Waves Revision

Waves - syllabus

According to the course handbook you should know about the following

Derivation of the one-dimensional wave equation and its application to transverse waves on a stretched string. D'Alembert's solution. Sinusoidal solutions and their complex representation. Characteristics of wave motion in one dimension: amplitude, phase, frequency, wavelength, wavenumber, phase velocity. Energy in a vibrating string. Travelling waves: energy, power, impedance, reflection and transmission at a boundary. Superposition of two waves of different frequencies: beats and elementary discussion of construction of wave packets; qualitative discussion of dispersive media; group velocity. Method of separation of variables for the one-dimensional wave equation; separation constants. Modes of a string with fixed end points (standing waves): superposition of modes, energy as a sum of mode energies.

Let's remind ourselves of the essentials, before looking at a few past problems

Waves - syllabus

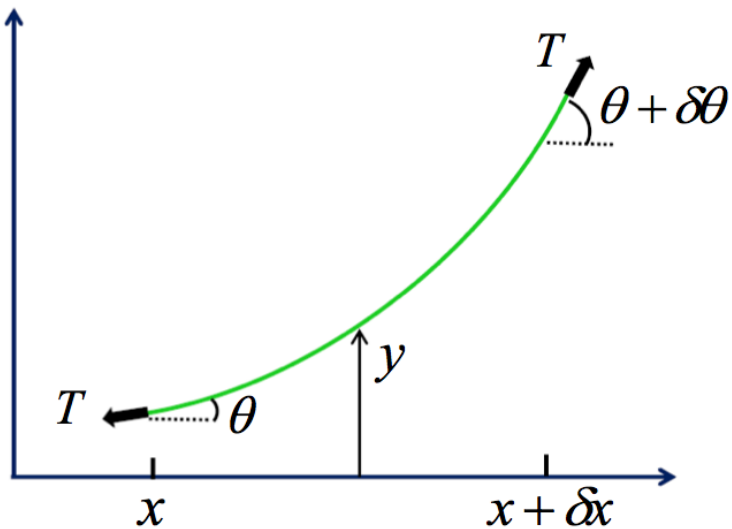
Focus first on the below topics

Derivation of the one-dimensional wave equation and its application to transverse waves on a stretched string. D'Alembert's solution. Sinusoidal solutions and their complex representation. Characteristics of wave motion in one dimension: amplitude, phase, frequency, wavelength, wavenumber, phase velocity. Energy in a vibrating string. Travelling waves: energy, power, impedance, reflection and transmission at a boundary. Superposition of two waves of different frequencies: beats and elementary discussion of construction of wave packets; qualitative discussion of dispersive media; group velocity. Method of separation of variables for the one-dimensional wave equation; separation constants. Modes of a string with fixed end points (standing waves): superposition of modes, energy as a sum of mode energies.

and illustrate by looking at Long Vacation 2011, Q9

Waves on a stretched string

Consider a segment of string of linear density ρ stretched under tension T



$\theta, \delta\theta$ small

Key point – means that no net horizontal force at first order, only vertical force, and we can make approximations such as $\sin\delta\vartheta \approx \delta\vartheta$

For full derivation see HT lecture notes or text book, *e.g.* French

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \left(\frac{\rho}{T}\right)\frac{\partial^2 y}{\partial t^2}}$$

This is the wave equation, which (anticipating solution) we can write

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{with} \quad c = \sqrt{\frac{T}{\rho}}}$$

d'Alembert solution of wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

y is a function of *x* and *t*.
Define new variables so that
y is now a function of *u* and *v*

$$u = x - ct$$
$$v = x + ct$$

With chain rule we can show

$$\frac{\partial^2 y}{\partial u \partial v} = 0 \qquad \Rightarrow y(u,v) = f(u) + g(v)$$

So general solution of wave equation is

$$\boxed{y(x,t) = f(x-ct) + g(x+ct)}$$

Here *f* and *g* are any functions of *(x-ct)* and *(x+ct)*, determined by initial conditions. A common scenario is that they are sinusoidal.

Note that $f(x-ct)$ describes *forward-going* wave and
$g(x+ct)$ describes a *backwards-going* one !

with *c* the (phase) velocity of the wave

Sinusoidal waves

A very common functional dependence for *f* and *g*...

