stable ^{4}He nuclei. The helium nuclei are perfectly safe radioactive containment vehicles. Here neutron decay is absolutely forbidden, in essence because the statistical phase space within the nuclear potential is degenerate. When the neutrons are confined to the nuclear potential, there are simply no available states to decay into. They are occupied by protons. It is rather like trying to use Oxford public transportation on a rainy day: SORRY, BUS FULL.

Back to our story. The neutrons are now scarfed up into He nuclei. This results in a firm prediction for the observed mass fraction of the Universe, since stellar nucleosynthesis does not change the number significantly. (By contrast, almost all of the much smaller heavier element abundances are dominated by stellar nucelosynthesis.) From what we have just seen, the mass fraction in helium will be

$$\frac{4m_p \times (n/2)}{m_p(n+p)} = \frac{2}{1 + p/n} \simeq 0.33 \tag{590}$$

Much more detailed, time-dependent calculations (pioneered by Peebles in 1965) give a number close to 0.27, but the essential physis is captured by our very simple estimate. There is not very much wiggle room here. We cannot use the precise value of this number, say, to determine retrospectively what sort of Universe we must live in; it is essnetially the same for any FRW model. The Big Bang Theory predicts 27% of the mass of the Universe is in the form of helium. And that, happily, turns out to be very close to what observations reveal.

8.11.3 Neutrino and photon temperatures

As the temperature slides from 5×10^9 K to below 10^9, the electron positron pairs annihilate into photons,

$$e^+e^- \to 2\gamma$$

but because the temperature falls, the photons do not maintain an equilibrium population of these e^+e^- pairs. In other words, there is a conversion of electrons and positrons into photons. But in thermal equilibrium, the number of photons at a given temperature is fixed. If you add more of them via e^+e^- annihilation, the extra photons force a new equilibrium, one at a higher temperature compatible with the increased photon number. The photons are, in effect, heated. By contrast, the relativistic neutrinos that are present march blithely along, enjoying the expansion of the Universe without a care. They don't care about anybody. This means that they are cooler than the photons! The question is, by how much?

This turns out to be a relatively simple problem, because if we fix our attention on a comoving volume of the Universe, the entropy in this volume is conserved by the conversion of electron-positron pairs into photons. For photons or relativistic electrons/positrons, the entropy per unit volume is a function only of the temperature. It is most easily calculated from a standard thermodynamic identity,

$$E + PV - TS + \cancel{\mu N} = 0,$$

for a gas with zero chemical potential μ. The latter is true of a relativistic gas that freely creates and annihilates its own particles. So for the entropy per unit volume s,

$$s \equiv \frac{S}{V} = \frac{E}{VT} + \frac{P}{T} \tag{591}$$

With ρc^2 denoting the energy per unit volume and $P = \rho c^2/3$ for a relativistic gas,

$$s = \frac{4\rho c^2}{3T} \tag{592}$$

With a relativistic energy density always proportional to T^4, the entropy per unit volume is proportional to T^3, and the entropy in a comoving volume R^3 is proportional to $(TR)^3$. It is conserved with the expansion, in essence because the entropy and particle number are proportional to one another, and particle number is conserved.

Before the e^+e^- annihilation, the entropy in volume R^3 in photons, electrons and positrons is

$$sR^3 = \frac{4a(TR)^3}{3}\left(1 + 2 \times \frac{7}{8}\right) = \frac{11a(TR)^3_{before}}{3} \tag{593}$$

Afterwards, and after the reestablishment of thermal equilibrium, it is all photons:

$$sR^3 = \frac{4a(TR)^3_{after}}{3} \tag{594}$$

This entire process conserves entropy sR^3: it conserves the total particles in photons plus e^+e^- pairs. It is therefore reversible: recompress the adiabatic expansion backwards to produce the pairs. In other words,

$$\frac{4(TR)^3_{after}}{3} = \frac{11(TR)^3_{before}}{3} \tag{595}$$

Now for an interesting point of physics. Whereas the muon-antimuon annihilation heated *all* relativistic populations, because the cross sections permitted this at this earlier time when the Universe was much more dense, the e^+e^- population annihilation ignores the neutrinos. The density has now fallen to a level that the neutrinos pass through everything, so their TR does not increase. The electron-positron pair annihilation must therefore produce different cosmic background photon and neutrino temperatures. The ratio of neutrino temperature T_ν to photon temperature T_γ is

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} = 0.7138 \tag{596}$$

This would correspond to a current value of $T_\nu = 1.95$ K — if the neutrinos remained a $\mu = 0$ relativistic population for all time. Which we now know they did not. At one time, people wondered whether there was any possible way to measure this. But a temperature of 1.95K corresponds to an energy kT of 1.7×10^{-4} eV, whereas the average mass per neutrino specie (the best we can measure at the current time) is about 0.1 eV, with a corresponding temperature of 1160 K. Neutrinos thus became "cold" at a redshift of $z \sim 400$. As we shall see, this is well after hydrogen recombined, but probably before galaxies formed. These neutrinos are a part, but only a very small part, of the dark matter in galaxies. Even if it had turned out that neutrinos had zero rest mass, it would have been impossible with present technology to measure a 2K neutrino background! The physics of this problem remains is enlightening, and the formal difference in neutrino and photon temperatures surprising. It merits discussion.

