Oxford Master Course in Mathematical and Theoretical Physics

Astrophysical Fluid Dynamics

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Most of the fundamental ideas of science are essentially simple, and may, as a rule, be expressed in a language comprehensible to everyone.

— Albert Einstein

1 Fundamentals

1.1 Opening Comment

We arose from the gas cloud that collapsed to form our solar system, and when, in the distant future, the sun enters its red giant phase, we shall all return to gas and dust. Little wonder, then, that the topic of astrophysical gasdynamics holds our fascination.

The behaviour of a gas subject to large-scale gravitational and magnetic forces is enormously rich and full of surprises. One of my goals in giving this course is to try to give you, the student encountering the topic for the first time, a sense of both the generality and the depth of the problems we are struggling with. Truly, there isn’t an area of modern astrophysics that is not touched in some way by the dynamical behaviour of gases. Astrophysical gas dynamics may be the most fundamental domain of astrophysics. It is impossible to understand star formation, stellar structure, planet formation, accretion discs, or anything in the early universe without a detailed knowledge of gas dynamics. It is an excellent way to begin a study of theoretical astrophysics.

1.2 Governing Equations

Although the ultimate fundamental objects are the atomic particles that comprise our gas, we shall work in the limit in which the matter is regarded as a nearly continuous fluid. The fact that this is not exactly a continuous fluid manifests itself in many ways, the most important of which is the equation of state of our gas, which depends upon the notion of atomic collisions. But more subtle effects are also present, like viscosity and thermal conduction, both of which are a consequence of finite mean free paths.

One of the most interesting features of astrophysical gases is that they are almost always magnetized. This allows modes of behaviour that are absent in
an ordinary gas (e.g. shear waves), sometimes with profound consequences, especially in rotating systems. The dynamics of magnetized gases is known as magnetohydrodynamics, or MHD for short, and we will have much to say about this topic. The ohmic resistivity of magnetized gas is another example of a collisional process involving individual particles, in this case charged particles.

I shall assume that the reader is familiar with the basic equations of standard hydrodynamics. If not, (s)he may review a standard textbook (my favorite is *Elementary Fluid Dynamics* by D.J. Acheson), or the set of extensive notes I have prepared for my course Hydrodynamics, Instability and Turbulence. Let us begin with a very brief review.

### 1.2.1 Mass Conservation

The statement of mass conservation is expressed by the equation:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
\]

(1)

Here \( \rho \) is the mass density and \( \mathbf{v} \) is the velocity field. The content of this equation is simply that if there is net mass influx into or mass outflux from a fixed volume, the mass within that volume must change accordingly. If the flow is divergence free, the density of an individual fluid element remains constant.

### 1.2.2 Newtonian Dynamics

Our second fundamental equation is a statement of Newton’s second law of motion, that applied forces cause acceleration in a fluid. The acceleration refers to an individual element of fluid, hence the time derivative is expressed as a total, or *Lagrangian* derivative, following the path of the element:

\[
\rho \frac{D \mathbf{v}}{Dt} = \rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \mathbf{F}
\]

(2)

where the right side is the sum of the forces on the fluid element.

The most fundamental force acting on a fluid is the pressure. We shall almost always be working with an ideal gas in this course, and the pressure is then given by the ideal gas equation of state

\[
P = \frac{\rho kT}{\mu}
\]

(3)
where $T$ is the temperature in Kelvins, $k$ is the Boltzmann constant $1.38 \times 10^{-23}$ J K$^{-1}$, and $\mu$ is the mass per particle. The quantity $kT/\mu$ has dimensions of velocity squared, and arises often enough that it is given its own name:

$$c_S^2 \equiv \frac{kT}{\mu}$$

(4)

where the subscript $S$ refers to “sound” for reasons that will become clear later. $c_S$ is the “isothermal sound speed.”

The pressure arises from the kinetic energy of the gas particles themselves, which must never be confused with fluid elements. A fluid element is small enough that it has uniquely defined dynamic and thermodynamic attributes (e.g. density and pressure), but large enough to contain a vast number of atoms. A fluid element has a well-defined entropy for example, an atom does not.

There is a very simple relationship between the pressure $P$ and internal energy density $\mathcal{E}$ of an ideal gas:

$$\mathcal{E} = \frac{P}{\gamma - 1}.$$  

(5)

Here $\gamma$ is the adiabatic index of the gas. It is equal to 5/3 for single particles, and 7/5 for diatomic molecules.

A pressure exerts a force only if it is not spatially uniform. For example, the pressure force in the $x$ direction on a slab of thickness $dx$ and area $dy\,dz$ is

$$[P(x - dx/2, y, z, t) - P(x + dx/2, y, z, t)]dy\,dz = -\frac{\partial P}{\partial x}dV$$  

(6)

There is nothing special about the $x$ direction, so the force per unit volume from a pressure is more generally $-\mathbf{\nabla}P\,dV$.

Other forces can be added on as needed. One force of obvious importance in astrophysics is gravity. The Newtonian gravitational acceleration $\mathbf{g}$ can always be derived from a potential function

$$\mathbf{g} = -\mathbf{\nabla}\Phi$$  

(7)

If the field is from an external source, then $\Phi$ is a given function of $\mathbf{r}$ and $t$, otherwise it must be computed along with the evolution of the fluid itself. We shall discuss the problems of self-gravity later in the course.

Another force that we must consider arises from the presence of a magnetic field. Magnetic fields allow a gas to behave in ways not allowed when the field vanishes, and the additional degrees of freedom imparted to a gas mean that magnetic forces can be very important even when the field appears
to be weak! To calculate the magnetic force per unit volume exerted by a magnetic field, start with the Maxwell equation
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \]  
(8)
The effects of the displacement current are negligible for nonrelativistic fluids, since they involve time delays associated with light propagation. Hence, the current density is determined by the magnetic field geometry:
\[ \mathbf{J} = \left( \frac{1}{\mu_0} \right) \nabla \times \mathbf{B} \]  
(9)
The Lorentz force per unit volume is \( \mathbf{J} \times \mathbf{B} \), assuming that the gas is everywhere locally neutral.

In the absence of dissipational processes, the equation of motion for a magnetized gas is therefore
\[ \rho \frac{\partial \mathbf{v}}{\partial t} + (\rho \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \rho \nabla \Phi + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} \]  
(10)

1.2.3 Viscosity

The fact that there is a finite distance between collisions of the mass particles of gas creates internal so-called viscous stresses in the flow. As a result, a velocity gradient tends to relax and become erased with time if undisturbed. To leading order, the forces arising from the velocity gradients are found to be linear in these gradients. This is reasonable, rather like the leading term in some sort of a Taylor expansion.

To represent this, we introduce index notation \( i, j, k \) which take on vector Cartesian components \( x, y, z \). A repeated index is summed over. Hence \( \nabla \cdot \mathbf{v} \) is denoted \( \frac{\partial v_i}{\partial x_i} \). The viscous stress tensor \( \sigma_{ij} \), dimensions of momentum per unit second per unit area, takes the form first introduced by Stokes,
\[ \sigma_{ij} = \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial v_k}{\partial x_k} \right) \]  
(11)

where \( \eta \) is a scalar that may be a function of density and pressure (more below). This is the most general linear superposition of velocity gradients that vanishes for solid body rotation (no shear) and for uniform expansion (no preferred direction for momentum flow). As an Exercise, you should prove this.

Notice that \( \sigma_{ij} = \sigma_{ji} \), i.e., the \( i \) component of the momentum being transported in the \( j \) direction is physically equivalent to the \( j \) component of
of the momentum being transported in the $i$ direction. The momentum being “carried” and the momentum doing the “transporting” are interchangeable.

Notice as well that the $\sigma_{ij}$ tensor has a zero trace: the sum of its diagonal elements $\sigma_{ii} = 0$. This is an indication that the deformations of the flow caused by the shear that affects the viscous stress are associated with no change in volume. There can be a viscous stress associated with the velocity divergence itself under unusual circumstances, but we will not pursue this in this course (see e.g. Landau & Lifschitz, *Fluid Mechanics*).

The parameter $\eta$ is known as the *dynamic viscosity* coefficient, with dimensions of mass per length per time. The viscosity is often ignorable in many applications, but is needed when small scales are important for dissipation. This may be the case for turbulent flow. In fact, astrophysicists love to represent the whole of turbulent flow phenomenologically with a fake “turbulent viscosity parameter,” as a way to account for the enhanced transport turbulence often produces. Avoid joining this crowd, but if you must, at least beware of drawing detailed mathematical conclusions this way.

The equation of motion with viscosity may be written, in a mixed vector-index notation as

\[ \rho \frac{\partial \mathbf{v}}{\partial t} + (\rho \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P - \rho \nabla \Phi + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{\partial \sigma_{ij}}{\partial x_j} \]

(12)

where $i$ is the component of all boldface vector quantities being selected.

In the limit of no gravity, no magnetic field, constant density and constant $\eta$, we obtain the so-called Navier-Stokes equation:

\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P' + \nu \nabla^2 \mathbf{v} \]

(13)

where $P' = P/\rho$ and $\nu = \eta/\rho$ is the *kinematic viscosity*. As an Exercise, you should prove this. If you can further prove that this equation, together with $\nabla \cdot \mathbf{v} = 0$, has no solutions developing singularities from nonsingular initial conditions (or much to everyone’s surprise find such a singular solution), you can win $1,000,000$ from the Clay Mathematics Institute\(^1\). Without viscosity this statement is false; there are singular solutions that develop from the “inviscid” equations, as we will see later in the course.

1.2.4 Energetics

The thermal energy behaviour of the gas is described by the internal energy loss equation, which is most conveniently expressed in terms of the entropy

\(^1\text{http://www.claymath.org/millenium-problems/navierstokes-equation} \)
per particle. The entropy is defined up to an (unimportant) additive constant, and is given by
\[ s = S_N = \frac{k}{\gamma - 1} \ln P \rho^{-\gamma} \]  
(14)
where \( N \) is the number of particles, \( \gamma \) is the adiabatic index (equal to \( 1 + 2/f \) where \( f \) is the number of degrees of freedom of a particle).

Exercise. Derive the above expression from \( dE = -PdV + TdS \), \( P = \rho kT/m = (\gamma - 1)\mathcal{E} \), \( E = \mathcal{E}V \).

The entropy of a fluid element is conserved unless there is a loss or gain from radiative processes, or the gas is heated by dissipation. If \( n \) is the number of particles per unit volume, then
\[ nT \frac{Ds}{Dt} = \frac{P}{\gamma - 1} \frac{D\ln P \rho^{-\gamma}}{Dt} = \text{volume heating rate} \equiv \dot{Q} \]  
(15)
If there are no radiative losses or gains and no dissipation, as is often the case when the fluid motions are too rapid for heat to escape, the fluid is said to be adiabatic and the right side of the above is zero. Note that the internal thermal energy is not conserved in an adiabatic fluid because of compression or expansion. As an exercise, the reader should show that \( c_2^s \) satisfies the equation
\[ \rho \frac{D}{Dt} \frac{c_2^s}{\gamma - 1} = -P \nabla \cdot \mathbf{v} \]  
(16)
for an adiabatic gas. (Use the entropy and mass conservation equations.) The temperature of a fluid element, like the density, remains fixed only if the motions are incompressible.

In the presence of “bulk” radiative losses, meaning that the photons can easily escape, a typical form for \( \dot{Q} \) might be
\[ \dot{Q} = n\Gamma - n^2 \Lambda(T) \equiv -\rho \mathcal{L} \]  
(17)
where \( \Gamma \) is an external heating rate, and the \( \Lambda \) term represents the effect of collisional losses (hence the \( n^2 \) dependence from binary collisions). In an early influential model of the interstellar medium, \( \Gamma \) was the heating rate due to cosmic rays and \( n^2 \Lambda \) the volumetric losses due to the excitation of CII lines. More generally, one uses the net loss function per unit mass \( \mathcal{L} \) as a measure of the departure from adiabatic behaviour. Typically, \( \mathcal{L} \) is written as a function of \( \rho \) and \( T \) (e.g. thermal bremsstrahlung losses are proportional to \( \rho^2 T^{1/2} \)), but any two thermodynamic variables will do.