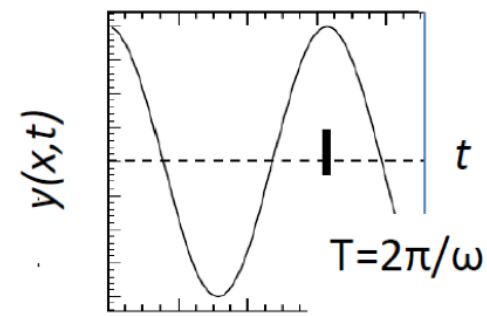
$$y(x,t) = f(x-ct) + g(x+ct)$$

...is sinusoidal. In this case it is usual to write:

$$\boxed{y(x,t) = A\cos(kx-\omega t) + B\cos(kx+\omega t)}$$

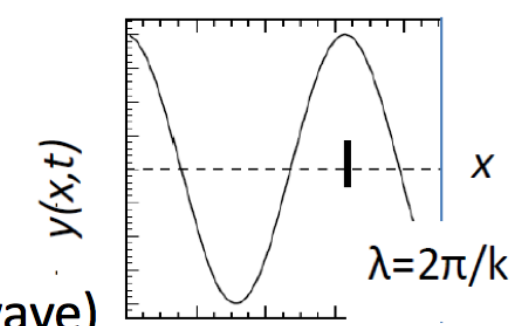
with *k* and *ω* (and *A* and *B*) constants

or *Asin(kx-ωt)* ... etc (choice doesn't matter, unless we are comparing one wave with another and then relative phases become important)



- **speed** of wave $\boxed{c = \omega / k}$
- **frequency** $\boxed{f = 1/T = \omega / 2\pi}$

  where ω is **angular frequency**



- **wavelength** $\boxed{\lambda = 2\pi / k}$

  where k is the **wave-number**

  (or **wave-vector** if also used to indicate direction of wave)

Notation choices

Sinusoidal solution $y(x,t) = A\cos(kx - \omega t)$ (writing here, for compactness, only the forward-going solution)

Using the relationships between *k*,*ω*, *λ* & *c* this can be expressed in many forms

$$y(x,t) = A\cos[k(x - ct)]$$

Also note that sometimes it is convenient to write $y(x,t) = A\cos(\omega t - kx)$

A very frequent approach is to use complex notation (we already made use of this when analysing normal modes, and you will have seen it in circuit analysis)

$$y(x,t) = \mathrm{Re}\left[A\exp[i(kx - \omega t)]\right]$$

or $y(x,t) = \mathrm{Im}\left[A\exp[i(kx - \omega t)]\right]$ if it's important to pick out sine function. Note that often the 'Re' or 'Im' is implicit, and it gets omitted in discussion.

Wave equation revisited – solving by separation of variables

We have already solved the wave equation using the d'Alembert approach

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Can also be solved by looking for solutions which have the 'separated' form

$$y(x,t) = X(x)T(t)$$

i.e. that factorise into functions that are separate functions of *x* and *t.*
This is just the situation that applies to standing waves !

$$\frac{\ddot{X}}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T}$$

set each side equal to some separation constant $-k^2$

This yields $X(x) = A\cos kx + B\sin kx$ and $T(t) = D\cos ckt + E\sin ckt$

with *A,B,D* and *E* constants defined by initial conditions

Energy and impedance for travelling wave on string

Energy stored in a mechanical wave

Integrate kinetic energy and potential energy densities over an integer number of wavelengths to show they contribute *equally* and give

$$\text{KE density} \equiv \frac{\mathrm{d}K}{\mathrm{d}x} = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2$$

$$\text{PE density} \equiv \frac{\mathrm{d}U}{\mathrm{d}x} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

Total energy in *n* wavelengths = $\frac{1}{2}\rho A^2\omega^2 n\lambda$

Power flow = $\frac{1}{2}TA^2\omega k$

Characteristic impedance

The characteristic impedance *Z* is defined as the applied driving force acting in the *y*-direction divided by the velocity of the string in the *y*-direction

$$Z = \frac{F_y}{v_y} = \frac{-T\dfrac{\partial y}{\partial x}}{\dfrac{\partial y}{\partial t}}$$

so with

$$y(x,t) = A\sin(kx - \omega t)$$

$$\Rightarrow Z = \frac{Tk}{\omega} = \frac{T}{c} = \sqrt{(T\rho)}$$

(Note sign on driving force which ensures *Z* positive for forward wave!)