8.11.4 Ionisation of Hydrogen

Between a few hundred seconds, the time of helium synthesis, and a few hundred thousand years, almost nothing happens to change the character of the Universe. Radiation remains the dominant source of energy and pressure, and the Universe simply expands with R scaling like $t^{1/2}$. The matter and the radiation remain tightly coupled, so that density fluctuations in the matter correspond also energy fluctuations in the radiation. But at some point, the Universe cools enough that hydrogen recombines and the matter is no longer an ionsed plasma, but a neutral gas. At what temperature does this occur? To answer this question, we need to use an interesting sort of variation of the Boltzmann equation, know as the Saha equation. It tells us the ionisation fraction as a function of T.

Recall the Boltzmann equation:

$$\frac{P_i}{P_j} = \frac{g_i}{g_j} \exp\left(-\frac{E_i - E_j}{kT}\right) \tag{597}$$

For simplicity we will consider a gas of pure hydrogen, and interpret this equation as follows. State j is the ground state of hydrogen with one electron, the 1s state. Let's set j to 0 so we think of neutrality. State i is the ionised state with a bare proton and a free electron with kinetic energy between E and $E + dE$. Set j to 1. The electron has two spin states available to it, whether bound or free, so this factor of 2 appears in both numerator and denominator, canceling in the process. Then g_0 is unity. We think of g_1 as the number of states available to the electron with energy in the range dE. Do you remember how to do that? In terms of momentum p (a scalar magnitude here), the number of states per electron around a shell of thickness dp in momentum space is

$$\frac{4\pi p^2}{n_e h^3}\,dp$$

where the factor $1/n_e$ represents the volume per electron and h is Planck's constant. This is derived in any standard text on statistical mechanics, but if this is not familiar, now is a good time to have a look at *Blundell and Blundell, Concepts of Thermal Physics.* The concept comes from the strategem of putting the electrons in big cube of volume V, counting the eigenstates (now discrete because of the box walls) for each electron, and then noticing that the artificial box appears in the calculation only as a volume per electron, which is just $1/n_e$. Drawing these threads together, (597) becomes, upon adding up all possible free electron states,

$$\frac{n_e n_p}{n_0} = \exp\left(-\frac{\Phi}{kT}\right)\frac{4\pi}{h^3}\int_0^\infty p^2\, e^{-p^2/2m_e kT}\,dp \tag{598}$$

where Φ is the ionisation potential of hydrogen, m_e the electron mass, n_e the electron density, n_p the proton density and n_0 the neutral hydrogen density. Note that we have interchanged the ratio of densities of p and H for their relative probabilities. Now

$$\int_0^\infty p^2\, e^{-p^2/2m_e kT}\,dp = (2m_e kT)^{3/2}\int_0^\infty x^2 e^{-x^2}\,dx = (2m_e kT)^{3/2}\times\frac{\pi^{1/2}}{4} \tag{599}$$

The Saha equation becomes

$$\frac{n_e n_p}{n_0} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{\Phi}{kT}\right) \tag{600}$$

The final step is to note that for pure hydrogen, $n_e = n_p$ and thus $n_e + n_0$ is the total hydrogen density n_H. This remains unchanged regardless of the ionisation state: if n_e goes up by 1, n_0 has gone down by 1. With $x = n_e/n_H$, our equation therefore becomes

$$\frac{x^2}{1-x} = \frac{1}{n_H}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{\Phi}{kT}\right) = \left(\frac{2.415\times 10^{21}T^{3/2}}{n_H}\right)\exp\left(-\frac{1.578\times 10^5}{T}\right) \tag{601}$$

in MKS units. The value $x = 0.5$ is attained at a redshift of about 1400, a temperature of 3800 K. This is remarkable, because it is much less than the formal Boltzmann ionisation temperature of 1.58×10^5K. At redshifts less than 1400, the Universe becomes transparent to photons, the energy density is already dominated by matter. Interestingly, the intergalactic medium seems to have been reionised shortly after galaxies were able to form at $z \sim 10$, presumably by the very radiation produced by the accretion process that gave rise to these galaxies. To pursue this active area of current astrophysical research would take us too far afield at this point. Now that we have a sense of the basics of helium nucleosynthesis and hydrogen recombination, we return to the very early Universe. It is there that we will learn about what seems to have been a key process for creating the Universe as we know it: inflation.

