Waves Revision
Waves - syllabus

According to the course handbook you should know about the following

Derivation of the one-dimensional wave equation and its application to transverse waves on a stretched string. D’Alembert’s solution. Sinusoidal solutions and their complex representation. Characteristics of wave motion in one dimension: amplitude, phase, frequency, wavelength, wavenumber, phase velocity. Energy in a vibrating string. Travelling waves: energy, power, impedance, reflection and transmission at a boundary. Superposition of two waves of different frequencies: beats and elementary discussion of construction of wave packets; qualitative discussion of dispersive media; group velocity. Method of separation of variables for the one-dimensional wave equation; separation constants. Modes of a string with fixed end points (standing waves): superposition of modes, energy as a sum of mode energies.

Let’s remind ourselves of the essentials, before looking at a few past problems
Waves - syllabus

Focus first on the below topics

Derivation of the one-dimensional wave equation and its application to transverse waves on a stretched string. D’Alembert’s solution. Sinusoidal solutions and their complex representation. Characteristics of wave motion in one dimension: amplitude, phase, frequency, wavelength, wavenumber, phase velocity. Energy in a vibrating string. Travelling waves: energy, power, impedance, reflection and transmission at a boundary. Superposition of two waves of different frequencies: beats and elementary discussion of construction of wave packets; qualitative discussion of dispersive media; group velocity. Method of separation of variables for the one-dimensional wave equation; separation constants. Modes of a string with fixed end points (standing waves): superposition of modes, energy as a sum of mode energies.

and illustrate by looking at Long Vacation 2011, Q9
Waves on a stretched string

Consider a segment of string of linear density $\rho$ stretched under tension $T$

Key point – means that no net horizontal force at first order, only vertical force, and we can make approximations such as $\sin \delta \theta \approx \delta \theta$

For full derivation see HT lecture notes or text book, e.g. French

This is the wave equation, which (anticipating solution) we can write

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$$

with

$$c = \sqrt{\frac{T}{\rho}}$$
d’Alembert solution of wave equation

\[ \frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \quad y \text{ is a function of } x \text{ and } t. \]
Define new variables so that
\[ y \text{ is now a function of } u \text{ and } v \]
\[ u = x - ct \]
\[ v = x + ct \]

With chain rule we can show
\[ \frac{\partial^2 y}{\partial u \partial v} = 0 \quad \Rightarrow \quad y(u, v) = f(u) + g(v) \]

So general solution of wave equation is

\[ y(x, t) = f(x - ct) + g(x + ct) \]

Here \( f \) and \( g \) are any functions of \((x - ct)\) and \((x + ct)\), determined by initial conditions. A common scenario is that they are sinusoidal.

Note that \( f(x - ct) \) describes forward-going wave and \( g(x + ct) \) describes a backwards-going one!

with \( c \) the (phase) velocity of the wave
Sinusoidal waves

A very common functional dependence for \( f \) and \( g \)...

\[ y(x, t) = f(x - ct) + g(x + ct) \]

...is sinusoidal. In this case it is usual to write:

\[ y(x, t) = A \cos(kx - \omega t) + B \cos(kx + \omega t) \]

or \( A \sin(kx-\omega t) \) ... etc (choice doesn’t matter, unless we are comparing one wave with another and then relative phases become important)

- **speed of wave** \( c = \omega / k \)
- **frequency** \( f = 1 / T = \omega / 2\pi \)
  where \( \omega \) is **angular frequency**
- **wavelength** \( \lambda = 2\pi / k \)
  where \( k \) is the **wave-number**
  (or **wave-vector** if also used to indicate direction of wave)
Notation choices

Sinusoidal solution \( y(x, t) = A \cos(kx - \omega t) \) (writing here, for compactness, only the forward-going solution)

Using the relationships between \( k, \omega, \lambda \) & \( c \) this can be expressed in many forms

\[ y(x, t) = A \cos[k(x - ct)] \]

Also note that sometimes it is convenient to write \( y(x, t) = A \cos(\omega t - kx) \)