To apply the energy equation to stellar interiors, the nature of radiative energy losses must be carefully assessed. If active convection is not taking
place, the dominant mode of energy loss is the transport of the radiation energy density. This may seem odd at first glance, because the energy density of a stellar interior is usually completely dominated by the thermal energy of the matter particles; only in the most massive stars does the radiation energy density become comparable. The reason that energy transport from radiation is more effective is that the mean free path for photon scattering is huge compared with any collision mean free path associated with the matter. In other words, the radiation energy leaks out much more rapidly from a stellar interior, and is thence radiated at the stellar surface very much like a blackbody: the energy loss per unit area of surface is \( \sigma T^4 \), where \( \sigma \) is the Stefan-Boltzmann constant and \( T \) is the surface temperature. (\( \sigma = 5.67 \times 10^{-8} \text{ J s}^{-1} \text{ m}^{-2} \text{ K}^{-4} \).)

A stellar interior is about as close to a perfect thermodynamic equilibrium that one can imagine, but it is not exactly perfect. It is, after all, hotter near the core than near the surface. It is this temperature gradient that is responsible for the drift of radiation out of the interior plasma into the surrounding space. We expect that the flux of energy caused by this gradient should be of order \( c\lambda \nabla(aT^4) \), where \( \lambda \) is the mean free path for photon scattering. In fact, the precise radiative flux \( \mathcal{F} \) is given by

\[
\mathcal{F} = -\frac{c\lambda}{3} \nabla(aT^4)
\]  

the factor of 3 being little more than a directional average. (See Martin Schwarzschild’s classic text, *The Structure and Evolution of the Stars*, for a careful derivation.)

The mean free path is given by elementary kinetic theory as

\[
\lambda = \frac{1}{n\sigma}
\]  

where \( n \) is the total of scattering particles and \( \sigma \) is an average cross section per particle (not to be confused with Stefan-Boltzmann constant!) It is convenient to work with the mass-related quantities. With \( \mu \) being a mass per particle, we define the mass density \( \rho = \mu n \) and the average cross section per particle \( \kappa = \sigma/\mu \), and write

\[
\mathcal{F} = -\frac{4acT^3}{3\rho\kappa} \nabla T
\]  

\( \kappa \) is called the opacity, generally a complicated function of the temperature, density and and abundances. At typical stellar temperatures, an approximation known as a Kramers law in which

\[
\kappa \propto \frac{\rho}{T^{3.5}}
\]
is often used (see e.g. *Principles of Stellar Evolution*, by D. Clayton, or good old Schwarzschild). At very high temperature however, the scattering is dominated by electron scattering, and $\kappa$ is a constant. Only the most massive stars come into this regime.

**Exercise.** Evaluate this constant for the case (a) of a purely hydrogenic gas; (b) a gas in which helium is 10% of the total number of baryons. Answers:

(a) 0.397 g cm$^{-2}$; (b) 0.6435 g cm$^{-2}$. Note the convenient cgs units! (DATA: Proton mass $= 1.6726 \times 10^{-24}$ g, electron cross section $= 6.6526 \times 10^{-25}$ cm$^2$, helium mass $= 6.6466 \times 10^{-23}$ g, electron mass $= 9.109 \times 10^{-28}$ g.)

**Exercise.** In a plasma, the ordinary nonradiative thermal heat is transported by electron Coulomb collisions at a flux (energy per area per time) of $F_C = -\chi T^{5/2} \nabla T$, where $\chi$ in cgs units is about $6 \times 10^{-7}$. (See Spitzer, *Physics of Fully Ionised Gases.*) Compare the radiative and Coulomb heat fluxes for a plasma at 1 gram per cm$^3$ and $T = 10^6$K.

In the presence of diffusive radiative losses (or any other sort of diffusive losses), the entropy equation becomes simply

$$ nT \frac{DS}{Dt} = \frac{P}{\gamma - 1} \frac{D \ln \rho}{Dt} = -\nabla \cdot \mathbf{F} \tag{22} $$

with $\mathbf{F}$ given by (20). In radiative equilibrium, the left hand side of this equation is identically zero, so that in a spherical star, the flux magnitude $F$ is proportional to $1/r^2$. In terms of the star’s luminosity $L$,

$$ F = -\frac{4acT^3}{3\rho \kappa} \frac{dT}{dr} = \frac{L}{4\pi r^2}. \tag{23} $$

### 1.3 The vector “v dot grad v”

The vector $(v \cdot \nabla)v$ is more complicated than it appears. In Cartesian coordinates, matters are simple: the $x$ component is just $(v \cdot \nabla)v_x$, and similar for $y$, $z$. But in cylindrical coordinates $(R, \phi, z)$, the radial $R$ component (say) of this vector is NOT $(v \cdot \nabla)v_R$. Rather, we must take care to write

$$ (v \cdot \nabla)v = v \cdot \nabla (v_R e_R + v_\phi e_\phi + v_z e_z) \tag{24} $$

where the $e_i$ are unit vectors in their respective directions. In Cartesian coordinates, these unit vectors would be constant, but in any other coordinate system they change with position. You should be able to show that

$$ \frac{\partial e_R}{\partial \phi} = e_\phi, \quad \frac{\partial e_\phi}{\partial \phi} = -e_R, \tag{25} $$
and that there are no other unit vector derivatives in cylindrical coordinates. Thus, the radial component of \((v \cdot \nabla) v\) is

\[ v \cdot \nabla v_R - \frac{v_R^2}{R}, \]

and the azimuthal component is

\[ v \cdot \nabla v_\phi + \frac{v_R v_\phi}{R}. \]

The extra terms are related to centripetal and Coriolis forces, though more work is needed to extract the latter...a piece of it still remains in the gradient term!

We will use both spherical and cylindrical coordinates throughout this course, as shown in figure 1.

1.4 Rotating Frames

It is often useful to work in a frame rotating at a constant angular velocity \(\Omega\), perhaps the frame in which an orbiting planet appears at rest around its star. The same rule that applies to ordinary point mechanics applies here as well: add

\[ -2\Omega \times v + R\Omega^2 e_R \]

to the applied forces operating on a fluid element (the right side of the equation). The first term is the Coriolis force, the second is the centrifugal force, \(\Omega\) is in the vertical direction, and all velocities are measured relative to the rotating frame of reference.
1.4.1 The Taylor-Proudman Theorem

Suppose that in the rotating frame the fluid velocity $v$ is much less than $R\Omega$. Suppose further that the density is constant and the pressure term $(1/\rho)\nabla P$ is an exact gradient. Then, the sum of the enthalpy, gravity, and centrifugal terms is expressible as a gradient, say $\nabla H$, and the steady-state fluid equation is simply

$$2\Omega \times v = \nabla H$$

(29)

Since $\Omega$ lies along the $z$ axis, the left hand side has no $z$ component, so neither does the right. But that means that $H$ is independent of $z$. Then the radial and azimuthal velocity components are independent of $z$ as well, that is, the flow is constant on cylinders! In the case of constant density, the mass conservation equation is $\nabla \cdot v = 0$. But the above equations of motion imply

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

(30)

so that $\partial v_z/\partial z = 0$, and $v_z$ is also independent of $z$. If there is no $v_z$ at the upper or lower boundaries, $v_z$ vanishes everywhere, and the motion is two dimensional! The fact that small motions in rotating systems are often independent of height is called the Taylor-Proudman theorem, and you will often hear “Taylor Columns” referred to in fluid mechanics talks. Now you know where they come from.

It may seem that our arguments go through even if the density is not constant when $P = P(\rho)$, since it is still possible to form an $H$ function. But now mass conservation cannot be so trivially satisfied, because of the $\nabla \cdot \rho$ term. With only three quantities $(v_x, v_y, \rho)$ we cannot satisfy four equations, so there must be a $v_z$ present, and the flow is no longer so simple.

1.5 The Indirect Potential

A common problem of interest is a gaseous disc around a dominant central body $M_\star$ in which a secondary body $M$ is embedded (or otherwise exerts an external gravitational force). With $M_\star$ as the origin, the acceleration on a fluid element at location $\mathbf{R}$ is NOT

$$\frac{d^2 \mathbf{R}}{dt^2} = -\frac{GM_\star \mathbf{R}}{R^3} - \frac{GMr}{r^3}.$$  

(NO!)  

(31)

(See figure [2] for vector definitions.) This is the acceleration that would be measured in an inertial frame. The right side of this equation is actually $d^2 \mathbf{X}/dt^2$, the acceleration at the same location from the point of view of the center-of-mass of $M_\star$ and $M$. Sitting on top of the central star, one is
no longer in an inertial frame: the star itself is accelerating. The correct acceleration in the star’s frame is given by adding a term $-\frac{GM R_M}{R^3}$ to the right side, i.e. minus the acceleration of the star itself. The correct equation is

$$\frac{d^2 R}{dt^2} = -\frac{GM_* R}{R^3} - \frac{GM r}{r^3} - \frac{GM R_M}{R_M^3} \quad \text{(YES!)} \quad (32)$$

The last term is known as the indirect term, derivable from the indirect potential $-\frac{GM}{R_M}$. The total perturbing potential due to $M$ is then

$$\Phi = -\frac{GM}{r} - \frac{GM}{R_M} = -\frac{GM}{|R - R_M|} - \frac{GM}{R_M} \quad (33)$$

Note that if $R_M \gg r$, then the indirect term may be neglected; it is important when $R_M$ is comparable to $r$. 

Figure 2: Central star $M_*$ surrounded by a disc with a perturbing mass $M$. CM denotes the center-of-mass; all $X$ vectors are measured relative to this point. The effect of working in the convenient but noninertial frame centered on $M_*$ can be included by adding an “indirect” potential to the equations of motion. See text.
1.6 Local Equations in Discs and Stars

It is possible to simplify the dynamics in a disc or star by working in a small neighbourhood around a point. Often, this is all that is necessary to reveal the critical dynamics of a flow, and simplifying the problem mathematically allows for deeper physical understanding.

Consider the case of a Keplerian disc, gas flow in a central potential $-\frac{GM}{r}$. In a frame rotating at the angular velocity $\Omega(R_0) \equiv \Omega_0$, where $R_0$ is particular cylindrical location, the added rotational force terms are $-2\Omega_0 \times v$ plus $R\Omega^2 e_R$. The radial force due to gravity may be written as $-R\Omega^2(R)$, where $\Omega(R)$ is the angular velocity of the gas as a function of $R$. At $R = R_0$ there is a force balance, but at a slightly different location $R = R_0 + x$ (x small), there is an imbalance given by $-x d\Omega^2/d\ln R = 3\Omega^2 x$:

$$(R_0 + x)[\Omega^2(R_0) - \Omega^2(R_0 + x)] \simeq -x \frac{d\Omega^2}{d\ln R}.$$ 

Dropping the “0” subscript, the local equation of motion in a Keplerian disc is given by

$$\frac{Dv_x}{Dt} + (v \cdot \nabla)v + 2\Omega \times v = -\frac{1}{\rho} \nabla P + 3\Omega^2 xe_R - z\Omega^2 e_z$$

(34)

**Exercise.** Where did the $-z\Omega^2$ term come from?

In taking the Lagrangian derivative, the $1/R$ curvature terms may be ignored, since we are working in a very small neighbourhood of a patch of disc, in essence taking the limit of $R \to \infty$ while $\Omega$ remains finite. Then we may use ordinary Cartesian coordinates with $dx, dy, dz$ replacing $dR, Rd\phi, dz$:

$$\frac{Dv_x}{Dt} - 2\Omega v_y = -\frac{1}{\rho} \frac{\partial P}{\partial x} + 3\Omega^2 x$$

(35)

$$\frac{Dv_y}{Dt} + 2\Omega v_x = -\frac{1}{\rho} \frac{\partial P}{\partial y}$$

(36)

$$\frac{Dv_z}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial z} - \Omega^2 z$$

(37)

Local forces (perhaps magnetic) may be added to the right side at will. These equations are sometimes referred to as the Hill system.

**Exercise.** The indirect potential should be ignored if we wish to study the effects of an embedded point mass in a disc in the local approximation. Why?

A similar, but somewhat simpler reduction is used for spherical stars or planets. Here, the centrifugal terms in $R\Omega^2$ are generally negligible. Now,
we may identify \( dx, dy, dz \) with \( r \, d\theta, r \sin \theta \, d\phi, \, dr \). The “\( \beta \)–plane” equations for motion on a spherical surface are

\[
\frac{Dv_x}{Dt} - f v_y = -\frac{1}{\rho} \frac{\partial P}{\partial x}
\]

(38)

\[
\frac{Dv_y}{Dt} + f v_x = -\frac{1}{\rho} \frac{\partial P}{\partial y}
\]

(39)

where \( f = 2\Omega \cos \theta \) is the “Coriolis parameter.” Note that \( f = 0 \) at the equator.

**Exercise.** Derive these equations.

An interesting two-dimensional oceanographic application is to add a tidal potential force on the right and use \( P = \rho g \zeta \), where \( \rho \) is the (here constant) density of water and \( \zeta = \zeta(x, y, t) \) is the (varying) height of the sea, relative to the equilibrium sea level.