8.11.5 Inflationary Models

We begin by posing two profound mysteries associated with classical FRW universes: the horizon problem and the flatness problem.

We have already encountered the first, the horizon problem, on page (111). The CMB is homogeneous to 1 part in 10^4 on all angular scales, yet the angular size of the horizon at the redshift of hydrogen recombination is of the order of the diameter of the full moon. How can we possibly understand this degree of homogeneity between regions that have never been in causal contact?

The second problem is known as the flatness problem. Consider the dyanmical equation of motion in the form

$$1 - \frac{8\pi G\rho}{H^2} \equiv 1 - \Omega^2 = \frac{2E}{\dot{R}^2} \tag{602}$$

Now if E happens to be zero, Ω is unity for all time. Fair enough. But the measured value of Ω, at least in terms of ordinary matter, was a number like 25%, including unseen dark matter, and only 5% for ordinary baryons. A number that is smaller than unity but not infinitesimally small. Now either Ω^2 is very close to unity if E is small and $\dot{R}$ large, or Ω is very small (proportional to $1/R\dot{R}$ throughout most of the vast history of a matter dominated universe. But Ω passes though through the value "a number less than but not very different from unity" during a tiny, fleeting moment of a universe's history. And this is the period we just so happen to be observing it? That certainly is a coincidence. We don't like coincidences like that.

To see how the concept of inflation can resolve both of these problems, consider the integral that is done on page (111) to calculate the horizon distance. The problem is that the horizon length at cosmic time t is very finite, of order $c/H(t)$. Formally, this distance is proportional to

$$\int \frac{dt'}{R(t')} = \int \frac{dR}{R\dot{R}} \tag{603}$$

Imagine that at very small R, the dominant behaviour of $\dot{R}$ is $\dot{R} \sim R^p$, where p is some number. If $p \geq 0$, then the integral diverges like $\ln R$ or R^{-p} at small R, and in this case divergence is good: it is *what we want.* Then the the horizon problem goes away because the horizon is unbounded! In essence a small patch of universe, small enough to communicate with itself completely, can rapidly grow to encompass an arbitrarily large segment of sky. With $p \geq 0$, $\ddot{R} \geq 0$ at small R so that the universe would be accelerating (at at worst not decelerating!). The problem with standard models is that they are radiation-dominated, $p = -1$, and highly decelerating. A matter-dominated universe, $p = -1/2$, is no help.

What instead appears to have happened is that the Universe, early in its history, went through a phase of exponential expansion with $\dot{R} \sim R$, $p = +1$. As we have seen, the Universe has begun such an inflationary period recently, at redshifts of order unity. Exponential expansion is the hallmark of a vacuum energy density ρ_V, with a corresponding pressure $P_V = -\rho_V c^2$. This rapid expansion makes an entire Universe from a once very tiny region that was in complete causal self-contact[15]. The rapid large expansion also has the effect of killing off the $2E/\dot{R}$ term in equation (622). In other words it resolves the flatness problem by, well, flattening the Universe! Think of being on patch of sphere and then having the radius expand by an *enormous* factor. The new surface would look very flat indeed. Dynamically,

[15]Note that the rest of the original universe is still hanging around! Inflationary models therefore lead naturally to the concept of a "multiverse," which in itself would help us to understand many otherwise mysterious cooperations between physical scales.

(622) shows us that Ω must then be equal to unity to great accuracy. This is just what observations show.

A nice story, but can we make a case for early inflation? If we are to understand the physics of the vacuum quantitatively, we need to learn a little about quantum field theory, the domain of physics where the vacuum makes a starring appearance in a leading role.

Let us start easy, with spin 0 particles. Spin 0 particles, so-called scalar fields, satisfy the Klein-Gordon equation. In Minkowski spacetime, it looks like

$$\Box\Phi - \mu^2\Phi = 0, \quad \mu^2 = (mc/\hbar)^2 \tag{604}$$

where as usual $\Box = \partial^\alpha\partial_\alpha$, m is the mass of the particle (the "quantum of the field"), and $\hbar$ is Planck's constant over 2π. Now when I solve the Einstein Field Equations, I need to know what the stress energy tensor $T_{\mu\nu}$ is. For ordinary stuff, this is not a problem:

$$T^{\mu\nu} = Pg^{\mu\nu} + (\rho + P/c^2)U^\mu U^\nu$$

The question is, what the heck is $T^{\mu\nu}$ for a *field* obeying equation (604)? What is the density? What is the pressure? We are not completely in the dark on this. Electrodynamics is also a field theory, and there is a perfectly good stress tensor for us. It is perfectly legitimate to put this stress tensor, like any other, into the right side of the Einstein Field Equations. With the help of our "4-potential" A_α (space components equal to the usual vector potential $\boldsymbol{A}$ and time component equal to minus the electrostatic potential), we define the tensor $F_{\alpha\beta}$:

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} \tag{605}$$

and

$$T^{\alpha\beta} = F^\alpha_\gamma F^{\beta\gamma} - \frac{1}{4}\eta^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} \tag{606}$$

(The texts of Jackson [1998] or W72 are good references if needed.) There is in fact only one conserved tensorial combination that is quadratic in the derivatives of the 4-potential. You're looking at it. The overall normalisation constant can be determined by looking at the interaction (the "work done") between the fields and the particles via the Lorentz equation of motion.

With this as background, it is a surprisingly simple matter to find the $T_{\alpha\beta}$ for the Klein-Gordon field, and even to pick out ρ and P. Multiply (604) by $\partial_\beta\Phi$. For ease of future generality, let's call the μ^2 term $dV(\Phi)/d\Phi$, and refer to it as "the potential derivative". For the K-G equation, $V = \mu^2\Phi^2/2$. We will shortly consider other forms. The equation becomes:

$$\frac{\partial\Phi}{\partial x^\beta}\frac{\partial^2\Phi}{\partial x^\alpha\partial x_\alpha} = \frac{dV(\Phi)}{d\Phi}\frac{\partial\Phi}{\partial x^\beta} = \frac{\partial V}{\partial x^\beta} = \frac{\partial(\eta_{\alpha\beta}V)}{\partial x_\alpha} \tag{607}$$

Integrate the left side by parts:

$$\frac{\partial\Phi}{\partial x^\beta}\frac{\partial^2\Phi}{\partial x^\alpha\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}\left(\frac{\partial\Phi}{\partial x^\alpha}\frac{\partial\Phi}{\partial x^\beta}\right) - \frac{\partial\Phi}{\partial x^\alpha}\frac{\partial^2\Phi}{\partial x^\beta\partial x_\alpha} \tag{608}$$

But the final term of (608) is

$$-\frac{\partial\Phi}{\partial x^\alpha}\frac{\partial^2\Phi}{\partial x^\beta\partial x_\alpha} = -\frac{1}{2}\frac{\partial}{\partial x_\alpha}\left(\eta_{\alpha\beta}\frac{\partial\Phi}{\partial x^\gamma}\frac{\partial\Phi}{\partial x_\gamma}\right) \tag{609}$$

Putting the last three equations together:

$$\frac{\partial T_{\alpha\beta}}{\partial x_\alpha} = 0, \quad \text{where} \;\; T_{\alpha\beta} = -\eta_{\alpha\beta}\left[\frac{1}{2}\frac{\partial \Phi}{\partial x^\gamma}\frac{\partial \Phi}{\partial x_\gamma} + V(\Phi)\right] + \left(\frac{\partial \Phi}{\partial x^\alpha}\frac{\partial \Phi}{\partial x^\beta}\right) \tag{610}$$

Notice that for T_{00}, the rest frame energy density component (ρc^2) is

$$T_{00} = \rho c^2 = \frac{1}{2c^2}\dot{\Phi}^2 + \frac{1}{2}|\boldsymbol{\nabla}\Phi|^2 + V \tag{611}$$

which really does look like an energy density. The first term is a kinetic energy density (the dot means time derivative), the second the effective potential energy density from the spring-like coupling that produces the simple harmonic motion of Φ, and the final term is the potential from an external driver. In textbooks on quantum fields, in Chapter 1 one begins by examining the quantum mechanics of a collection of masses on springs, because that problem is not just similar to, but is practically identical with, the problem of the excitation of massless spin 0 particles. The Hamiltonian density used for this problem is precisely our expression (611) for ρc^2.

To get the rest frame pressure P in terms of Φ, just read off $T_{xx} = P$ (or any other Cartesian component) assuming $|\partial_x \Phi|^2 = |\boldsymbol{\nabla}\Phi|^2/3$ since there is no preferred direction:

$$P = \frac{1}{2c^2}\dot{\Phi}^2 - \frac{1}{6}|\boldsymbol{\nabla}\Phi|^2 - V \tag{612}$$

As a check, note that in the massless boson $V = 0$ limit, $(\dot{\Phi}/c)^2 = |\boldsymbol{\nabla}\Phi|^2$ (from $\omega^2/c^2 = k^2$) and $P = \rho c^2/3$, which is the correct equation of state[16]. When Φ is a constant in space and time, $P = -\rho c^2$, the vacuum equation of state.