A very frequent approach is to use complex notation (we already made use of this when analysing normal modes, and you will have seen it in circuit analysis)

\[ y(x, t) = \text{Re}[A \exp[i(kx - \omega t)]] \]

or \( y(x, t) = \text{Im}[A \exp[i(kx - \omega t)]] \) if it’s important to pick out sine function. Note that often the ‘Re’ or ‘Im’ is implicit, and it gets omitted in discussion.
Wave equation revisited – solving by separation of variables

We have already solved the wave equation using the d’Alembert approach

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}
\]

Can also be solved by looking for solutions which have the ‘separated’ form

\[y(x, t) = X(x)T(t)\]

i.e. that factorise into functions that are separate functions of \(x\) and \(t\). This is just the situation that applies to standing waves!

\[
\frac{\ddot{X}}{X} = \frac{1}{c^2} \frac{\ddot{T}}{T}
\]

set each side equal to some separation constant \(-k^2\)

This yields \(X(x) = A \cos kx + B \sin kx\) and \(T(t) = D \cos ckt + E \sin ckt\) with \(A, B, D\) and \(E\) constants defined by initial conditions
Energy and impedance for travelling wave on string

Energy stored in a mechanical wave

Integrate kinetic energy and potential energy densities over an integer number of wavelengths to show they contribute *equally* and give

Total energy in $n$ wavelengths $= \frac{1}{2} \rho A^2 \omega^2 n\lambda$

**KE density** $\frac{dK}{dx} = \frac{1}{2} \rho \left( \frac{\partial y}{\partial t} \right)^2$

**PE density** $\frac{dU}{dx} = \frac{1}{2} T \left( \frac{\partial y}{\partial x} \right)^2$

Power flow $= \frac{1}{2} T A^2 \omega k$

Characteristic impedance

The characteristic impedance $Z$ is defined as the applied driving force acting in the $y$-direction divided by the velocity of the string in the $y$-direction

$Z = \frac{F_y}{v_y} = -T \frac{\partial y}{\partial x}$

so with

$y(x,t) = A \sin(kx - \omega t)$

$\Rightarrow Z = \frac{T k}{\omega} = \frac{T}{c} = \sqrt{(T \rho)}$

(Note sign on driving force which ensures $Z$ positive for forward wave!)
9. Two identical long strings are attached to a point mass \( M \). The strings are stretched along the \( x \)-axis and are under tension \( T \). The equilibrium position of the mass is at the origin. The mass is now displaced slightly in the transverse direction \( y \) and subsequently released. Show that

\[
M \left[ \frac{\partial^2 y_2}{\partial t^2} \right]_{x=0} = T \left[ \frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x} \right]_{x=0},
\]

where \( y_1 \) and \( y_2 \) represent displacements of the string at \( x \leq 0 \) and \( x \geq 0 \), respectively. [6]

Show that the amplitude reflection coefficient for a wave incident on the mass is

\[
r = \frac{-ip}{1 + ip}
\]

where \( p = \frac{\omega^2 M}{2Tk} \), \( k \) is the wavenumber \( 2\pi/\lambda \), and \( \omega \) is the angular frequency. What is the transmission coefficient? [10]

Sketch the variation of the phase change on reflection as a function of \( M \) for fixed \( \omega \) and \( T \). [4]
Waves at boundaries

This question a good opportunity to remind ourselves what happens to waves at boundaries, so let’s answer it in a more general way than is being asked by first allowing strings to be different (e.g. different densities)

So we must allow for reflected and transmitted waves. To satisfy boundary conditions all waves must have same frequencies, but their velocity and wave-vector will depend on which string they are on

Incident: $\text{Re}[A \exp[i(\omega t - k_1 x)]]$

Reflected: $\text{Re}[A' \exp[i(\omega t + k_1 x)]]$

Transmitted: $\text{Re}[A'' \exp[i(\omega t - k_2 x)]]$

Complex notation convenient for these problems. Pay attention to signs!
Write down boundary conditions

1. Equation of motion for mass

\[ M \frac{\partial^2 y_{1\text{or}2}}{\partial t^2} \bigg|_{x=0} = (-T \sin \vartheta_1 - T \sin \theta_2) \bigg|_{x=0} \]

\[ \Rightarrow M \frac{\partial^2 y_{1\text{or}2}}{\partial t^2} \bigg|_{x=0} \approx T \left( \frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x} \right) \bigg|_{x=0} \]