Long Vacation 2011, Q9

9. Two identical long strings are attached to a point mass M. The strings are stretched along the x-axis and are under tension T. The equilibrium position of the mass is at the origin. The mass is now displaced slightly in the transverse direction y and subsequently released. Show that

$$M\left[\frac{\partial^2 y_2}{\partial t^2}\right]_{x=0} = T\left[\frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x}\right]_{x=0},$$

where y_1 and y_2 represent displacements of the string at $x \leq 0$ and $x \geq 0$, respectively. [6]

Show that the amplitude reflection coefficient for a wave incident on the mass is

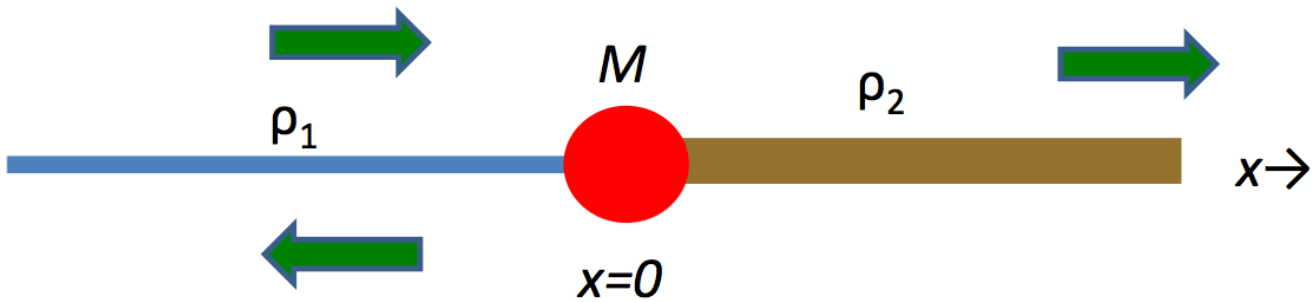
$$r = \frac{-ip}{1+ip}$$

where $p = \frac{\omega^2 M}{2Tk}$, k is the wavenumber $2\pi/\lambda$, and ω is the angular frequency. What is the transmission coefficient? [10]

Sketch the variation of the phase change on reflection as a function of M for fixed ω and T. [4]

Waves at boundaries

This question a good opportunity to remind ourselves what happens to waves at boundaries, so let's answer it in a more general way than is being asked by first allowing strings to be different (*e.g.* different densities)



So we must allow for reflected and transmitted waves. To satisfy boundary conditions all waves must have same frequencies, but their velocity and wave-vector will depend on which string they are on

Incident $\mathrm{Re}\left[A\exp\left[i(\omega t - k_1 x)\right]\right]$

Reflected $\mathrm{Re}\left[A'\exp\left[i(\omega t + k_1 x)\right]\right]$

Transmitted $\mathrm{Re}\left[A''\exp\left[i(\omega t - k_2 x)\right]\right]$

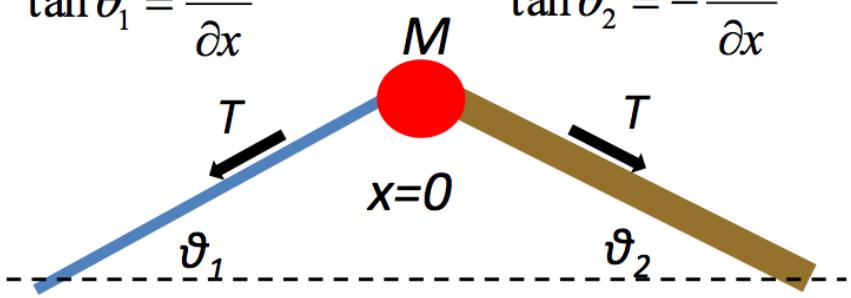
Complex notation convenient for these problems. Pay attention to signs!