**Exercise.** Show that the equation of mass conservation for this problem is

\[
\frac{\partial \zeta}{\partial t} + \frac{\partial (hv_x)}{\partial x} + \frac{\partial (hv_y)}{\partial y} = 0
\]

(40)

where \( h \) is the total height of the sea. (Note that to leading order, \( h \) can be replaced by undisturbed sea depth \( h_0(x, y) \). In what sense do I mean "leading order"?) The set of three equations, with the additional tidal forcing terms, are known as the Laplace tidal equations.

### 1.7 Manipulating the Fluid Equations

For a particular problem, working in cylindrical or spherical coordinates is often the most convenient, but for proving general theorems or identities, Cartesian coordinates are usually the simplest to use. There is a formalism that makes working with the fluid equations much easier in this case.

As in our brief introductory discussion of viscosity, we will let the index \( i, \, j, \, o r \, k \) represent Cartesian component \( x, \, y, \, o r \, z \). Hence \( v_i \) means the \( i \)th component of \( v \), which may any of the three depending upon what value \( i \) is chosen. So \( v_i \) is a way to write \( v \). The gradient operator \( \nabla \) is written \( \partial_i \), in a way that should be self-explanatory.

As before, if an index appears twice, it is understood that it is to be summed over all the values \( x, \, y, \, \text{and} \, z \). Hence

\[
A \cdot B = A_i B_i = A_x B_x + A_y B_y + A_z B_z,
\]

(41)
and
\[(v \cdot \nabla)v = (v_i \partial_i)v_j \quad (42)\]

In this last example \(i\) is a dummy index, and the vector component is represented by \(j\). The dynamical equation of motion in this notation is
\[\rho \partial_t + (v_i \partial_i)v_j = -\partial_j P - \rho \partial_j \Phi \quad (43)\]

Sometimes the “rot” (or “curl”) operator is needed. For this, we introduce the Levi-Civita symbol \(\epsilon^{ijk}\). It is defined as follows:

- If any of the \(i, j,\) or \(k\) are equal to one another, then \(\epsilon^{ijk} = 0\).
- If \(ijk = 123, 231,\) or \(312\), the so-called even permutations of \(123\), then \(\epsilon^{ijk} = +1\).
- If \(ijk = 132, 213,\) or \(321\), the so-called odd permutations of \(123\), then \(\epsilon^{ijk} = -1\).

The reader should be able to convince him(her)self that
\[\nabla \times A = \epsilon^{ijk} \partial_i A_j \quad (44)\]

Here, the vector component is represented by the index \(k\). Don’t forget to sum over \(i\) and \(j\)! \(\epsilon^{ijk}\) is used in the ordinary cross product as well:
\[A \times B = \epsilon^{ijk} A_i B_j \quad (45)\]

Notice that
\[A \cdot (B \times C) = \epsilon^{ijk} A_i B_k C_j \quad (46)\]

which proves that any even permutation of the vectors on the left side of the equation must give the same value, and an odd rearrangement gives the same value with the opposite sign.

A double cross product looks complicated:
\[A \times (B \times C) = \epsilon^{lkm} A_l (\epsilon^{ijk} B_i C_j) = \epsilon^{mlk} \epsilon^{ijk} A_l B_i C_j \quad (47)\]

The last equality follows because \(mlk\) is an even permutation of \(lkm\). This looks unpleasant, but fortunately there is an identity that saves us:
\[\epsilon^{mlk} \epsilon^{ijk} = \delta_{mi} \delta_{lj} - \delta_{mj} \delta_{li} \quad (48)\]

where \(\delta_{ij}\) is the Kronecker delta function (equal to zero if \(i\) and \(j\) are different, unity if they are the same). The proof of this is left as an exercise for the
reader, who should be convinced after a few simple explicit examples. With this identity, our double cross product becomes

\[ A \times (B \times C) = B_mA_jC_j - C_mB_iA_i = B(A \cdot C) - C(A \cdot B). \] (49)

Our final example is to derive an expression for

\[ A \times (\nabla \times B) = \epsilon^{ijk}A_i(\epsilon^{lmj}\partial_lB_m) = \epsilon^{kij}\epsilon^{lmj}(A_i\partial_lB_m) \] (50)

Using our identity (48), this becomes

\[ (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})(A_i\partial_lB_m) = A_i\partial_kB_i - A_i\partial_lB_k = A_i\partial_kB_i - (A \cdot \nabla)B \] (51)

One consequence of this result is a representation of \( A_i\partial_kB_i \) in any coordinate system:

\[ A_i\partial_kB_i = A \times (\nabla \times B) + (A \cdot \nabla)B \] (52)

Another particular important application of (51) is to the Lorentz force expression

\[ (\nabla \times B) \times B = -\frac{1}{2} \nabla B^2 + (B \cdot \nabla)B \] (53)

The first term on the right side has the form of a magnetic pressure gradient; the second behaves like a tension force. It depends on the derivative of \( B \) along its length, and if the magnitude of \( B \) remains fixed, the force must be perpendicular to \( B \) itself. The effect of this tension force is profound, allowing a magnetized gas to support shear waves (known as Alfvén waves) that ordinarily do not exist in a fluid.

1.8 The Conservation of Vorticity

A quantity of great interest to fluid dynamicists is the vorticity, \( \omega = \nabla \times v \). We will see that it is intimately related to the angular-momentum-like circulation element \( v \cdot dl \) integrated around a closed fluid loop. Under some interesting circumstances, this circulation integral is conserved.

We start with the following identity, which follows immediately from the results of the previous section:

\[ v \times (\nabla \times v) = \frac{1}{2} \nabla v^2 - (v \cdot \nabla)v \] (54)

Using this result to replace \( (v \cdot \nabla)v \) in the inviscid, unmagnetised dynamical equation of motion results in

\[ \frac{\partial v}{\partial t} + \frac{1}{2} \nabla v^2 - v \times \omega = -\frac{1}{\rho \phi} \nabla P - \nabla \Phi. \] (55)
Exercise. If \( P = P(\rho) \) and \( dH = dP/\rho \), show that

\[
\frac{v^2}{2} + H + \Phi
\]

is constant along a velocity flow streamline. This is called the Bernoulli constant, \( B \). \( B \) need not be the same constant on every streamline!

If we take the curl of equation (55), and remember that the curl of the gradient vanishes, we find

\[
\frac{\partial \omega}{\partial t} - \nabla \times (v \times \omega) = \frac{1}{\rho^2} \left( \nabla \rho \times \nabla P \right)
\]

Let us once again consider the case where either \( \rho \) is constant, or when \( P \) is a function only of \( \rho \). Then the right hand side vanishes, and:

\[
\frac{\partial \omega}{\partial t} - \nabla \times (v \times \omega) = 0.
\]

To interpret this physically, consider a closed curve, a loop, frozen into the fluid. The integral \( \int v \cdot dl \) around the loop is by Stokes’ theorem \( \int \omega \cdot dA \), taken over an area that is bounded by the loop. This is the vorticity flux. How does the vorticity flux change with time as the fluid evolves? The answer is contained within (57).

We consider more generally an equation of the form

\[
\frac{\partial A}{\partial t} = v \times (\nabla \times A) + \nabla \Phi
\]

where \( \Phi \) is a potential function. The curl of this equation leads back to equation (57) for the special case \( A = v \), but we retain generality here. Expanding the double cross product on the right and regrouping leads to

\[
\frac{DA_i}{Dt} = v_j \partial_j A_i + \partial_i \Phi
\]

where \( D/Dt \) is the usual Lagrangian derivative and we have switched to index notation. Next, consider the change in the line integral of the vector field \( A \) over a closed loop moving with the fluid:

\[
\frac{D}{Dt} \int A \cdot dl = \int \left[ \frac{DA}{Dt} \cdot dl + A \cdot \frac{Ddl}{Dt} \right]
\]
Notice the strange derivative of the line element $dl$! The Lagrangian derivative of the line element as it moves through the fluid is just the difference between the velocity field at the two endpoints of the segment $dl$:

$$\frac{Ddl}{Dt} = (dl \cdot \nabla) v$$

(61)

or

$$\frac{Ddl_i}{Dt} = dl_i \partial_i v_j$$

(62)

From equation (59)

$$\frac{DA}{Dt} \cdot dl = dl_i v_j \partial_i A_j + dl_i \partial_i \Phi,$$

(63)

and we have just seen that

$$A \cdot \frac{Ddl}{Dt} = A_j dl_i \partial_i v_j.$$  

(64)

Adding these last two equations gives

$$\frac{D}{Dt} (A \cdot dl) = dl_i \partial_i (\Phi + v_j A_j)$$

(65)

In equation (60) we thus have a perfect gradient integrated over a closed curve, hence the integral must vanish. The line integral $\int A \cdot dl$ is conserved with the fluid. In particular when $A = v$, the velocity circulation integral along with the vorticity flux surface integral are conserved in the Lagrangian sense, moving with the fluid. We shall see very soon that the same is true for the magnetic vector potential and the magnetic flux.

The fact that the integral $\int v \cdot dl$ around any closed curve embedded in the fluid remains constant as the fluid flows is known as vorticity conservation. Another way to say the same thing is that the field lines of vorticity $\omega$ are “frozen” into the fluid. Once again, this is not a general fluid result, but depends upon the fact that either $\rho$ is constant or that $P$ and $\rho$ are directly functionally related. If the pressure gradient can push with the same force along a surface of varying inertial response, vorticity will surely be generated.

With the help of our $\varepsilon^{ijk} \varepsilon^{lmk}$ identity and just a little work, it is straightforward to show that

$$\frac{\partial \omega}{\partial t} - \nabla \times (v \times \omega) = 0$$

(66)

is the same as

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla) \omega = + (\omega \cdot \nabla) v - \omega \nabla \cdot v$$

(67)
Mass conservation may be written
\[ \frac{D \ln \rho}{Dt} = -\nabla \cdot \mathbf{v}, \]  
so that our equation becomes
\[ \frac{D\omega}{Dt} - \omega \frac{D \ln \rho}{Dt} = (\omega \cdot \nabla)\mathbf{v}, \]  
or
\[ \frac{D}{Dt} \left( \frac{\omega}{\rho} \right) = \frac{1}{\rho} (\omega \cdot \nabla)\mathbf{v} \]  
In strictly two-dimensional flow, this is a very powerful constraint. Then \( \omega \) has only a \( z \) component, and the right side must vanish. The density \( \rho \) may be replaced by the surface density \( \Sigma \) on the left side of the equation: multiply by \( \rho^2 \), expand the \( D/Dt \) derivative, integrate over height, and refold the \( D/Dt \) derivative. We find that
\[ \frac{D}{Dt} \left( \frac{\omega}{\Sigma} \right) = 0 \]  
This is known as the conservation of potential vorticity. In this case, potential vorticity (PV in the parlance) labels fluid elements. PV is extremely useful in the study of two-dimensional turbulence, and in studying long wavelength wave propagation in planetary atmospheres.

**Exercise.** Consider rotational flow, with the velocity \( \mathbf{v} \) having only a \( \phi \) component \( v_\phi \). In general, \( v_\phi \) could depend upon \( R \) and \( z \), but show that if vorticity conservation holds, under steady conditions \( v_\phi \) cannot depend upon \( z \). This is known as von Zeipel’s theorem.
The nation that controls magnetism will control the universe.

— Dick Tracy, created by Chester Gould

2 Magnetohydrodynamics (MHD)

2.1 Magnetic Forces

Astrophysical gases are almost always at least partially ionized. This is not too surprising: a glass of distilled water is ionized at the level of one part in $10^7$, and salty sea water is much more ionized: it is a very good conductor. A medium can be almost entirely neutral and still behave like a good conductor. All but the coolest and densest astrophysical gases (e.g., protostellar discs) are electrodynamically active.

The Lorentz force per unit volume in the gas is

$$ F = \rho_e E + J \times B $$

where $\rho_e$ is the charge density, $E$ is the electric field, $J$ is the current density, and $B$ is the magnetic field. The gases of interest here are all electrically neutral, so that $\rho_e = 0$. This means that the only part of the Lorentz force that affects the gas is the magnetic part.