2. Strings stay stuck to mass

\[ y_1(0,t) = y_2(0,t) \]

Apply to expressions for incident, reflected and transmitted waves

(1) \[ A''(ik_2T - m \omega^2) = ik_1TA - ik_1TA' \]

(2) \[ A'' = A + A' \]
Evaluate \( r \equiv \frac{A'}{A} \) and \( t \equiv \frac{A''}{A} \)

\[
r \equiv \frac{A'}{A} = \frac{T(k_1 - k_2) - im\omega^2}{T(k_1 + k_2) + i\omega^2 m}
\]

\[
t \equiv \frac{A''}{A} = \frac{2kT}{T(k_1 + k_2) + i\omega^2 m}
\]

Specialising to case where strings are identical and so \( k_1 = k_2 = k \) gives

\[
r = \frac{-im\omega^2}{2kT + i\omega^2 m} = \frac{-ip}{1 + ip}
\]

and

\[
t = \frac{2kT}{2kT + i\omega^2 m} = \frac{1}{1 + ip}
\]

with \( p = \frac{m\omega^2}{2kT} \)

(Aside: one is often asked about transmitted and reflected energy. So make sure you remember wave power \( = \frac{1}{2} T\omega k \text{(Amplitude)}^2 \))

Phase of \( r = -\frac{\pi}{2} - \tan^{-1}(p) \)

- \( m \) low, phase is \(-\pi/2\)
- \( m \) high, phase is \(-\pi\)

but this case is slightly artificial, as no reflection in case \( m=0! \)
3. Consider a taut string of mass per unit length $\rho_1$ which carries transverse wave pulses with amplitude of the generic form $y = F(x - v_1 t)$. These pulses are incident upon a point $I$ where the string connects to a second string of mass per unit length $\rho_2$. Given that energy is conserved as the incident pulses are partially transmitted and partially reflected, show that the amplitude reflection and transmission coefficients $r$ and $\tau$ obey the following relation:

$$r^2 + \frac{v_1}{v_2} \tau^2 = 1,$$

where $v_2$ is the speed at which the transmitted pulses travel on the second string. \[8\]

$$r \equiv \frac{A'}{A} = \frac{T(k_1 - k_2) - i m \omega^2}{T(k_1 + k_2) + i \omega^2 m} \quad t \equiv \frac{A''}{A} = \frac{2k_1 T}{T(k_1 + k_2) + i \omega^2 m}$$

And remember $v = \omega / k$. 

Waves - syllabus

Now let’s consider

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with reference to questions: TT 2009, Q6; TT 2010, Q6; TT 2012, Q8
Wave packets and group velocity

A single wave cannot transmit information. To do that we need a wave packet. Any wave packet can be formed from a sum of single waves. Simplest example:

Sum together two waves which differ by $2\delta \omega$ and $2\delta k$ in angular frequency and wave-number, respectively:

$$y_1 = A \sin[(k + \delta k)x - (\omega + \delta \omega)t]$$
$$y_2 = A \sin[(k - \delta k)x - (\omega - \delta \omega)t]$$

$$\Rightarrow y = y_1 + y_2 = 2A \cos(\delta k x - \delta \omega t) \sin(kx - \omega t)$$

Describes envelope – so envelope moves with velocity $\frac{\delta \omega}{\delta k}$ and indeed

Group velocity $v_g = \frac{d\omega}{dk}$ while phase velocity $v_p = \frac{\omega}{k}$
Dispersion

Dispersion is when there is not a linear relationship between $\omega$ and $k$.

Two consequences:

1. Phase velocity, $\omega/k$, depends on $\omega$ and $k$.
   e.g. Light in medium $m$ has refractive index $n$ and velocity $c_m$, where $c_m = c/n$
   That’s why a prism splits light.