Long Vacation 2011, Q9

Write down boundary conditions

$$\sin\theta_1 \approx \tan\theta_1 \qquad \sin\theta_2 \approx \tan\theta_2$$

$$\tan\theta_1 = \frac{\partial y_1}{\partial x} \qquad \tan\theta_2 = -\frac{\partial y_2}{\partial x}$$



1. Equation of motion for mass

$$M\left.\frac{\partial^2 y_{1\,\mathrm{or}2}}{\partial t^2}\right|_{x=0} = \left.(-T\sin\vartheta_1 - T\sin\theta_2)\right|_{x=0}$$

$$\Rightarrow M\left.\frac{\partial^2 y_{1\,\mathrm{or}2}}{\partial t^2}\right|_{x=0} \approx T\left.\left(\frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x}\right)\right|_{x=0}$$

2. Strings stay stuck to mass

$$y_1(0,t) = y_2(0,t)$$

Apply to expressions for incident, reflected and transmitted waves

(1)→ $$A''(ik_2T - m\omega^2) = ik_1TA - ik_1TA'$$

(2)→ $$A'' = A + A'$$

Long Vacation 2011, Q9

Evaluate $r \equiv A'/A$ and $t \equiv A''/A$

$$r \equiv \frac{A'}{A} = \frac{T(k_1 - k_2) - im\omega^2}{T(k_1 + k_2) + i\omega^2 m} \qquad t \equiv \frac{A''}{A} = \frac{2k_1 T}{T(k_1 + k_2) + i\omega^2 m}$$

Specialising to case where strings are identical and so $k_1 = k_2 = k$ gives

$$\boxed{r = \frac{-im\omega^2}{2kT + i\omega^2 m} = \frac{-ip}{1+ip} \quad \text{and} \quad t = \frac{2kT}{2kT + i\omega^2 m} = \frac{1}{1+ip} \quad \text{with} \quad p = \frac{m\omega^2}{2kT}}$$

(Aside: one is often asked about transmitted and reflected energy.
So make sure you remember wave power $= \frac{1}{2} T\omega k (\text{Amplitude})^2$)

Phase of $r = -\frac{\pi}{2} - \tan^{-1}(p)$

- m low, phase is $-\pi/2$ ← but this case is slightly artificial, as no reflection in case $m=0$!
- m high, phase is $-\pi$

Sep 2012, Q3

3. Consider a taut string of mass per unit length ρ_1 which carries transverse wave pulses with amplitude of the generic form $y = F(x - v_1 t)$. These pulses are incident upon a point I where the string connects to a second string of mass per unit length ρ_2. Given that energy is conserved as the incident pulses are partially transmitted and partially reflected, show that the amplitude reflection and transmission coefficients r and τ obey the following relation:

$$r^2 + \frac{v_1}{v_2}\tau^2 = 1\,,$$

where v_2 is the speed at which the transmitted pulses travel on the second string. [8]

$$r \equiv \frac{A'}{A} = \frac{T(k_1 - k_2) - im\omega^2}{T(k_1 + k_2) + i\omega^2 m} \qquad t \equiv \frac{A''}{A} = \frac{2k_1 T}{T(k_1 + k_2) + i\omega^2 m}$$

And remember v=ω/k

Waves - syllabus

Now let's consider

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with reference to questions: TT 2009, Q6; TT 2010, Q6; TT 2012, Q8