The big idea now is to upgrade from $\eta_{\alpha\beta}$ to $g_{\mu\nu}$ as per the usual GR prescription, and then use *this* form of $T_{\mu\nu}$ in the cosmological equations during the earliest phase of the Universe. What is the connection between Φ and "real life," the classical limit? This consists of rapidly oscillating Φ, both in space and time. We then really do get back to a uniform gas, since we then average over these rapid oscillations, just as in the classical limit of any quantum mechanical problem. But in the earliest Universe, we consider near vacuum conditions in which this averaging is not appropriate. Indeed, to a first approximation, we shall ignore any spatial structure in Φ, allowing only temporal behaviour! That is far from classical behaviour. The reason for explicitly identifying a ρ and P from our stress tensor is that it is now a relatively easy matter to solve the Field Equations with our quantum field $T_{\mu\nu}$.

The fundamental dynamical equation in the absence of curvature is

$$H^2 = \frac{8\pi G\rho}{3} \tag{613}$$

where $H = \dot{R}/R$. We need a second equation to know how ρ and P depend upon R. This is the energy conservation equation, (522):

$$\dot{\rho} + 3H\left(\rho + \frac{P}{c^2}\right) = 0. \tag{614}$$

[16] There is a discrepancy between these formulae (which agree with HEL06, and Kolb & Turner [1990]) and those in Weinberg's 2008 text *Cosmology*. Weinberg's result $P = \rho c^2$ (from his equations B.66 and B.67, page 527) in the limit $V = 0$ seems incorrect to me. The discrepancy does not affect the discussion here, with $\boldsymbol{\nabla}\Phi = 0$

Remember that this equation comes from $\partial^\mu T_{\mu\nu} = 0$. But this amounts to solving the Klein-Gordon equation itself, since the way we formed our stress tensor (610) was by contracting the K-G equation with a 4-gradient of Φ. So all relevant equations are embodied in (613) and (614), with (611) and (612) for Φ.

Right. Now then. About $V(\Phi)$. What is this $V((\Phi)$? It helps to have a concrete mechanical model. If I have a one-dimensional collection of (concrete!) masses on springs, and ϕ_n is the lateral displacment of mass n, the equation of motion is:

$$\ddot{\phi}_n = -k(\phi_n - \phi_{n-1}) - k(\phi_n - \phi_{n+1}) \tag{615}$$

Now when n is very large and the masses closely spaced with a small separation Δx, I can take the limit

$$\phi_n - \phi_{n-1} \simeq \Delta x \phi'_{n-1/2}$$

so that

$$-k(\phi_n - \phi_{n-1}) - k(\phi_n - \phi_{n+1}) \to -k\Delta x(\phi'_{n-1/2} - \phi'_{n+1/2}) \tag{616}$$

where the spatial x derivative ϕ' is formally defined halfway between the integer n's. A second use of this limit brings us to

$$-k\Delta x(\phi'_{n-1/2} - \phi'_{n+1/2}) = k(\Delta x)^2 \phi''_n \tag{617}$$

or

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} \tag{618}$$

where $c^2 = k(\Delta x)^2$ becomes the velocity of a propagating ϕ disturbance as k gets large and Δx small. A three-dimensional extension of this argument would introduce nothing new, so this is a mechanical analogue of the standard wave equation. But it is not the Klein-Gordon equation. Where is V?

Do the same problem, but this time hang the masses from strings of length l in a gravitational field g while they slide back and forth on their connecting springs, as in figure [18]. Then, our final partial differential equation becomes

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{g}{l}\phi \tag{619}$$

This is the Klein-Gordon equation. The final term is an external coupling between the displacement ϕ (or the "field" Φ) and some external interaction. Note: an external *interaction.* In the K-G equation, the coefficient representing this interaction g/l is called "mass." This sets up Richard Feynman's famous quotation: "*All mass is interaction.*" Inertial mass is the price you pay to jiggle a field against some kind of external coupling, a foreign entanglement if you will.

We could imagine putting our concrete masses in some kind of a more complicated external force. Maybe $g\phi/l$ is only the first term in a Taylor series of, say, $g \sin\phi/l$. (Can you think of mechanical system that would, in fact, have this property?) The point here is that as long as the equilibrium $\phi = 0$ null displacement state is one that is stable, any small deviation from $\phi = 0$ will be generically linear in the interaction, and $V(\phi)$ (or $V(\Phi)$) *quadratic.* This is why we think of V in terms of a potential function. We speak casually of the field being a "ball sitting at the bottom of a potential well." The oscillation frequency is, in essence, the mass of the field particle.

So in the real world, what is giving rise to this so-called interaction? Where is the external pendulum force coming from that is affecting the Φ field? The answer is that is coming from the vacuum itself.