We have already encountered the Lorentz force in our discussion of the equation of motion for a magnetized gas:

$$ J \times B = \frac{1}{\mu_0} (\nabla \times B) \times B $$

With our $\varepsilon_{ijk} \varepsilon_{lmk}$ identity, we have already seen that

$$ (\nabla \times B) \times B = -\frac{1}{2} \nabla B^2 + (B \cdot \nabla)B $$

Thus, the dynamical equation of motion for a magnetized gas is

$$ \rho \frac{Dv}{Dt} = -\nabla \left( P + \frac{B^2}{2\mu_0} \right) - \rho \nabla \Phi + \left( \frac{B}{\mu_0 \cdot \nabla} \right) B $$
The first magnetic term on the right clearly behaves like a sort of pressure. Magnetic fields lines of force do not like to be squeezed any more than gas molecules do.

The \((B \cdot \nabla)B\) term is less obvious. It corresponds to a sort of magnetic tension. Notice that it vanishes when the magnetic field does not change along its own direction. On the other hand, when there are such changes, the resulting force acts in the direction of restoring the field line back to an unstretched position. In fact, this can be made quantitative: there is a magnetic analogue to waves propagating along an ordinary string that is under tension. In the case of “magnetic strings,” these waves are called Alfvén waves.

2.2 Induction Equation

Having introduced the magnetic field, we need to know how it evolves as the flow changes. The magnetic field adds one more variable to our problem (well, three actually, since there are three components of \(B\)), so we need more equations. The motion of the gas causes the charged particles to move, the ions and electrons respond differently to the applied forces, currents form, these currents in turn generate new fields that affect the currents all over that change the fields ... Help. It seems like a complicated mess!

Fortunately there is a great simplifying principle to save us: in a perfect conductor, the electric field vanishes. Actually, what we should say is that in the rest frame of the conductor, the electric field locally vanishes. In a frame in which the conductor (in our case a fluid element of conducting gas) moves, the total Lorentz force (not the electric field) must vanish. In other words,

\[
E + v \times B = 0. \tag{76}
\]

So even though we have assumed conditions for charge neutrality, there must be an electric field: \(-v \times B\). But if the divergence of this electric field does not vanish, then there must be a local charge density, and charge neutrality cannot hold, which looks like a contradiction. Well, maybe it just turns out that \(\nabla \cdot (v \times B) = 0\). Guess what? The divergence doesn’t vanish. In a moment, we’ll come back and explain why this is not really a contradiction after all, but for the time being let us nervously continue.

Faraday’s law of induction is

\[
\frac{\partial B}{\partial t} = -\nabla \times E \tag{77}
\]

and with \(E = -v \times B\), this becomes

\[
\frac{\partial B}{\partial t} = \nabla \times (v \times B) \tag{78}
\]

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This is the extra set of equations we need to determine the magnetic field. By knowing how the spatial gradients of $B$ are behaving, we may compute how the field evolves in time, thanks to the powerful constraint that the Lorentz force on the charge carriers must vanish.

### 2.3 Self-consistency

Why don’t we have a contradiction with the fact that $\nabla \cdot E$ is not zero? The answer is that while not zero, it is in fact small. Small?? That answer is not good enough. How small? Well, of order $v^2/c^2$ ($c$ is the speed of light), which, as we will see, is precisely of the same order as the displacement current that was also neglected.

To estimate $\nabla \cdot (v \times B)$, assume that any magnetic field gradients are as large as they can be (of order $\mu_0 J$), and that $J$ is also as large as it can be, of order the ion charge density times $v$, $\rho_i v$ (it could be smaller since it is proportional to the difference between ion and electron velocities). Then

$$\nabla \cdot (v \times B) \sim v \mu_0 J \sim \frac{\rho_i v^2}{\epsilon_0 c^2}. \quad (79)$$

That answer, that the divergence of the electric field is of order $v^2/c^2$ times the ion charge density, is good enough. Not only is it permitted to ignore the divergence of the electric field, it is required! We have already not included the displacement current, and this too is a correction of order $v^2/c^2$. In this case, if $L$ is a characteristic length and $\partial/\partial t \sim v/L$, then

$$\epsilon_0 \frac{\partial E}{\partial t} \sim \epsilon_0 \mu_0 v E L \sim \epsilon_0 \mu_0 \frac{v^2 B}{L} \sim \epsilon_0 \mu_0 \frac{v^2 J}{L} \quad (80)$$

which is indeed of order $(v^2/c^2)\mu_0 J$. Corrections of order $v^2/c^2$ are relativistic, and we must ignore them to be self-consistently nonrelativistic!

Notice something quite remarkable: the magnetic field satisfies the same equation as the vorticity. In particular, equation (78) can be recast in the form of equation (58), by “uncurling” it! That means everything we learned about vorticity, in particular that it is frozen in to the fluid, also holds for the magnetic field. Magnetic flux, $\int B \cdot dA$, is conserved as the area moves with the fluid. But unlike the case of vorticity conservation, which depended upon a restrictive relationship between $P$ and $\rho$, magnetic flux conservation depends only upon there being no dissipation (i.e., electrical resistance) in the gas. This is generally an excellent approximation.
A Summary of the Dissipationless Equations of Motion

From now on, we shall drop the subscript “0” on $\mu_0$, and write $\mu$.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (81)$$

$$\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla \left( P + \frac{B^2}{2\mu} \right) - \rho \nabla \Phi + \frac{1}{\mu} (\mathbf{B} \cdot \nabla) \mathbf{B} \quad (82)$$

$$\frac{P}{\gamma - 1} \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \ln \rho^{\gamma-1} = 0 \quad (83)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \quad (84)$$

2.4 MHD Fundamentals

Now then, Dimitri ... 
— President Merkin Muffley to Premier Kissoff, Dr. Strangelove

In this section, a detailed derivation of the fundamental MHD equations is presented. The discussion will be more technical here than in most of the rest of the course, but it is very important to see how the basic governing equations of the subject arise, and much of this material is not so easy to find outside of specialized treatments. I hope the reader will have the patience to read carefully through this section, but it may be skipped the first time through without loss of continuity.

In astrophysics, we are very often interested in the MHD behaviour of a gas that is almost entirely neutral but is still a good conductor. This may seem like contradictory, since a neutral gas has no charge carriers, but the key word is “almost.” Even a very small population of charge carriers will make the gas magnetized and highly conducting, as we will shortly see.

A typical environment is a gas cloud consisting of neutral particles (predominantly $H_2$ molecules), electrons, and ions. Each species (denoted by subscript $s$) is separately conserved, and obeys the mass conservation equation

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{v}_s) = 0 \quad (85)$$
where $\rho_s$ is the mass density for species $s$ and $v_s$ is the velocity. The symbols of the flow quantities (e.g. $v$, $\rho$, etc.) for the dominant neutral species will henceforth be presented without subscripts.

So far, everything is simple. The dynamical equations become more coupled, however, since we need to include interactions between the different species. The dynamical equation for the neutral particles is

$$\rho \frac{\partial v}{\partial t} + \rho (v \cdot \nabla) v = -\nabla P - \rho \nabla \Phi - p_{nI} - p_{ne} \quad (86)$$

where $P$ is the pressure of the neutrals, $\Phi$ the gravitational potential and $p_{nI}$ ($p_{ne}$) is the momentum exchange rate between the neutrals and the ions (electrons).

The ion equation is

$$\rho_I \frac{\partial v_I}{\partial t} + \rho_I (v_I \cdot \nabla) v_I = eZn_I (E + v_I \times B) - \nabla P_I - \rho_I \nabla \Phi - p_{In} \quad (87)$$

and the electron equation is

$$\rho_e \frac{\partial v_e}{\partial t} + \rho_e (v_e \cdot \nabla) v_e = -en_e (E + v_e \times B) - \nabla P_e - \rho_e \nabla \Phi - p_{en} \quad (88)$$

The subscript $I$ ($e$) refers to the ions (electrons). When not in a subscript but used in an equation, $e$ is the fundamental charge of a proton, i.e. it is always positive. The electron charge is always $-e$. The momentum exchange rate $p_{In}$ is precisely $-p_{nI}$, and the same holds for $p_{en}$. (Why?) The quantity $Z$ is the mean charge per ion, $n$ is a number density, and the fluid is neutral in bulk, $eZn_I = en_e$.

The key point is that for the charge carriers, all terms proportional to the mass densities $\rho_I$ and $\rho_e$ are small compared with the Lorentz force and momentum exchange rates. Hence, to a very good approximation,

$$0 = eZn_I (E + v_I \times B) - p_{In} \quad (89)$$

$$0 = -en_e (E + v_e \times B) - p_{en} \quad (90)$$

Adding these two equations and using bulk neutrality leads to ,

$$0 = eZn_I (v_I - v_e) \times B - p_{In} - p_{en} \quad (91)$$

But $eZn_I (v_I - v_e)$ is just the current density $J$, so that

$$p_{In} + p_{en} = J \times B \quad (92)$$
Using this in the neutral equation leads to
\[
\frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P - \rho \nabla \Phi + \mathbf{J} \times \mathbf{B} \tag{93}
\]
Remarkably, the net Lorentz force appears unmodified in the equation for the neutrals.

For a sparsely ionized fluid, departures from ideal MHD appear mainly in the induction equation. To relate \( \mathbf{E} \) and \( \mathbf{B} \) it is best to use the electron force equation, since the ions may be more closely locked to the neutrals. Thus
\[
\mathbf{E} = -\mathbf{v}_e \times \mathbf{B} - \frac{p_{\text{en}}}{en_e} = -[\mathbf{v} + (\mathbf{v}_e - \mathbf{v}_I) + (\mathbf{v}_I - \mathbf{v})] \times \mathbf{B} - \frac{p_{\text{en}}}{en_e} \tag{94}
\]
Now matters start to get very detailed. I present these details in the following section, but for purposes of this course I view this material as entirely optional. Having made the details available to you, however, I feel free to make a quick summary of the results, leaving it for you to read the next section if you wish more explanation.

The term \( \mathbf{v}_e - \mathbf{v}_I \) is \(-J/en_e\).

The term \( \mathbf{v}_I - \mathbf{v} \) is related to \( p_{\text{ln}} \) by an equation of the form
\[
p_{\text{ln}} = \gamma \rho \rho_I (\mathbf{v}_I - \mathbf{v})
\]
where \( \gamma \) is a coefficient that may be calculated from knowledge of the interaction cross sections. (See equation (104).) But \( p_{\text{ln}} \) is simply related to \( \mathbf{J} \times \mathbf{B} \) from equation (92), because \( p_{\text{ln}} \) is in fact dominant over \( p_{\text{en}} \). Ultimately, the reason is that the ions are more massive than the electrons.

The final term proportional to \( p_{\text{en}} \) represents ohmic dissipation. We denote the electrical conductivity by \( \sigma_{\text{cond}} \).

Putting all of this together leads to the full induction equation:
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[ \mathbf{v} \times \mathbf{B} - \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0 en_e} \right] + \left[ \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0 \gamma \rho \rho_I} \right] - \frac{\nabla \times \mathbf{B}}{\mu_0 \sigma_{\text{cond}}} \tag{95}
\]

The Details....

Let us examine matters a little more closely. (Please don’t worry about every last detail. My purpose here is to give you a feeling for all that goes into a calculation like this, and to be able to understand the nomenclature that you
will encounter in the literature. You don’t have to become an expert in the minutiae of interstellar kinetic theory for this course!) \( \mathbf{p}_{nl} \) takes the form

\[
\mathbf{p}_{nl} = n\mu_{nl}(\mathbf{v} - \mathbf{v}_I)\nu_{nl}
\]

where \( n \) is the number density of neutrals, and \( \mu_{nl} \) is the reduced mass of an ion–neutral particle pair,

\[
\mu_{nl} \equiv \frac{m_im_n}{m_I + m_n},
\]

\( m_I \) and \( m_n \) being the ion and neutral mass respectively. \( \nu_{nl} \) is the collision frequency of a neutral with a population of ions,

\[
\nu_{nl} = n_I\langle \sigma_{nl}w_{nl} \rangle.
\]

In equation (98), \( n_I \) is the number density of ions, \( \sigma_{nl} \) is the cross section for neutral-ion collisions, and \( w_{nl} \) is the relative velocity between a neutral particle and an ion. The angle brackets represent an average over all possible relative velocities in the thermal population of particles. Notice that equation (96) has the dimensions of a force per unit volume, and that it is proportional to the velocity difference between the species: if there is no difference in their mean velocities, two population of particles cannot exchange momentum.

Why does the reduced mass \( \mu_{nl} \) appear? Because the reduced mass always appears in any interaction between two individual particles: in the center of mass frame the equations reduce to a single particle equation with the particle mass equal to the reduced mass. In an elastic one-dimensional collision, for example, if \( v \) is initial relative velocity of the two interacting particles, then the momentum exchange is \( 2\mu_{12}v \), where \( \mu_{12} \) is the reduced mass. (Show this.)