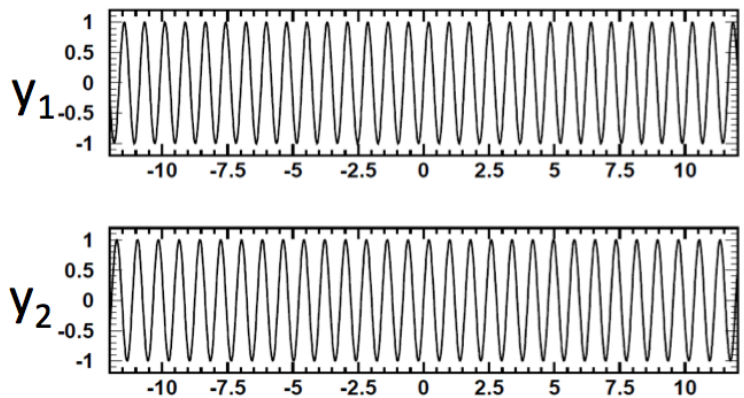
Wave packets and group velocity

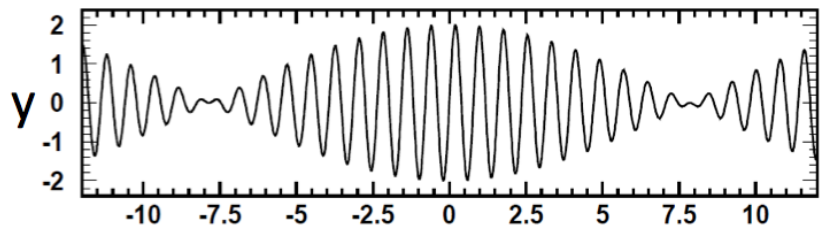
A single wave cannot transmit information. To do that we need a wave packet. Any wave packet can be formed from a sum of single waves. Simplest example:

Sum together two waves which differ by 2δω and 2δk in angular frequency and wave-number, respectively:

$$y_1 = A\sin[(k+\delta k)x-(\omega+\delta\omega)t]$$
$$y_2 = A\sin[(k-\delta k)x-(\omega-\delta\omega)t]$$

$$\Rightarrow y = y_1 + y_2 = 2A\cos(\delta k\, x - \delta\omega\, t)\sin(kx-\omega t)$$





$$2A\cos(\delta k\, x - \delta\omega\, t)\sin(kx-\omega t)$$

Describes envelope – so envelope moves with velocity $\dfrac{\delta\omega}{\delta k}$ and indeed

Group velocity $v_g = \dfrac{\mathrm{d}\omega}{\mathrm{d}k}$ while phase velocity $v_p = \dfrac{\omega}{k}$

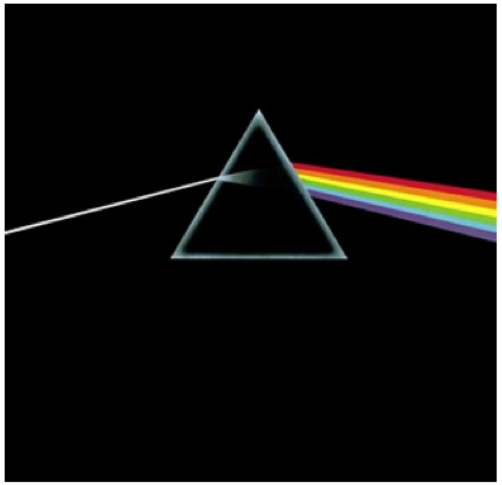
Dispersion

Dispersion is when there is not a linear relationship between ω and k.

Two consequences:

1. Phase velocity, ω/k, depends on ω and k.
   e.g. Light in medium m has refractive index n and velocity c_m, where $c_m = c/n$
   That's why a prism splits light.

2. Group velocity ≠ phase velocity



If $v_g = \dfrac{\mathrm{d}\omega}{\mathrm{d}k}$ and $v_p = \dfrac{\omega}{k}$ it follows that

$$v_g = v_p + k\frac{dv_p}{dk} \quad \text{and} \quad v_g = v_p - \lambda\frac{dv_p}{d\lambda} \quad \text{and} \quad v_g = \frac{c}{n}\left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right)$$

(more important that you can derive these, rather than learn them!)

Dispersion question: TT 2009, Q6

6. The phase velocity v of light travelling through a gas at a wavelength λ is given by

$$\frac{c^2}{v^2} = A + \frac{B}{\lambda^2} - D\lambda^2 \qquad (1)$$

where A, B, c and D are constants. Show that the group velocity v_g is given by

$$v_g = \frac{v^3}{c^2}(A - 2D\lambda^2)\,. \qquad [5]$$

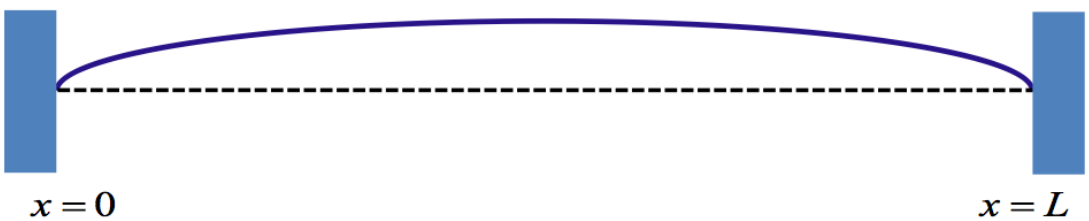
Now $\dfrac{\mathrm{d}\omega}{\mathrm{d}k} = \dfrac{\mathrm{d}\omega}{\mathrm{d}\lambda}\dfrac{\mathrm{d}\lambda}{\mathrm{d}k}$ and $\dfrac{\mathrm{d}\lambda}{\mathrm{d}k} = -\dfrac{2\pi}{k^2}$