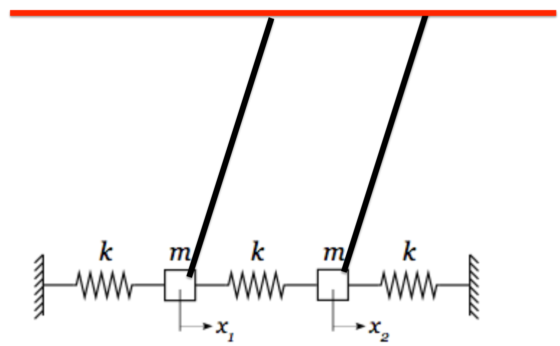

Figure 18: Mechanical analogue of the Klein-Gordon equation. Spring-like coupling between adjacent masses gives rise to the wave equation, while the pendula produce an acceleration directly proportional to the displacement, not the ∂_x derivative thereof. This g/l force is exactly analogous to the mass term in the KG equation, which evidently arises from similar "external couplings" of the scalar field Φ. (Only two masses are shown instead of an infinite continuum.)

Effective Potentials in Quantum Field Theory

One thing you have to understand about the quantum vacuum: *it is a jungle out there.*

The vacuum is full of fluctuations in the varied collection of harmonic oscillators that we are pleased to call particles. Depending upon circumstances, the Klein-Gordon equation, representing only free massive particles, might not capture the dynamics of the scalar field in question. The scalar field could interact with all these other fields, and since interaction is mass, perhaps in the process create an effective new mass coefficient. We could easily imagine a non-linear "pendulum," coupling to other fields. Then, the mass constant μ^2 is not, in fact, a constant, but would depend on the field strength Φ. In the simplest case, μ^2 would depend additively upon Φ^2, so that only the magnitude of Φ, not its sign, affects the distortion of μ^2. Then, the resulting potential $V(\Phi)$ would take the form

$$V(\Phi) = \mu^2\Phi^2/2 + \beta\Phi^4 \tag{620}$$

In fact, for decades before the notion of the inflation became popular, precisely this potential had been in wide use amongst particle physicists for completely different reasons. If for some reason μ^2 was despite its form a *negative* quantity, and β positive, then $\Phi = 0$ is not a stable vacuum solution. The true, stable vacuum state would be the global minimum, $\Phi^2 = |\mu^2/4\beta|$. Locally with respect to this minimum, the potential once again appears stable-quadratic, and we are back to Klein-Gordon...but with another mass coefficient. For understanding our Universe, it is not the destination of the new equilibrium that is important, it is the journey we take to reach it: down the potential slide! This is from whence inflation may orginate—assuming, of course, that we start off at or near the top of the slide.

I know, I know. Try not to be put off by the vagueness of all this. *What if? Maybe... assuming... We imagine that...* It is better than it looks. The real $V(\Phi)$ may well be a

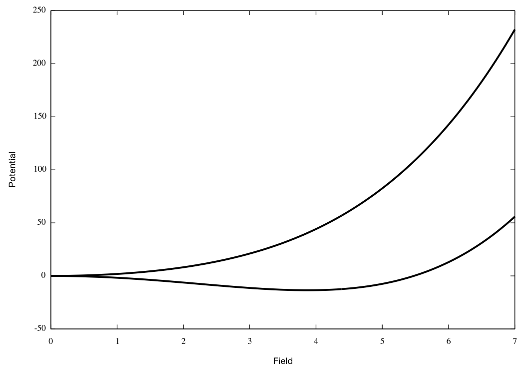

Figure 19: $V(\Phi)$ potential functions of the form (620). The upper curve corresponds to $\mu^2 > 0$, the lower to $\mu^2 < 0$. If there is a scalar field described by the $\mu^2 < 0$ potential in the early Universe, it could trigger an episode of inflation. Before cosmological applications, these potential functions were used to describe ferromagnets, and to explain how symmetry is broken in fundamental particle physics.

complicated function (lots of interactions), but we have long ago landed in a stable local parabolic minimum of V, so things now look deceptively, Klein-Gordonly, simple. The idea here is that it is OK to grope in the dark a bit to try to understand what type of physics might in principle lie beyond the usual theories. But within reason, of course. As long as certain ground rules like Lorentz invariance are respected, there is considerable freedom in choosing the form of the interaction. As noted, the type of quartic potential (620) was already well-known to particle physicists, who had borrowed it from condensed matter physicists, who had in turn used it (in the form of a thermodynamic potential) to describe ferromagnets. This analogy has proven to be enormously useful in particle physics. (The Higgs Boson, for example.) Just as a ferromagnet can spontaneously magnetise itself, so too can certain types of particle spontaneously acquire mass. These physically distinct processes turn out to have very similar mathematics.