For neutral-ion scattering, we may approximate the cross section \( \sigma_{nl} \) to be geometrical, which means that the quantity in angle brackets will be proportional to \( \mu_{nl}^{-1/2} \). The order of the subscripts has no particular significance in either the cross section \( \sigma_{nl} \), reduced mass \( \mu_{nl} \), or relative velocity \( w_{nl} \). But \( \nu_{nl} \) does differ from \( \nu_{nl} \): the former is proportional to the neutral density \( n \), the latter to the ion density \( n_I \).

Putting all these definitions together gives

\[
\mathbf{p}_{nl} = mn_I\mu_{nl}\langle \sigma_{nl}w_{nl} \rangle(\mathbf{v} - \mathbf{v}_I)
\]

In accordance with Newton’s third law, this is symmetric with respect to the interchange \( n \leftrightarrow I \), except for a change in sign, \( \mathbf{p}_{nl} = -\mathbf{p}_{ln} \). All of these considerations hold, of course, for electron-neutral scattering as well. Explicitly, we have

\[
\mathbf{p}_{ne} = mn_e\mu_{ne}\langle \sigma_{ne}w_{ne} \rangle(\mathbf{v} - \mathbf{v}_e) \simeq mn_e\langle \sigma_{ne}w_{ne} \rangle(\mathbf{v} - \mathbf{v}_e).
\]
The gas is assumed to be locally neutral, so that \( n_e = Z n_i \) where \( Z \) is the number of ionizations per ion particle. In a weakly ionized gas, \( Z = 1 \). The reduced mass \( \mu_{ne} \) is very nearly equal to the electron mass \( m_e \). The collision rates are given by (see Draine, Roberge, & Dalgarno 1983 ApJ 264, 485 for yet more details) (note, cgs units!):

\[
\langle \sigma_{nl} w_{nl} \rangle = 1.9 \times 10^{-9} \text{ cm}^3 \text{ s}^{-1}
\]

\[
\langle \sigma_{ne} w_{ne} \rangle = 10^{-15} \left( \frac{128 kT}{9 \pi m_e} \right)^{1/2} = 8.3 \times 10^{-10} T^{1/2} \text{ cm}^4 \text{ s}^{-1}
\]

The electron-neutral collision rate is just the ion geometric cross section times an electron thermal velocity. (The peculiar factor of \((128/9 \pi)^{1/2}\) is a detail of the averaging procedure.) But the ion-neutral collision rate is temperature independent, much more beholden to long range induced dipole interactions, and significantly enhanced relative to a geometrical cross section assumption. Even if the ion-neutral rate were determined only by a geometrical cross section, \( |p_{ne}| \) would exceed \( |p_{nl}| \) by a factor of order \((m_e/\mu_{nl})^{1/2}\). In fact, the dipole enhancement of the ion-neutral cross section makes this factor larger still.\(^2\)

In the astrophysical literature, it is common to write the ion-neutral momentum coupling in the form

\[
p_{In} = \rho I \gamma (v_I - \mathbf{v}),
\]

where \( \gamma \) is the so-called drag coefficient,

\[
\gamma \equiv \frac{\langle \sigma_{nl} w_{nl} \rangle}{m_I + m_n}
\]

and we will use this notation from here on. Numerically, \( \gamma = 3 \times 10^{13} \text{ cm}^3 \text{ s}^{-1} \text{ g}^{-1} \) for astrophysical mixtures (Draine, Roberge, & Dalgarno 1983).

We come next to the ions and electrons. The dynamical equations for the ions and electrons are

\[
\rho_I \frac{\partial v_I}{\partial t} + \rho_I v_I \cdot \nabla v_I = -\nabla P_I - \rho_I \nabla \Phi + Z n_I (E + v_I \times B) - p_{In}
\]

and

\[
\rho_e \frac{\partial v_e}{\partial t} + \rho_e v_e \cdot \nabla v_e = -\nabla P_e - \rho_e \nabla \Phi - e n_e (E + v_e \times B) - p_{en}.
\]

\(^2\)I should be a little bit more careful. The statement that \( |p_{ne}| \) is larger than \( |p_{nl}| \) by a factor of \((m_e/\mu_{nl})^{1/2}\) assumes that the velocity differences \( \mathbf{v} - \mathbf{v}_e \) and \( \mathbf{v} - \mathbf{v}_I \) do not introduce any mass dependencies, which is generally true.
respectively. \( e \) will always denote the positive charge of a proton, the absolute value of the electron charge, \( 1.602 \times 10^{-19} \) Coulombs or \( 4.803 \times 10^{-10} \) esu.\(^3\)

For a weakly ionized gas, the Lorentz force and collisional terms dominate in each of the latter two equations. Comparison of the magnetic and inertial forces, for example, shows that the latter are smaller than the former by the ratio of the proton or electron gyroperiod to a macroscopic flow crossing time. Thus, to an excellent degree of approximation,

\[
Zen_I (E + v_I \times B) - p_{In} = 0, \tag{107}
\]

and

\[
- en_e (E + v_e \times B) - p_{en} = 0. \tag{108}
\]

The sum of these two equations gives

\[
J \times B = p_{In} + p_{en} \tag{109}
\]

where charge neutrality \( n_e = Zn_I \) has been used, and we have introduced the current density

\[
J \equiv en_e (v_I - v_e). \tag{110}
\]

The equation for the neutrals becomes

\[
\begin{aligned}
\rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v &= - \nabla P - \rho \nabla \Phi + J \times B \\
\end{aligned}
\tag{111}
\]

Due to collisional coupling, the neutrals are subject to the magnetic Lorentz force just as though they were a gas of charged particles. It is not the magnetic force per se that changes in a neutral gas. As we shall presently see, it is the inductive properties of the gas.

Let us return to the force balance equations for the electrons:

\[
- en_e (E + v_e \times B) - p_{en} = 0. \tag{112}
\]

After division by \( -en_e \), this may be expanded to

\[
E + [v + (v_e - v_I) + (v_I - v)] \times B + \frac{m_e \nu_{en}}{e} ([v_e - v_I] + [v_I - v]) = 0,
\tag{113}
\]

where we have introduced the collision frequency of an electron in a population of neutrals:

\[
\nu_{en} = n \langle \sigma_{ne} w_{ne} \rangle. \tag{114}
\]

We have written the electron velocity \( v_e \) in terms of the dominant neutral velocity \( v \) and the key physical velocity differences of our problem. It has

\(^3\)Beware: esu units are still commonly used in the astrophysical literature! You should become comfortable with them.
already been noted that in equation (109), \( p_{en} \) is small compared with \( p_{In} \), provided that the velocity difference \( |v_e - v| \) is not much larger than \( |v_I - v| \). As we argued earlier, the \( p_{en} \) term in equation (109) is small relative to \( p_{In} \):

\[
J \times B \simeq p_{In} = n n_I \mu n I (v_I - v) \nu_{II}.
\]  

(115)

It then follows that the final term in equation (113)

\[
\frac{m_e \nu_{en}}{e} (v_I - v),
\]

which is proportional to \( J \times B \), becomes small compared with the third term

\( (v_e - v_I) \times B \),

which also proportional to \( J \times B \), by a factor of order \( (m_e/\mu_{In})^{1/2} \). These simplifications allow us to write the electron force balance equation as

\[
E + \nu \times B - \frac{J \times B}{en_e} - \frac{J}{\sigma_{cond}} + \frac{(J \times B) \times B}{\gamma \nu p I} = 0,
\]

(116)

where the electrical conductivity has been defined as

\[
\sigma_{cond} \equiv \frac{e^2 n_e}{m_e \nu_{en}}
\]

(117)

The associated resistivity \( \eta \) is

\[
\eta = \frac{1}{\mu_0 \sigma_{cond}},
\]

(118)


\[
\eta = 0.0234 \left( \frac{n}{n_e} \right) T^{1/2} \text{ m}^2 \text{ s}^{-1}
\]

(119)

Equation (116) is a general form of Ohm’s law for a moving, multiple fluid system.

Next, we make use of two of Maxwell’s equations. The first is Faraday’s induction law:

\[
\nabla \times E = -\frac{\partial B}{\partial t}.
\]

(120)
We substitute $E$ from equation (116) to obtain an equation for the self-
induction of the magnetized fluid,
\[
\frac{\partial B}{\partial t} = \nabla \times \left[ v \times B - \frac{J \times B}{en_e} + \frac{(J \times B) \times B}{\gamma \rho \rho_I} - \frac{J}{\sigma_{\text{cond}}} \right]
\] (121)

It remains to relate the current density $J$ to the magnetic field $B$. This
is accomplished by the second Maxwell equation,
\[
\mu_0 J = \nabla \times B + \frac{\partial E}{\partial t}
\] (122)

The final term in the above is the displacement current, and it may be
ignored. Indeed, since we have not, and will not, use the “Gauss’s Law”
equation
\[
\nabla \cdot E = (\varepsilon / \varepsilon_0)(Zn_I - n_e),
\] (123)
we must not include the displacement current. In Appendix B, we show
that departures from charge neutrality in $\nabla \cdot E$ and the displacement cur-
rent are both small terms that contribute at the same order: $v^2/c^2$. These
must both be self-consistently neglected in nonrelativisite MHD. (The final
Maxwell equation $\nabla \cdot B = 0$ adds nothing new. It is automatically satisfied
by equation (120), as long as the initial magnetic field satisfies this divergence
free condition.) These considerations imply
\[
J = \frac{1}{\mu_0} \nabla \times B
\] (124)

for use in equation (121).

To summarize, the fundamental equations of a weakly ionized fluid are
mass conservation of the dominant neutrals (eq.[85])
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0,
\] (125)
the equation of motion (eq. [111] with [124])
\[
\rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v = -\nabla P - \rho \nabla \Phi + \frac{1}{\mu_0} (\nabla \times B) \times B,
\] (126)
and the induction equation (eq. [121] with [118] and [124])
\[
\frac{\partial B}{\partial t} = \nabla \times \left[ v \times B - \frac{(\nabla \times B) \times B}{\mu_0 en_e} + \frac{[(\nabla \times B) \times B] \times B}{\mu_0 \gamma \rho \rho_I} - \frac{\nabla \times B}{\mu_0 \sigma_{\text{cond}}} \right]
\] (127)
It is only natural that the reader should be a little taken aback by the sight of equation (127). Be assured that it is rarely, if ever, needed in full generality: almost always one or more terms on the right side of the equation may be discarded. When only the induction term $\mathbf{v} \times \mathbf{B}$ is important, we refer to this regime as ideal MHD. The three remaining terms on the right are the nonideal MHD terms.

To get a better feel for the relative importance of the nonideal MHD terms in equation (121), we denote the terms on the right side of the equation, moving left to right, as $I$ (induction), $H$ (Hall), $A$ (ambipolar diffusion), and $O$ (Ohmic resistivity). We will always be in a regime in which the presence of the induction term is not in question. More interesting is the relative importance of the nonideal terms. The explicit dependence of $A/H$ and $O/H$ in terms of the fluid properties of a cosmic gas has been worked out by Balbus & Terquem (2001):

$$\frac{A}{H} = Z \left( \frac{9 \times 10^{12} \text{ cm}^{-3}}{n} \right)^{1/2} \left( \frac{T}{10^3 \text{ K}} \right)^{1/2} \left( \frac{v_A}{c_S} \right)$$  \hspace{1cm} (128)

and

$$\frac{O}{H} = \left( \frac{n}{8 \times 10^{17} \text{ cm}^{-3}} \right)^{1/2} \left( \frac{c_S}{v_A} \right)$$  \hspace{1cm} (129)

Here $n$ is the total number density of all particles, $T$ is the kinetic temperature, $v_A$ is the so-called Alfvén velocity (much more about this quantity will come later!),

$$v_A = \frac{B}{\sqrt{\mu_0 \rho}}$$  \hspace{1cm} (130)

and $c_S$ is the isothermal speed of sound,

$$c_S^2 = 0.429 \frac{kT}{m_p}$$  \hspace{1cm} (131)

where $k$ is the Boltzmann constant and $m_p$ the mass of the proton. The coefficient 0.429 corresponds to a mean mass per particle of $2.33 m_p$, appropriate to a molecular gas with a 10% helium admixture.

As reassurance that the fully general nonideal MHD induction equation is not needed for our purposes, note that equations (128) and (129) imply that for all three nonideal MHD terms to be comparable, $T \approx 10^8$ K! Obviously this is not a weakly ionized regime. In figure (2), we plot the domains of relative dominance of the nonideal MHD terms in the $nT$ plane.