and (1) → $c^2 4\pi^2 = A\omega^2\lambda^2 + B\omega^2 - D\omega^2\lambda^4$ so $\dfrac{\mathrm{d}\omega}{\mathrm{d}\lambda} = -\dfrac{\omega}{\lambda}\dfrac{(A-2D\lambda^2)}{c^2}v^2$

hence result

Standing waves: TT 2010, Q6

6. A uniform string of length L which is fixed at both ends has a mass per unit length μ and tension T. Show that transverse standing waves along the string can arise as a result of superposition of two sinusoidal waves travelling in opposite directions. Derive an expression for the displacement of transverse standing waves ...



Sum right- and left-going travelling waves of same amplitude and frequency:

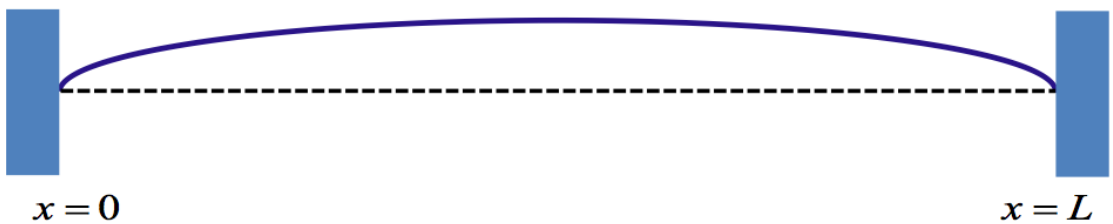
$$y(x,t) = A\sin(kx - \omega t) + A\sin(kx + \omega t)$$
$$= 2A\sin kx \cos \omega t$$

factorised spatial & temporal dependence

Standing waves: every point on the string moves with a certain time dependence (*cosωt*), but the amplitude depends on its position along the string (*sinkx*)

Standing waves: TT 2010, Q6

... and hence find the frequencies of the normal modes. What is the tension in a violin string of length 325 mm and mass 125 mg, tuned to 660 Hz? [You may assume that the velocity of travelling waves is given by $v = \sqrt{T/\mu}$.]



Ends of string being fixed determine boundary conditions: $y(0,t) = y(L,t) = 0$

$$y(x,t) = 2A\sin kx\cos\omega t \quad \Rightarrow k_n L = n\pi \quad \text{and} \quad \omega_n = n\sqrt{\frac{T}{\mu}}\frac{\pi}{L}$$

angular frequency of normal modes

and putting in numbers, with *n=1*, gives *T=71* N

Standing waves: TT 2012 Q8

8. An elastic, horizontal string with tension T and mass per unit length ρ is held fixed at both ends $x = 0$ and $x = L$. At $t = 0$, the string is displaced transversally along the y direction in such a way that:

$$y(x,0) = \sin\frac{\pi x}{L} - \frac{2}{3}\sin\frac{3\pi x}{L}\,,$$

(a) Calculate the total energy of the string at $t = 0$. [5]

(b) The string is now released and starts to oscillate. Derive the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} - c^2\frac{\partial^2 y(x,t)}{\partial x^2} = 0\,,$$

describing small amplitude transverse waves on the string, and a formula for the wave speed c. [5]

(c) Solve this wave equation to obtain the transverse displacement $y(x,t)$ of the string at time t. [4]

(d) Do you think the string will ever go back to its original $t = 0$ shape? If so, at what time t_1 will it happen for the first time? [2]

(e) Calculate the kinetic energy associated with each standing wave found to be a solution of the wave equation in part (c) as a function of time t. [4]