2. Group velocity $\neq$ phase velocity

   If $v_g = \frac{d\omega}{dk}$ and $v_p = \frac{\omega}{k}$ it follows that

   $$ v_g = v_p + k \frac{dv_p}{dk} \quad \text{and} \quad v_g = v_p - \lambda \frac{dv_p}{d\lambda} \quad \text{and} \quad v_g = \frac{c}{n} \left(1 + \frac{\lambda}{n} \frac{dn}{d\lambda}\right) $$

   (more important that you can derive these, rather than learn them!)
6. The phase velocity \( v \) of light travelling through a gas at a wavelength \( \lambda \) is given by

\[
\frac{c^2}{v^2} = A + \frac{B}{\lambda^2} - D\lambda^2
\]

where \( A, B, c \) and \( D \) are constants. Show that the group velocity \( v_g \) is given by

\[
v_g = \frac{v^3}{c^2}(A - 2D\lambda^2).
\]

Now \( \frac{d\omega}{dk} = \frac{d\omega}{d\lambda} \frac{d\lambda}{dk} \) and \( \frac{d\lambda}{dk} = -\frac{2\pi}{k^2} \)

and (1) \( c^2 4\pi^2 = A\omega^2 \lambda^2 + B\omega^2 - D\omega^2 \lambda^4 \) so

\[
\frac{d\omega}{d\lambda} = -\frac{\omega (A - 2D\lambda^2)}{c^2} \lambda^2
\]

hence result
Standing waves: TT 2010, Q6

6. A uniform string of length $L$ which is fixed at both ends has a mass per unit length $\mu$ and tension $T$. Show that transverse standing waves along the string can arise as a result of superposition of two sinusoidal waves travelling in opposite directions. Derive an expression for the displacement of transverse standing waves:

\[ y(x, t) = A \sin(kx - \omega t) + A \sin(kx + \omega t) \]
\[ = 2A \sin kx \cos \omega t \]

Standing waves: every point on the string moves with a certain time dependence ($\cos \omega t$), but the amplitude depends on its position along the string ($\sin kx$).
Standing waves: TT 2010, Q6

... and hence find the frequencies of the normal modes. What is the tension in a violin string of length 325 mm and mass 125 mg, tuned to 660 Hz? [You may assume that the velocity of travelling waves is given by \( v = \sqrt{\frac{T}{\mu}} \).]

\[
y(x,t) = 2A \sin kx \cos \omega t \quad \Rightarrow k_n L = n\pi \quad \text{and} \quad \omega_n = n\sqrt{\frac{T}{\mu}} \frac{\pi}{L}
\]

Ends of string being fixed determine boundary conditions: \( y(0,t) = y(L,t) = 0 \)

and putting in numbers, with \( n=1 \), gives \( T = 71 \text{ N} \)
8. An elastic, horizontal string with tension $T$ and mass per unit length $\rho$ is held fixed at both ends $x = 0$ and $x = L$. At $t = 0$, the string is displaced transversally along the $y$ direction in such a way that:

$$y(x, 0) = \sin \frac{\pi x}{L} - \frac{2}{3} \sin \frac{3\pi x}{L},$$

(a) Calculate the total energy of the string at $t = 0$. 
(b) The string is now released and starts to oscillate. Derive the wave equation

$$\frac{\partial^2 y(x, t)}{\partial t^2} - c^2 \frac{\partial^2 y(x, t)}{\partial x^2} = 0,$$

describing small amplitude transverse waves on the string, and a formula for the wave speed $c$.

(c) Solve this wave equation to obtain the transverse displacement $y(x, t)$ of the string at time $t$.

(d) Do you think the string will ever go back to its original $t = 0$ shape? If so, at what time $t_1$ will it happen for the first time?

(e) Calculate the kinetic energy associated with each standing wave found to be a solution of the wave equation in part (c) as a function of time $t$. 