Our emphasis of the relative ordering of the nonideal terms in the induction equation should not obscure the fact that ideal MHD is often an
Figure 3: Parameter space for nonideal MHD. The curves correspond to the case $v_A/c_S = 0.1$. (From Kunz & Balbus 2004, MNRAS, 348, 355.)
excellent approximation, even when the ionization fraction is $\ll 1$. For example, the ratio of the ideal inductive term to the ohmic loss term is given by the Lundquist number

$$\ell = \frac{v_A H}{\eta}$$  \hspace{1cm} (132)

where $H$ is a characteristic gradient length scale. To orient ourselves, let us consider the case of a protostellar disc and set $H = 0.1 R$, where $R$ is the radial location in the disc. (This would correspond to $H$ being about the vertical thickness of the disc.) Then $\ell$ is given by

$$\ell \simeq 2.5 (n_e/n)(v_A/c_S) R_{cm},$$

$R_{cm}$ being the radius in centimeters. In other words, the critical ionization fraction at which $\ell = 1$ is about

$$(n_e/n)_{crit} = 0.4 (c_S/v_A) R_{cm}^{-1} \sim 10^{-13} (c_S/10 v_A)$$

at $R = 1$ AU. The actual ionization fraction at this location may be above or below this during the course of the solar systems evolution, but the point worth noting here is that $R_{cm}$ is a large number for a protostellar disc! Ionization fractions far, far below unity can render an astrophysical gas a near perfect electrical conductor. It therefore makes a great deal of sense to begin by examining the behaviour of an ideal MHD fluid.

Exercise. Show that the Lorentz force may be written

$$\mathbf{J} \times \mathbf{B} = \partial_i \left( \frac{B_i B_j}{\mu} - \delta_{ij} \frac{B^2}{2 \mu} \right) \equiv \partial_i T^L_{ij}. \hspace{1cm} (133)$$

Exercise. Show that the Newtonian self-gravity force may be written

$$-\rho \nabla \Phi = \partial_i \left( - \frac{g_i g_j}{4G\pi} + \delta_{ij} \frac{g^2}{8G\pi} \right) \equiv \partial_i T^N_{ij}. \hspace{1cm} (134)$$

where $g_i = -\partial_i \Phi$. (Hint: $\partial_i \partial_j \Phi = 4\pi G \rho$.)

Exercise. Show that the inertial terms in the equation of motion be written

$$\rho \partial_t v_i + \rho v_j \partial_j v_i + \partial_t P = \partial_t (\rho v_i) + \partial_i (\rho v_i v_j + \delta_{ij} P) \equiv \partial_t (\rho v_i) + \partial_i T^R_{ij}, \hspace{1cm} (135)$$

which defines the Reynolds stress $T^R_{ij}$.

Exercise. Show that the equation of motion may be written

$$\rho \partial_t v_i + \partial_i T_{ij} = 0, \hspace{1cm} (136)$$

where $T_{ij} = T^L_{ij} + T^N_{ij} + T^R_{ij}$ is the energy-momentum stress tensor. This form of the equation of motion is most readily generalized when relativity becomes important.
... I deduced that the forces which keep the Planets in their Orbs must [be] reciprocally as the squares of their distances from centres about which they revolve: & thereby compared the force requisite to keep the Moon in her Orb with the force of gravity at the surface of the earth, & found them answer pretty nearly.

— Sir Isaac Newton

We can lick gravity, but sometimes the paperwork is overwhelming.

— Dr. Werner von Braun

3 Gravity

3.1 Legendre Expansion

The gravitational field a distance \( r \) from a point mass \( M \) located at the origin is

\[
g = -\frac{GM}{r^3},
\]

(137)

or more generally,

\[
g = -\frac{GM(r - r')}{|r - r'|^3}
\]

(138)

if the point mass is located at \( r' \). This field is derivable from a potential function \( \Phi \),

\[
\Phi = -\frac{GM}{|r - r'|}, \quad g = -\nabla \Phi
\]

(139)

Gravity is a linear theory, and the fields (and thus the potentials) from an extended mass distribution superpose. Therefore, in general,

\[
\Phi = -GM \int \frac{\rho(r')d^3r'}{|r - r'|},
\]

(140)
with

$$|\mathbf{r} - \mathbf{r}'| = (r^2 - 2rr'\cos \theta + r'^2)^{1/2}$$  \hspace{1cm} (141)

where $\theta$ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. For $r \gg r'$,

$$(r^2 - 2rr'\cos \theta + r'^2)^{-1/2} = r^{-1}\left(1 - \frac{2r'\cos \theta + r'^2}{r^2}\right)^{-1/2}$$  \hspace{1cm} (142)

We are very often interested in the potential at great distances from the source, $r \gg r'$. The last two terms in $r'/r$ are small, so we define

$$\delta = -\frac{2r'}{r} \cos \theta + \frac{r'^2}{r^2} \ll 1.$$  \hspace{1cm} (143)

Then,

$$\left(1 - \frac{2r'}{r} \cos \theta + \frac{r'^2}{r^2}\right)^{-1/2} = r^{-1}(1 + \delta)^{-1/2} = r^{-1}\left[1 - \frac{\delta}{2} + \frac{3\delta^2}{8} + \ldots\right]$$  \hspace{1cm} (144)

Expanding $\delta$ and retaining terms through order $(r'/r)^2$, we find

$$r^{-1}\left(1 - \frac{2r'}{r} \cos \theta + \frac{r'^2}{r^2}\right)^{-1/2} = \frac{1}{r} \left[1 + \left(\frac{r'}{r}\right) \cos \theta + \left(\frac{r'}{r}\right)^2 \frac{1}{2}(3\cos^2 \theta - 1) + \ldots\right].$$  \hspace{1cm} (145)

The expansion consists of powers of $(r'/r)$ multiplied by a polynomial in $\cos \theta$. These latter are denoted $P_l(\cos \theta)$ and are known as Legendre polynomials. Their properties are discussed very clearly in Jackson’s text, *Classical Electrodynamics*. The most important of these for our purposes is that the $P_l$ are orthogonal when integrated over spherical solid angles:

$$\int P_l(\cos \theta) P_{l'}(\cos \theta) d\Omega = \frac{4\pi}{2l + 1}\delta_{ll'}$$  \hspace{1cm} (146)

Because of the symmetry between $\mathbf{r}$ and $\mathbf{r}'$, in general we must have

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r>}\right)^l P_l(\cos \theta),$$  \hspace{1cm} (147)

where $r>$ ($r<$) is the greater (lesser) of $r$ and $r'$. 
3.2 Gauss’s Law

A remarkable property of a $1/r^2$ force law is that

$$\int g \cdot dA = -4\pi GM$$

(148)

where the surface integral is over any volume containing a total mass $M$. To prove this, we show that it is true for a single point mass. Then by superposition, it is true for any distribution.

Note that $|g \cdot dA|$ is just the product of $g$ with the area $dA*$, the projection of $dA$ parallel to $-g$. For a point mass, $g$ is radial, and the projected area of $dA*$ is precisely that of a small portion of a sphere centered on the mass point, at the same radius $r$ as $dA$. Since this small spherical area is $r^2d\Omega$, where $d\Omega$ is the solid angle subtended by the area at the point mass, and $g = -GM/r^2$,

$$g \cdot dA = -GM d\Omega$$

(149)

This is independent of $r$, and the integral of the whole area just adds up the total solid angle subtended by the volume enclosing the mass: $4\pi$. This proves Gauss’s law for a point mass. By linear superposition, it is therefore true for any mass distribution interior to the surface.
3.3 Poisson Equation

We have shown
\[ \int \mathbf{g} \cdot d\mathbf{A} = -4\pi G \int \rho \, dV \]  
(150)
for any mass distribution inside the volume \( V \). Using the divergence theorem,
\[ \int \nabla \cdot \mathbf{g} \, dV = -4\pi G \int \rho \, dV \]  
(151)
and since \( \mathbf{g} = -\nabla \Phi \),
\[ \int \nabla^2 \Phi \, dV = 4\pi G \int \rho \, dV \]  
(152)
The volume is arbitrary, hence we conclude
\[ \nabla^2 \Phi = 4\pi G \rho, \]  
(153)
which is known as the Poisson equation. It allows us to compute the gravitational potential, and hence the gravitational forces, from a given distribution of mass. It must added to the fluid equations of motion to calculate the evolution of a self-gravitating system.

Exercise. Show that the Lorentz force may be written
\[ \mathbf{J} \times \mathbf{B} = \partial_i \left( \frac{B_i B_j}{\mu} - \delta_{ij} \frac{B^2}{2\mu} \right) \]  
(154)
and that for a self-gravitating gas the Newtonian gravitational force has a very similar form:
\[ -\rho \nabla \Phi = -\partial_i \left( \frac{g_i g_j}{4\pi G} - \delta_{ij} \frac{g^2}{8\pi G} \right) \]  
(155)
where \( g_i = -\nabla_i \Phi \), \( g^2 = g_i g_i \).

The quantities inside the \( \partial_i \) operators are known respectively as the Maxwell and gravitational stress tensors. They play a key role in momentum and energy transport in magnetic and self-gravitating systems.

3.4 Gravitational Tidal Forces.

As an illustration of how the expansion of the potential function can be used, let us calculate the height of the tides that are raised on the earth by the
We define the $z$ axis to be along the line joining the centers of the earth and the moon. The distance between the centers will be $r$, and a point on the earth’s surface will be at a vector location $r + s$ relative to the center of the moon. Let $s = (x, y, z)$ in Cartesian coordinates with origin at the center of the earth. Note that

$$\frac{1}{|r + s|} = (r^2 + s^2 + 2rs \cos \theta)^{-1/2}$$

so we need to keep track of the sign, which is different from our $r, r'$ expansion. We regard $r$ as fixed, and calculate forces by taking the gradient with respect to $x, y, and z$. We have, with $r \gg s$,

$$\frac{GM_m}{|r + s|} = -\frac{GM_m}{r} \left[ 1 - \frac{s \cos \theta}{r} + \left(\frac{s}{r}\right)^2 P_2(\cos \theta) + \ldots \right]$$

where $M_m$ is the mass of the moon. Differentiating with respect to $z = s \cos \theta$ gives, to first approximation

$$-\frac{\partial \Phi}{\partial z} = -\frac{GM_m}{r^2}$$

which looks familiar: it is the Newtonian force acting between the centers of the two bodies, directing along the line joining them. It is not the tidal force, which comes in at the next level of approximation. The tidal potential is:

$$\Phi \text{ (tidal)} = -\frac{GM_m s^2}{r^3} P_2(\cos \theta)$$

And the tidal force is, after carrying out the gradient operation,

$$\mathbf{g} \text{ (tidal)} = -\nabla \Phi = \frac{GM_m}{r^3}(-x, -y, 2z)$$
Tidal forces try to squeeze matter along the directions perpendicular to the line joining the bodies, and try to stretch matter along the direction parallel to this line. Note that we speak here only of the forces; the resulting displacements can be much more complex. Not only are they sensitive to local surface features in the oceans, there are also time delays in the response of the displacement, due to the presence of dissipation.

Let us assume, however, that the new shape of the earth has adjusted so that the surface now follows an equipotential of the earth’s gravitational field plus that of the moon. Let $\Phi_1$ be the potential function of the earth’s unperturbed spherical field. Let $\Phi_2$ be the new potential function in the presence of the moon’s potential, differing slightly from $\Phi_1$ at a given location. The tidal force causes a displacement of the original spherical equipotential surfaces by an amount $\xi$, and this is what we wish to calculate. Let $\Phi_1(s) = \Phi$ be constant on a sphere of radius $s$. The new equipotential surface with this (constant) value of $\Phi$ is $\Phi_2(r + \xi)$, where $\xi$ is the small displacement caused by the moon. It is this quantity that we wish to calculate. If an equipotential surface of $\Phi_2$ has the same value as an equipotential surface of $\Phi_1$, but only after the surface has been displaced by $\xi$, then

$$\Phi = \Phi_2(s + \xi) = \Phi_2(s) + \xi \cdot \nabla \Phi_2 = \Phi_1(s)$$

where $s$ is the radius of the earth. But at the same location $s$:

$$\Phi_2(s) - \Phi_1(s) = \Phi \text{ (tidal)},$$

and to leading order we may replace $\Phi_2$ with $\Phi_1$ in the term proportional to $\xi$. Then

$$-\xi \cdot \nabla \Phi_1 = \Phi \text{ (tidal)}$$

which states the physically very sensible result that the work done against the gravitational force of the earth in distorting the surface is provided by the additional tidal potential energy. Writing the potential functions explicitly:

$$\xi_s \frac{GM_e}{s^2} = \frac{GM_m s^2}{r^3} P_2(\cos \theta)$$

or

$$\xi_s = s \frac{M_m}{M_e} \left( \frac{s}{r} \right)^3 P_2(\cos \theta)$$

This works out to be

$$\xi_s = 0.32 P_2(\cos \theta) \text{ meters}$$

for the earth-moon system. Notice how extremely sensitive the height of the tidal displacement is to the separation distance $r$. When the moon was a factor of 2 closer to the earth, as it is believed to have been on a timescale of $10^9$ years ago, the tidal forces were almost an order of magnitude larger.
### 3.5 The Virial Theorem

The Virial Theorem is one of the most useful theorems in astrophysical gasdynamics. Basically, it is an integral form of the equation of motion in full generality. When the dominant balance is between two forces, the theorem states that the associated energies must be comparable in strength. We shall use Cartesian index notation in our proof.