In a ferromagnet, the equilibrium state minimises the Gibbs Free Energy G, as is usual for thermodynamical systems at fixed temperature and pressure. G is a function of the magnetisation M, and typically takes the form of (620):

$$G = \alpha M^2 + \beta M^4 \tag{621}$$

At high temperatures, above some critical T_c, both α and β are positive. Below T_c, however, α changes sign, and it becomes energetically more favourable — G attains a smaller value— when M^2 is finite and equal to $-\alpha/2\beta$. The system, in other words, becomes spontaneously magnetised. A similar mathematical arrangement also produces the phenomenon of superconductivity. And in particle physics, this is the core of the argument for how a class of field particles acquires mass under conditions that are otherwise mysterious.

The slow roll inflationary scenario

The essence of the so-called slow roll inflationary is easily grasped. Start by using (611) and (612) in (613) and (614). This leads to the equations

$$H^2 = \frac{8\pi G}{3c^2}\left(V + \frac{\dot{\Phi}^2}{2c^2}\right) \tag{622}$$

and

$$\ddot{\Phi} + 3H\dot{\Phi} + c^2\frac{dV}{d\Phi} = 0 \tag{623}$$

The idea is that the gross form of the potential $V(\Phi)$ itself changes while the Universe expands and cools, going from the top form of figure (19) early on to the bottom form as things cool, much as a ferromagnet's free energy does when the temperature changes from $T > T_c$ to $T < T_c$. Moreover, if V is very flat, so that $dV/d\Phi$ is small, then $\dot{\Phi}$ is also small by equation (623) and there is an extended period when (622) is simply

$$H = \frac{\dot{R}}{R} \simeq \sqrt{\frac{8\pi G V_0}{3c^2}} \tag{624}$$

where V_0 is the (approximately) constant of $V(\Phi)$. R grows exponentially,

$$R \propto \exp\left(\sqrt{\frac{8\pi G V_0}{3c^2}}t\right) \tag{625}$$

and the Universe enters its inflationary phase. Φ meanwhile grows slowly, but it does grow, and eventually, after many e-foldings, the inflation stops when the minimum of $V(\Phi)$ is reached. It was Alan Guth, a particle physicist, who put together this picture in 1980, and brought to the fore the concept of inflation as a phase of the history of the early Universe. In particular, he argued that the peculiar model potentials then in widespread use to understand ferromagnets and symmetry breaking in particle physics, might also be relevant to fundamental problems in cosmology.

Exercise. The slow-roll equations of inflationary cosmology, from (622) and (623), are

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G V}{3c^2}, \quad \dot{\Phi} = -\left(\frac{c^5}{24\pi G V}\right)^{1/2}\frac{dV}{d\Phi}.$$

What are the conditions for their validity? Next, try a potential of the form

$$V(\Phi) = V_0(1 - \epsilon^2\Phi^2)^2,$$

where ϵ is small and V_0 constant. Plot V as a function of Φ. Show that this V satisfies the slow-roll constraints, and solve the above differential equations for $R(t)$ and $\Phi(t)$ with no further approximations.

The theoretical arguments for the mechanism of inflation are not based on fundamental theory (at least not yet). They are what physicists call "phenomenological." That means they are motivated by the existence of an as yet unexplained phenomenon, and rely on the detailed mathematical exploration of a what-if theory to see how the ideas might lead to the behaviour in question. Perhaps the theory, if framed carefully, will explain something it wasn't specifically designed to do. That would be encouraging! Inflation models have this property, which is why there are very attractive.

Here is an example. During the period of inflation, small fluctuations go through two types of behaviour in sequence. At first, they oscillate, like a sound wave. But as the Universe rapidly expands, at some point the peak of a wave and the trough of a wave find themselves outside of each other's horizons! A wave can't possibly oscillate coherently under those conditions, so the disturbance remains "frozen."

But the rapid inflation eventually stops while the Universe is still practically a newborn baby. Then, the expansion no longer is accelerating, but decelerating, and enough time