Standing waves: TT 2012 Q8

Let’s rearrange question so we can discuss the relevant topics more clearly

8. An elastic, horizontal string with tension $T$ and mass per unit length $\rho$ is held fixed at both ends $x = 0$ and $x = L$. At $t = 0$, the string is displaced transversally along the $y$ direction in such a way that:

$$y(x, 0) = \sin \frac{\pi x}{L} - \frac{2}{3} \sin \frac{3\pi x}{L},$$

Going from general solution to specific solution through applying initial conditions and monitoring subsequent evolution with time

(c) Solve this wave equation to obtain the transverse displacement $y(x, t)$ of the string at time $t$. [4]

(d) Do you think the string will ever go back to its original $t = 0$ shape? If so, at what time $t_1$ will it happen for the first time? [2]

Energy of system

(a) Calculate the total energy of the string at $t = 0$. [5]

(e) Calculate the kinetic energy associated with each standing wave found to be a solution of the wave equation in part (c) as a function of time $t$. [4]
Standing waves: TT 2012 Q8

Know from separation of variables that a solution to wave equation is

\[ y(x,t) = (A\cos kx + B\sin kx)(C\cos ct + D\sin ct) \]

Or e.g. d’Alambert

\[ y(x,t) = A\sin(kx - \omega t) + B\sin(kx + \omega t) \]

and we also have four boundary conditions:

1. String initially at rest, i.e. \( \frac{\partial y}{\partial t} = 0 \) for all \( x \) \( \Rightarrow D = 0 \)
2. \( y(0,t) = 0 \) \( \Rightarrow A = 0 \)
3. \( y(L,t) = 0 \) \( \Rightarrow kL = n\pi \) where \( n \) any integer. This is the eigenvalue eqn. and discretises \( k \). Each value of \( n \) corresponds to a normal mode.
4. Form of initial displacement involves normal modes 1 and 3. From these we fix coefficient of mode 1 to be 1 and 3 to be \(-2/3\), and all others 0.

\[
\begin{align*}
\text{hence} \quad y(x,t) &= \sin \frac{\pi x}{L} \cos \frac{\pi ct}{L} - \frac{2}{3} \sin \frac{3\pi x}{L} \cos \frac{3\pi ct}{L} \\
\text{This first returns to initial displacement when} \quad t &= \frac{2L}{c}
\end{align*}
\]
Energy of standing waves

Normal mode \( n \) for our string, with given boundary conditions:

\[
y_n(x, t) = F_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}
\]

Calculate kinetic energy, \( K_n \), and potential energy, \( U_n \), for each mode

\[
K_n = \int_0^L \frac{1}{2} \rho \left( \frac{\partial y_n}{\partial t} \right)^2 \, dx \\
U_n = \int_0^L \frac{1}{2} T \left( \frac{\partial y_n}{\partial x} \right)^2 \, dx
\]

Evaluate and sum

\[
E_n = K_n + U_n = \frac{\rho L F_n^2 \omega_n^2}{4} \quad \text{with} \quad \omega_n = n \sqrt{\frac{T}{\rho L}} \pi
\]

What about the case when several normal modes are excited (as in question)?

Note that all cross-terms have vanished due to the orthogonality of sines

\[
E = \sum_{n=1}^{\infty} E_n
\]

\[i.e. \text{ all these terms are zero} \]

\[
\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx \quad \text{with} \quad n \neq m
\]

So total energy is weighted sum of all the excited normal modes
Standing waves: TT 2012 Q8

(a) Calculate the total energy of the string at $t = 0$.

(e) Calculate the kinetic energy associated with each standing wave found to be a solution of the wave equation in part (c) as a function of time $t$.

Initial energy of string is all in PE

$$U(t = 0) = \int_{0}^{L} \frac{1}{2} T \left( \frac{\partial y(t = 0)}{\partial x} \right)^2 \, dx \quad \Rightarrow \quad U(t = 0) = \frac{5T\pi^2}{4L}$$

This result makes sense as it equals total energy of system as calculated from

$$E = \sum_{n=1}^{\infty} E_n \quad \text{and} \quad E_n = \frac{\rho LF_n^2 \omega_n^2}{4}$$

Kinetic energy of each standing wave, i.e. kinetic energy of each mode

$$K_n = \int_{0}^{L} \frac{1}{2} \rho \left( \frac{\partial y_n}{\partial t} \right)^2 \, dx \quad \Rightarrow \quad K_n = A_n^2 \frac{\rho}{4L} (n\pi c)^2 \sin^2 \frac{n\pi ct}{L}$$

where $A_n$ is the amplitude coefficient for mode $n$ (here $A_1=1$ and $A_3=-2/3$ and others $=0$)