Begin with

\[
\frac{Dv_i}{Dt} = -\partial_i P - \partial_i \left( \frac{B^2}{2\mu} \right) - \rho \partial_i \Phi + \frac{B_i}{\mu} \partial_j B_j
\]

where

\[
\Phi(r) = -G \int \frac{\rho(r') d^3r'}{|r-r'|}
\]

is the gravitational potential the system. Note that

\[
-\partial_i \Phi = -G \int \frac{\rho(r') (r_i - r_i') d^3r'}{|r-r'|^3}
\]

Multiply the equation of motion by \(r_i\) and sum over \(i\),

\[
\rho r_i \frac{Dv_i}{Dt} = -r_i \partial_i P - r_i \partial_i \left( \frac{B^2}{2\mu} \right) - \rho r_i \partial_i \Phi + r_i \frac{B_i}{\mu} \partial_j B_j
\]

and then integrate over a fixed volume \(V\). For the pressure integral,

\[
- \int r_i \partial_i P \, dV = - \int \partial_i (r_i P) \, dV + 3 \int P \, dV
\]

\[
= - \int P r \cdot dA + 3 \int P \, dV
\]

\[
= - \int P r \cdot dA + 2 \int U_{\text{therm}} \, dV
\]

where \(U_{\text{therm}} = (3/2)P\) is the thermal energy density.

The integral involving the potential is

\[
\int \rho r_i \frac{\partial \Phi}{\partial r_i} \, d^3r = G \int \frac{\rho(r) \rho(r')}{|r-r'|^3} \, d^3r' \, d^3r
\]

If we switch the labels \(r\) and \(r'\), we obtain

\[
\int \rho r_i \frac{\partial \Phi}{\partial r_i} \, d^3r = G \int \frac{\rho(r) \rho(r') (r_i - r_i')}{|r-r'|^3} \, d^3r' \, d^3r
\]
Adding and dividing by 2:
\[ \int \rho \frac{\partial \Phi}{\partial r_i} d^3r = \frac{G}{2} \int \frac{\rho(r)\rho(r')}{|r-r'|} d^3r d^3r' \equiv -V. \] (175)
i.e., this is just minus the gravitational potential energy \( V \). (The factor of \( 1/2 \) is present because each pair of interacting fluid elements occurs twice in the integration, but should only be counted once.)

On to the magnetic integrals:
\[ \int r_i \partial_i \left( \frac{B^2}{2\mu} \right) d^3r = \int \frac{(B^2/2\mu)\cdot dA}{r} - 3 \int \frac{(B^2/2\mu) d^3r}, \] (176)
where we have integrated by parts and used the divergence theorem. And
\[ \int \frac{r_i \partial_i (B_i B_j)}{\mu} d^3r = \int \left( r \cdot \frac{B}{\mu} \cdot dA \right) - \int \frac{\delta_{ij}}{\mu} B_i B_j d^3r \]
\[ = \int \left( r \cdot \frac{B}{\mu} \cdot dA \right) - \int \frac{B^2}{\mu} d^3r. \] (177)
The sum of all the terms on the right side of our equation is then
\[ 2E_{\text{therm}} + V + M - \int \left( P + \frac{B^2}{2\mu} \right) \cdot dA + \frac{1}{\mu} \int (r \cdot B) B \cdot dA \] (178)
where
\[ E_{\text{therm}} = \int U_{\text{therm}} d^3r, \quad M = \int \frac{B^2}{2\mu} d^3r \] (179)
are the total thermal and magnetic energies.

For the left side of the virial equation we start with the following identity:
\[ \int \rho \frac{DQ}{Dt} d^3r = \int \rho \left( \frac{\partial Q}{\partial t} + \mathbf{v} \cdot \nabla Q \right) d^3r = \int \frac{\partial (\rho Q)}{\partial t} d^3r + \int \nabla \cdot (\rho \mathbf{v} Q) d^3r \] (180)
where the second equality follows from mass conservation. The last integral can be converted to a surface integral of the flux \( \rho \mathbf{v} Q \) over a bounding area. If we choose the surface so that the velocity vanishes at this surface, then this integral vanishes, and we shall make this assumption. We then have:
\[ \int \rho \frac{DQ}{Dt} d^3r = \int \frac{\partial (\rho Q)}{\partial t} d^3r = \frac{d}{dt} \int \rho Q d^3r \] (181)
With this in hand, we perform the following manipulations:

\[
\int \rho r_i \frac{Dv_i}{Dt} d^3r = \int \rho \frac{D(r_i v_i)}{Dt} d^3r - \int \rho v_i \frac{Dr_i}{Dt} d^3r = \frac{d}{dt} \int \rho r_i v_i d^3r - \int \rho v^2 d^3r = \frac{d}{dt} \int \rho r_i \frac{Dv_i}{Dt} d^3r - 2KE = \frac{d^2}{dt^2} \int \rho r_i v_i d^3r - 2KE\\
\]

where \( I \) is \( \int \rho r^2 d^3r \) and \( KE \) denotes the total kinetic energy of the fluid. The Virial theorem is then:

\[
\frac{1}{2} \ddot{I} = 2KE + 2E_{\text{therm}} + V + M - \int \left( P + \frac{B^2}{2\mu} \right) \mathbf{r} \cdot d\mathbf{A} + \frac{1}{\mu} \int (\mathbf{r} \cdot \mathbf{B}) \mathbf{B} \cdot d\mathbf{A} \tag{184}\]

where the velocity is assumed to vanish over the bounding surface. The Virial theorem shows that when a dominant steady-state balance is present between two effects—pressure and gravity, say—the two associated energies are comparable. In particular, for a star in hydrostatic equilibrium,

\[
E_{\text{therm}} = -\frac{1}{2}V, \quad E_{\text{total}} = E_{\text{therm}} + V = \frac{V}{2} \tag{185}\]

since the pressure vanishes at the surface, and magnetic fields are generally negligible for stellar hydrostatic equilibrium.

### 3.6 The Lane-Emden Equation

Consider the hydrostatic equilibrium of a star with a very simple equation of state: \( P = K \rho^\gamma \). Special cases of interest include \( \gamma = 1 \) (isothermal spheres), \( \gamma = 6/5, 2 \) (elementary analytic solutions), \( \gamma = 5/3, 4/3 \) (nonrelativistic, relativistic white dwarfs). Our fundamental equation

\[
-\nabla \Phi = \frac{1}{\rho} \nabla P \tag{186}\]
becomes
\[-\nabla^2 \Phi = -4\pi G \rho = \nabla \cdot \left( \frac{1}{\rho} \nabla \left( \frac{r}{K \rho^7} \right) \right) = \nabla \cdot \left( \frac{1}{\rho} \nabla K \rho^7 \right) \] (187)

Assuming spherical geometry and carrying through the differentiation,
\[
\frac{K}{r} \frac{1}{\gamma - 1} \frac{d}{d \ln \rho} \left( \frac{d \rho^{\gamma - 1}}{d \rho} \right) = -4\pi G \rho 
\] (188)

For the case of an isothermal gas,
\[
K \frac{1}{r^2} \frac{d}{d r} r^2 \left( \frac{d \ln \rho}{d r} \right) = -4\pi G \rho
\] (189)

Next, we introduce the polytropic index,
\[
n = \frac{1}{\gamma - 1}
\] (190)

and let the density \( \rho \) be written
\[
\rho = \rho_c \theta^n
\] (191)

where \( \rho_c \) is the central density. If we further rescale the length \( r \) as \( r = a \xi \), with \( a \) a length to be determined, our equation becomes
\[
\frac{1}{\xi^2} \frac{d}{d \xi} \xi^2 \frac{d \theta}{d \xi} = -\left[ \frac{4\pi G \rho_c^{1/(\gamma - 1)} (\gamma - 1)a^2}{K \gamma} \right] \theta^n
\] (192)

Obviously, we should choose \( a \) so that the constant in square brackets is unity, or
\[
a^2 = \frac{K(n + 1) \rho_c^{1/(\gamma - 1) - 1}}{4\pi G}
\] (193)

The resulting equation:
\[
\frac{1}{\xi^2} \frac{d}{d \xi} \xi^2 \frac{d \theta}{d \xi} = -\theta^n
\] (194)

is known as the Lane-Emden equation.
3.7 Solution properties of the Lane-Emden Equation

To solve the Lane-Emden equation, we need boundary conditions at the origin $\xi = 0$. Let us return to the original first order equation of hydrostatic equilibrium. Since we are in spherical symmetry, this may be written

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r}$$

(195)

where $M(r)$ is the mass interior to radius $r$. We have made use of Gauss’s theorem, which assures us that the mass exterior to $r$ contributes nothing to the force at $r$, and that the mass interior to $r$ acts as though it were concentrated at $r = 0$. When $r$ is very small, $M(r) \approx 4\pi \rho r^3/3$, and thus $dP/dr \to 0$. But since $P \sim \theta^{1+n}$, it follows immediately that the $\xi$ gradient of $\theta$ must also vanish at the origin. Our two boundary conditions on $\theta$ at $\xi = 0$ are

$$\theta(0) = 1, \quad \theta'(0) = 0.$$ 

(196)

It is now a simple matter to show at all solutions to the Lande-Emden equation must have the same asymptotic form as $\xi \to 0$:

$$\theta(\xi) \to 1 - \frac{\xi^2}{6} + ...$$

(197)

It is natural to ask whether there are solutions that extend to infinity, and if so, what their asymptotic form is. This can be done at once by looking for solutions of the $\theta = A\xi^p$, and solving for $A$ and $p$ by demanding self-consistency. We have

$$\frac{1}{\xi^2} \frac{d}{d\xi} \xi^2 d\theta = p(p+1)A\xi^{p-2} = \theta^n = -A^n\xi^{np}$$

(198)

This requires

$$p = \frac{2}{1-n}, \quad A^{n-1} = \frac{2(n-3)}{(n-1)^2}$$

(199)

In fact, this is an exact solution, not just an approximate one. But it is not the exact solution we want, because it does not satisfy the boundary conditions at the origin. Moreover, there is a subtle flaw in our analysis. The case $p = -1, n = 3$ breaks down, because what we thought was the leading order term on the right side of the Lane-Emden equation actually vanishes. We cannot exclude a solution with asymptotic behaviour $\theta \sim 1/\xi$ on the basis of our analysis; we simply have to retain the next order terms!

In fact, notice that a function of the form $(1 + \xi^2/3)^{-1/2}$ satisfies both the boundary conditions at $\xi = 0$, and has the asymptotic form $\theta \sim A/\xi$.
Amazingly enough, this turns out to be an exact solution of the equation, the only known analytic solution of a nonlinear form of the Lane-Emden equation satisfying all the boundary conditions at $\xi = 0$. (The cases $n = 0$ and $n = 1$ reduce to elementary cases, which you should solve explicitly.) The solution
\[ \theta = (1 + \xi^2/3)^{-1/2} \]
corresponds to $n = 5$ (show!). All solutions with $n \geq 5$ extend to infinity, those with $n < 5$ are finite.

**Exercise.** Expand equation (200) through terms of order $1/\xi^3$ to show that it satisfies the Lane-Emden equation only to leading order, with $n = 5$.

### 3.8 Polytrope Masses

The mass interior to radius $r$ is given by
\[ M(r) = -\frac{r^2}{\rho G} \frac{dP}{dr} = -\frac{Kr^2}{G} \left( 1 + \frac{1}{n} \right) \rho^{1/n} \frac{d\ln \rho}{dr} \]
(201)

Translating this to $\theta$ and $\xi$ variables gives:
\[ M(r) = \left[ \frac{K^3(n + 1)^3}{4\pi G^3} \right]^{1/2} \rho_e^{(3-n)/2n} \xi \frac{d\theta}{d\xi} \]
(202)

The total mass of a polytrope can be evaluated once the surface values of $\xi$ and $\theta'(\xi)$ are known: no further integration is needed to determine the mass once $\theta$ is tabulated. Notice something remarkable: for $n = 3$, the mass is a universal constant, independent of whatever is chosen for the central density $\rho_c$!