Standing waves: TT 2012 Q8

Let's rearrange question so we can discuss the relevant topics more clearly

8. An elastic, horizontal string with tension T and mass per unit length ρ is held fixed at both ends $x = 0$ and $x = L$. At $t = 0$, the string is displaced transversally along the y direction in such a way that:

$$y(x, 0) = \sin\frac{\pi x}{L} - \frac{2}{3}\sin\frac{3\pi x}{L},$$

Going from general solution to specific solution through applying initial conditions and monitoring subsequent evolution with time

(c) Solve this wave equation to obtain the transverse displacement $y(x, t)$ of the string at time t. [4]

(d) Do you think the string will ever go back to its original $t = 0$ shape? If so, at what time t_1 will it happen for the first time? [2]

Energy of system

(a) Calculate the total energy of the string at $t = 0$. [5]

(e) Calculate the kinetic energy associated with each standing wave found to be a solution of the wave equation in part (c) as a function of time t. [4]

Standing waves: TT 2012 Q8

Know from separation of variables that a solution to wave equation is

$$y(x,t) = (A\cos kx + B\sin kx)(C\cos kct + D\sin kct)$$

Or e.g. d'Alambert $y(x,t) = A\sin(kx - \omega t) + B\sin(kx + \omega t)$

and we also have four boundary conditions:

1. String initially at rest, *i.e.* $\partial y / \partial t = 0$ for all *x* $\Rightarrow D = 0$
2. *y(0,t)=0* $\Rightarrow A = 0$
3. *y(L,t)=0* $\Rightarrow kL = n\pi$ where *n* any integer. This is the eigenvalue eqn. and discretises *k*. Each value of *n* corresponds to a normal mode.
4. Form of initial displacement involves normal modes 1 and 3. From these we fix coefficient of mode 1 to be *1* and 3 to be *-2/3*, and all others *0*.

hence

$$\boxed{y(x,t) = \sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} - \frac{2}{3}\sin\frac{3\pi x}{L}\cos\frac{3\pi ct}{L}}$$

This first returns to initial displacement when $\boxed{t = 2L/c}$

Energy of standing waves

Normal mode *n* for our string, with given boundary conditions:

$$y_n(x,t) = F_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

Calculate kinetic energy, K_n, and potential energy, U_n, for each mode

$$K_n = \int_0^L \frac{1}{2}\rho\left(\frac{\partial y_n}{\partial t}\right)^2 \mathrm{d}x \qquad U_n = \int_0^L \frac{1}{2}T\left(\frac{\partial y_n}{\partial x}\right)^2 \mathrm{d}x$$

Evaluate and sum $\boxed{E_n = K_n + U_n = \frac{\rho L F_n^{\,2}\omega_n^{\,2}}{4}}$ with $\omega_n = n\sqrt{\frac{T}{\rho}}\frac{\pi}{L}$

What about the case when several normal modes are excited (as in question)?

$$\boxed{E = \sum_{n=1}^{\infty} E_n}$$

Note that all cross-terms have vanished due to the orthogonality of sines

i.e. all these terms are zero

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\mathrm{d}x \quad \text{with } n \neq m$$

So total energy is weighted sum of all the excited normal modes

Standing waves: TT 2012 Q8

(a) Calculate the total energy of the string at $t = 0$. [5]

(e) Calculate the kinetic energy associated with each standing wave found to be a solution of the wave equation in part (c) as a function of time t. [4]

Initial energy of string is all in PE

$$U(t=0) = \int_0^L \frac{1}{2} T \left(\frac{\partial y(t=0)}{\partial x} \right)^2 \mathrm{d}x \quad \Rightarrow \boxed{U(t=0) = \frac{5T\pi^2}{4L}}$$

This result makes sense as it equals total energy of system as calculated from

$$E = \sum_{n=1}^{\infty} E_n \quad \text{and} \quad E_n = \frac{\rho L F_n^{\,2} \omega_n^{\,2}}{4}$$

Kinetic energy of each standing wave, *i.e.* kinetic energy of each mode

$$K_n = \int_0^L \frac{1}{2} \rho \left(\frac{\partial y_n}{\partial t} \right)^2 \mathrm{d}x \quad \Rightarrow \boxed{K_n = A_n^{\,2} \frac{\rho}{4L} (n\pi c)^2 \sin^2 \frac{n\pi c t}{L}}$$

where A_n is the amplitude coefficient for mode n (here $A_1=1$ and $A_3=-2/3$ and others $=0$)