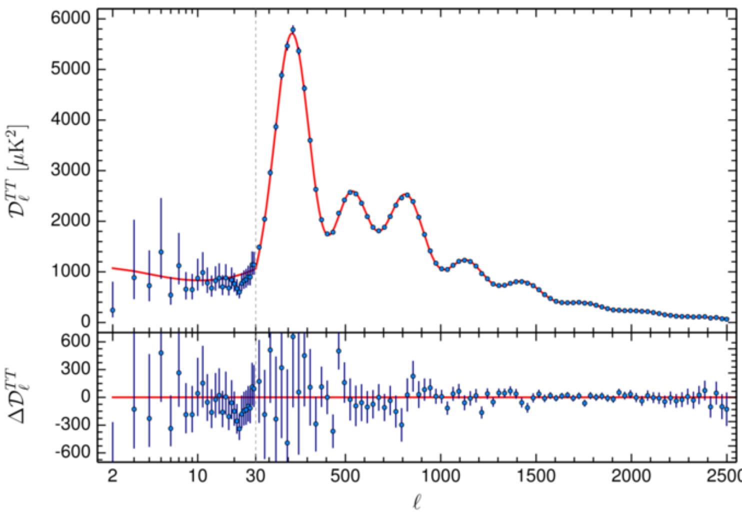

Figure 20: *Top:* The temperature power spectrum of fluctuations (known as "TT spectrum" in the literature) as a function of angular scale from recent Planck data. (The bottom plot is a measure of the discrepancy between the data and a model, and does not concern us here.) The excellent model fit in the top plot is strong evidence for inflation.

passes so that the horizon once again can encompass a wavelength. The oscillation can then restart. Suppose that between this (very early) moment of restart when the oscillation starts with zero velocity, and the time of hydrogen recombination (several hundred thousand years later), when we see the imprint of the fluctuation in the radiation, there is one full contraction of the oscillation. (Or one full expansion as well.) This half-period of oscillation correponds to a particular wavelength. For this particular wavelength we would expect to see a peak when we plot the spectrum of fluctuations at all wavelengths. And then another peak for the wavelength of full expansion. And so on. The so-called power spectrum[17] in figure (19) shows a sequence of peaks on certain angular scales on the sky. In order to exist as well-defined entitites, instead of just a smear, these peaks need to have ocillations recommence at nearly the same time. This is possible only because of inflation. During inflation, the rapid expansion prevents the random oscillations that would otherwise be present. It then allows the dance to begin again from zero velocity. This happens very early on, once the rapid expansion has slowed and the oscillations "enter the horizon." The coherent release at only very slightly different very early times (for different wavelengths) is what makes these peaks possible: their mere existence is powerful evidence for inflation.

Let us recap. The concept of an early inflationary period of the Universe explains both why there is such uniformity in the CMB temperature across the sky and why the Universe is flat. Indeed, it explains why there is an FRW metric in the first place. But there is subtlety as well in the predictions of inflation, including statistical predictions of where the power spectrum should have its peaks. There is much more that, alas, we don't have time to go

[17]If $T(\theta, \phi)$ is the CMB temperature as a function of angle, and $\mathcal{T}$ is its fourier transform in wavenumber space, the power spectrum is $|\mathcal{T}|^2$.

into in this course. There are firm predictions for how large scale structure in the Universe evolves—how galaxies cluster—from the intial seed fluctuations. These rely on the notion of *Cold Dark Matter* taking part in the gravitational response. Cold Dark Matter (CDM) is cold in the sense that it responds readily to gravitational perturbations, forming the bulk of the mass distribution in galaxies and clusters. Though we don't know very much about CDM, there are reasons to think that it may be some kind of weakly interacting massive particle, and there are searches underway to try to detect such particles via their (perhaps) more easily found decay products. But it all must begin with some kind of inflationary process. There really is no other explanation for how vast streches of the Universe could even have been in causual contact. Inflation is a powerful, unifying concept without which we can not make sense of even the most basic cosmological observations. And don't forget: the Universe is inflating right now! We are living through mild inflation that will, with time, become much more dramatic.

But why? From whence? What is the underlying imperative? While there are some promising ideas afoot, we still don't really know how and why inflation occured. Maybe somebody reading these notes will settle the matter.[18]

8.11.6 A Final Word

Astrophysics can be a very a messy and speculative business. But every once and a while, something truly outstanding is accomplished. The development of stellar structure and evolution is one such triumph. This led to a new field of science: nuclear astrophysics, and ultimately a precision theory for the origin of the chemical elements. We figured out where atoms come from and even how to make them ourselves, a stunning achievement. Another milestone is the blossoming of the theory of black holes, brilliantly confirmed in the last year by the LIGO detection of gravitational radiation from merging black hole binaries. Surely the development of precision cosmology, the discovery and construction of a model of the Universe must rank as one of the great advances in science, on a par with the Crick-Watson DNA model, not just technical but transformative. We have taken the full measure of the entire Universe. What we thought of a generation ago as the stuff of the Universe is only 5% of the stuff of the Universe, and the concept of inflation suggests that a incomprehensibly vast multiverse is a viable description of the true reality. To be sure, important questions remain, and thank goodness for that. But we know the age, we know the dynamics, and we know the gross history of the Universe. That is a breathtaking accomplishment, one that will happen only once in our existence as a species.

End of notes February 16, 2017.

[18] I'm quite serious: given the historical track record, when this problem is solved, it would not be at all surprising to me that it is by someone who was an Oxford undergraduate.