This is a result of tremendous significance for astrophysics. For white dwarf stars, the Lane-Emden equation is not simply an approximation to help us understand more complex and realistic models. A polytropic equation of state is in fact exact for a gas of degenerate electrons in either the nonrelativistic limit, in which case $P \sim \rho^{5/3}$ and $n = 1.5$, or in the extreme relativistic limit, in which case $P \sim \rho^{4/3}$ and $n = 3$. The pressure from the nuclei in the white dwarf, is nondegenerate, and negligible compared with the degenerate electron pressure.

Imagine constructing white dwarfs by slowly increasing the value of the central density $\rho_c$. At first, when the the gas in nonrelativistic, the larger the value of $\rho_c$, the larger the value of the stellar mass $M$. Note however that the radius of the star, which is proportional to $a$, gets smaller and smaller. At some point the electrons become so confined that they become a
relativistic degenerate gas, and $n \to 3$. The mass of the star can increase no more! This limiting mass is called the Chandrasekhar Mass, and it may be evaluated by calculating the value of $\xi^2 \theta = 2.01824$ at the surface of an $n = 3$ polytrope (notice that the $n = 5$ polytrope has the value 1.732 evaluated at the “surface” $\xi \to \infty$), and by using $K = 4.935 \times 10^9$ in SI units for an extreme relativistic degenerate gas. One finds

$$M_{Ch} = 1.44 M_\odot$$

(203)

It is the existence of this upper mass limit, whose value depends upon the very simple solution of the $n = 3$ Lane-Emden equation, that leads to the production of neutron stars and black holes.


It is a remarkable fact that even though we cannot express the solution to the Lane-Emden equation in a closed analytic form for arbitrary $\gamma$, there is a very simple expression for the gravitational potential energy as a function of mass $M$, radius $R$ and of course $\gamma$. The calculation is very igneous.

The Potential Energy $U$ is given by

$$U = -\int_0^R \frac{G M_r}{r} 4\pi \rho r^3 dr$$

(204)

where $M_r$ is shorthand for $M(r)$, the mass within radius $r$. We then perform a series of substitutions and integrations by parts:

$$U = -\int_0^R \frac{G M_r \rho}{r^2} 4\pi r^3 dr = \int_0^R \frac{dP}{dr} 4\pi r^3 dr = -3 \int_0^R \left( \frac{P}{\rho} \right) 4\pi \rho r^2 dr$$

(205)

Continuing with $dM_r/dr = 4\pi \rho r^2$

$$U = -3 \int_0^R \left( \frac{P}{\rho} \right) \frac{dM_r}{dr} dr = 3 \int_0^R \frac{d}{dr} \left( \frac{P}{\rho} \right) M_r dr$$

(206)

Notice that we have yet to use anything but hydrostatic equilibrium by way of physics! Now, for the first time, we use the fact that we have a polytrope. From $P = K \rho^\gamma$ follows

$$d \left( \frac{P}{\rho} \right) = \left( \frac{\gamma - 1}{\gamma} \right) \frac{dP}{\rho},$$

49
and we then have
\[ U = 3 \left( \frac{\gamma - 1}{\gamma} \right) \int_0^R \frac{M_r dP}{\rho} \frac{1}{dr} = -3 \left( \frac{\gamma - 1}{\gamma} \right) \int_0^R \frac{GM^2}{r^2} dr \]  
(207)
A last integration by parts gives
\[ U = 3 \left( \frac{\gamma - 1}{\gamma} \right) \int_0^R GM^2 \frac{d}{dr} \frac{1}{r} dr = 3 \left( \frac{\gamma - 1}{\gamma} \right) \left( \frac{GM^2}{R} + 2U \right) \]  
(208)
where we have written \( U \) for \(- \int GM_r dM_r/r \) on the right. We now have a simple equation involving only \( U, GM^2/R, \) and \( \gamma \). It’s solution is:
\[ U = -\frac{3(\gamma - 1)GM^2}{5\gamma - 6} \frac{R}{R} \]  
(209)
Notice the recovery of the well-known constant density solution,
\[ U = -\frac{3 GM^2}{5} \frac{R}{R} \]  
(constant density)
in the limit \( \gamma \to \infty \). We can also make sense of why the case \( n = 5 \) in section 3.7 marks the division between finite and and infinite extent of the polytrope: it corresponds to the the gravitational potential energy becoming infinite.

### 3.10 Effects of Rotation

We conclude our discussion of polytropes by examining a few simple rotating systems. In the presence of rotation, the equation for hydrostatic equilibrium becomes:
\[ -\frac{1}{\rho} \nabla P - \nabla \Phi = -R\Omega^2 e_R \]  
(210)
If \( P = P(\rho) \), the vorticity conservation equation implies that \( \Omega = \Omega(R) \). (This is von Zeippel’s theorem, proved in an earlier exercise.) Then
\[ R\Omega^2(R) = \nabla \left[ \int R R' \Omega^2(R') dR' \right] = \nabla \Phi_\Omega \]  
(211)
which defines the velocity potential function \( \Phi_\Omega \). We see that all terms are derivable from a potential function. The equation of hydrostatic equilibrium becomes
\[ H + \Phi - \Phi_\Omega = C = \text{CONSTANT}, \]  
(212)
where \( H \) is the enthalpy function
\[ H \equiv \int \frac{dP}{\rho} = \frac{\gamma \gamma}{\gamma - 1} \frac{P}{\rho} \]  
(213)
3.10.1 Example 1: Rotating Liquid

As a first example, consider the case of a constant density, uniformly rotating liquid in a constant vertical gravitational field \( g \). (Constant density corresponds to \( \gamma \to \infty \).) Then \( \Phi = gz \) and \( \Phi_0 = R^2 \Omega^2 / 2 \). If \( z_0 \) is the surface height of the liquid at the center \( R = 0 \), then \( C = gz_0 \), assuming that the pressure vanishes at the liquid surface. The solution is

\[
\frac{P}{\rho} = g(z_0 - z) + \frac{1}{2} \rho^2 \Omega^2, \tag{214}
\]

and the \( P = 0 \) surface of the liquid is given by

\[
z = z_0 + \frac{R^2 \Omega^2}{2g}. \tag{215}
\]

The primary astronomical application of this equation is that it tells us how to make parabolic mirrors! Liquid glass is spun in huge ovens, and its surface acquires a perfectly parabolic shape. When silvered, the parabolic mirror reflects all incoming parallel rays onto a single point, the focus. In this way, an image is formed. Even a small segment of a parabolic mirror has this property. Very large telescopes can be constructed (and controled) by using many small mirrors.

3.10.2 Example 2: Sub-Keplerian Disks

The Keplerian rotation law of planetary motion is

\[
v_\phi^2 = \frac{GM}{R} \tag{216}
\]

In a gas disc, this law is modified by gas pressure. But if we assume that the rotation law follows the same functional form (linearly proportional to central mass), the modification may be parameterized as

\[
v_\phi^2 = \frac{GM \cos \beta}{R} \tag{217}
\]

where \( \beta \) is a free parameter. Such a profile is said to be sub-Keplerian, with the additional support provided by a pressure gradient. As before, \( v_\phi \) depends only upon cylindrical radius \( R \) for a polytropic gas. With \( \Phi = -GM/r \) and \( \Phi_0 = -GM \cos \beta/R \), the integrated equation of hydrostatic equilibrium is now:

\[
\frac{\gamma}{\gamma - 1} \frac{P}{\rho} - \frac{GM}{r} + \frac{GM \cos \beta}{R} = \mathcal{H}_\infty \tag{218}
\]
where $H_\infty$ is the (constant) enthalpy at $R = \infty$. Recall that $R = r \sin \theta$, where $\theta$ is the usual spherical angle measured from the positive $z$ axis. We define the latitude angle $\lambda = \pi/2 - \theta$. Then, equipotential surfaces (as well as isobaric and isochoric\footnote{Isochoric means constant density.} surfaces) are given by the equation

$$\left(\frac{\cos \beta}{\cos \lambda} - 1\right) = Cr$$

(219)

where the constant $C$, a combination of enthalpy terms, may be either positive or negative. When $C > 0$, then $\lambda > \beta$ everywhere, and the surface is unbounded in spherical radius, extending to the cylinder

$$R_{\text{max}} = r \cos \lambda = (\cos \beta)/C$$

(220)

When $C = 0$, the surface is a cone $\lambda = \beta$. Note that all surfaces approach this cone as $r \to 0$. Since these are surfaces of constant density, this result gives us a geometrical interpretation of what was originally a dynamical parameter: $\beta$ is the opening angle of the disc wedge. Finally, when $C < 0$, then $\lambda < \beta$ everywhere, and the surface is bounded, reaching a maximum radial extent of

$$r_{\text{max}} = |C|^{-1}(1 - \cos \beta)$$

(221)

at $\lambda = 0$. These surfaces, which form the interior surfaces of constant density in the disc, are shaped like drops of water.

Since gas can move freely along equipotential surfaces, the unbounded surfaces suggest the possibility of subsonic winds and outflows from discs, with a small enough velocity to allow the assumption of hydrostatic equilibrium. Whether this occurs in reality depends upon how easily the escaping gas can acquire angular momentum to orbit at its asymptotic cylinder $R = R_{\text{max}}$.

The bounded surfaces correspond to the body of the disc itself. Their shape suggests that slow but significant radial mixing might occur in the disc, if gas can flow along these surfaces and exchange angular momentum as needed. The protostellar disc that formed the solar system offers some evidence that mixing has occurred because dust grains far from the sun seem to have been exposed to the higher temperatures of the inner regions. This possibility of radial mixing in discs is currently being explored by computer simulations.

Our results are summarized in figure 3.

3.11 Self-gravity and Rotation

The simplest problem with self-gravity is that of a uniformly rotating, constant density cylinder. An application of Gauss’s law shows that at a distance
Figure 6: Equipotential disc contours for the case when $\Omega$ is 0.93 of its Keplerian value, corresponding to an opening wedge angle of $\beta = 30^\circ$, marked by the line marked OUTFLOW. (The colour scale on the right gives contour information for $c_S^2 \equiv \gamma P/\rho$ as shown, but it may be ignored if you do not have a colour printer!) The constant $H_\infty$ is the enthalpy at infinity, $\gamma/({\gamma - 1})$ times $P_\infty/\rho_\infty$. Open contours become very closely packed near the $\rho = 0$ boundary, are not drawn.
From the rotation axis, the gravitational field is
\[ g_R = -\frac{2G\mu(R)}{R} \] (222)
where \( \mu(R) \) is the mass per unit length interior to \( R \),
\[ \mu(R) = \int_0^R 2\pi\rho R\,dR = \pi R^2 \rho. \] (223)
This gives
\[ g_R = -2\pi G\rho R \] (224)
and an associated potential of
\[ \Phi = \pi G\rho R^2. \] (225)
The potential integration constant is not important here, since it can be absorbed into the integration constant of the hydrostatic equilibrium equation, which is
\[ \frac{P}{\rho} + \pi G\rho R^2 - \frac{1}{2} R^2 \Omega^2 = \text{CONSTANT} = \frac{P_0}{\rho} \] (226)
where \( P_0 \) is the central density. Our solution is then
\[ \frac{P}{\rho} = \frac{P_0}{\rho} + \frac{1}{2} R^2 \Omega^2 \left( 1 - \frac{2\pi G\rho}{\Omega^2} \right) \] (227)
When self-gravity dominates over rotation, the pressure goes to zero at a finite radius, \( R_{P=0} \):
\[ [R_{P=0}]^2 = \frac{2P_0}{\rho(2\pi G\rho - \Omega^2)} \] (228)
On the other hand, when rotation dominates, we find that the pressure rises sharply away from the center:
\[ \frac{P}{\rho} = \frac{P_0}{\rho} + \frac{1}{2} R^2 \Omega^2 \] (229)
There is interesting application of this simple formula, which is to the central region of a hurricane, where the pressure can be very, very low. (At large rotation velocities the earth’s Coriolis force is negligible; we also assume that the compressibility of air is unimportant.) The characteristic feature of a hurricane is a central low pressure “eye”, surrounded by an “eye wall,” in which the pressure rises sharply. This is reproduced in our quadratically growing solution for the pressure profile. In the exterior region of the hurricane, which is not reproduced in the above formula, significant departures from solid body rotation occur, and the pressure decreases outwards.