Chapter 5

Waves II

5.1 Reflection & Transmission of waves

Let us now consider what happens to a wave travelling along a string which no longer has a single uniform density, but has a step change in density at $x = 0$, with the string essentially extending from $-\infty < x < 0$ with a density of $\rho_1$ and for $0 < x < \infty$ with a density of $\rho_2$.

If the wave travels from the left-hand of the string to the right, then we can write,

$$y(x, t) = A \sin(\omega t - k_1 x),$$

(5.1)

with the negative $k_1 x$ implying that the wave is travelling to the right. In this case $k_1$ contains the relevant information about the density of the string. Remembering that $k = \omega/c$, and that $c_{1,2} = \sqrt{T/\rho_{1,2}}$, thus $k_{1,2} \propto \sqrt{\rho_{1,2}}$.

We also know that although the density is no longer uniform, the tension in the string is uniform throughout, otherwise there would be a non-zero horizontal acceleration somewhere.

The wave moves to the right along the string towards $x = 0$, at $x = 0$ two things could happen, (i) the wave could be reflected resulting in a wave travelling to the left, and (ii) the wave could be transmitted across the boundary and continue moving to the right. Therefore, for the wave to the left of $x = 0$ we can write it as the sum of the incident and reflected waves,

$$y(x, t) = A \sin(\omega t - k_1 x) + A' \sin(\omega t + k_1 x).$$

(5.2)
For the transmitted wave, we just have the component moving to the right at $x > 0$:

$$y(x, t) = A'' \sin(\omega t - k_2 x) \quad (5.3)$$

where we now have $k_2$ which contains the information about the density of the string.

### 5.1.1 Boundary Conditions

We can now apply some boundary conditions to determine how the amplitude of the transmitted and reflected waves depends on the density of the string.

We know that the string is continuous across the boundary, so that

$$y_1(0, t) = y_2(0, t)$$

We also know that the tension throughout the string is also constant, implying that the vertical tension to the left of the boundary is balanced by the vertical component of the tension to the right of the boundary. Therefore, from Eq. 4.2,

$$F_y = T \frac{\delta y}{\delta x} = \tan \delta \theta \approx T \delta \theta,$$

therefore,

$$\frac{\partial y_1}{\partial x}(0, t) = \frac{\partial y_2}{\partial x}(0, t). \quad (5.4)$$

So applying these boundary conditions at $x = 0$ we find, with the fact that the string is continuous,

$$A \sin \omega t + A' \sin \omega t = A'' \sin \omega t$$

$$\Rightarrow A + A' = A'' \quad (5.5)$$

and that we have balanced vertical tension,

$$-k_1 A \cos \omega t + k_1 A' \cos \omega t = -k_2 A'' \cos \omega t$$

$$\Rightarrow k_1 (A - A') = k_2 A''. \quad (5.6)$$

We can rewrite these equations in terms of reflection and transmission coefficients, which are just the ratios of the amplitudes of the reflected and transmitted waves to the incident wave respectively.
\[
\frac{r}{A} \equiv \frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2}
\] (5.7)

\[
\frac{t}{A} \equiv \frac{A''}{A} = \frac{2k_1}{k_1 + k_2}.
\] (5.8)

5.1.2 Particular cases

Given these reflection and transmission coefficients, we can consider some specific cases,

- \(k_1 = k_2\)

  \(r = 0, t = 1\) as you would expect, the string is just a single uniform density and there is no reflection, only transmission.

- \(k_1 < k_2\)

  \(A'\) is negative and we can write down the equation for the reflected wave as \(-|A'| \sin(\omega t + k_1 x) = |A'| \sin(\omega t + k_1 x + \pi)\), i.e. there is a phase change at the boundary as we move from a less dense to a more dense string.

- \(k_1 > k_2\)

  the \(A'\) is positive, i.e. we don’t get the phase change in this case where \(\rho_1 > \rho_2\).

- \(k_2 \to \infty\) (or \(\rho_2 \to \infty\))

  in this case \(r = \frac{A'}{A} \to -1\), i.e. full reflection with a phase change and no transmitted wave. This is unsurprising as it is just the same as the second string being immovable, i.e. having the string attached to a brick wall at \(x = 0\).

5.2 Power flow at a boundary

From the last section, we have the reflection and transmission coefficients,
\[ r = \frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2} \]  \hspace{1cm} (5.9)

\[ t = \frac{A''}{A} = \frac{2k_1}{k_1 + k_2} \]  \hspace{1cm} (5.10)

In Sec. 4.6 we showed that the power to generate a wave, was given by

\[ P = \frac{1}{2} T \omega k A^2 \]  \hspace{1cm} (5.11)

So the ratios of the reflected to incident power, \( R_r \) and the transmitted to incident power, \( R_t \), are given by

\[ R_r = \frac{\frac{1}{2} k_1 T \omega A'^2}{\frac{1}{2} k_1 T \omega A'^2} = \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^2 \]
\[ R_t = \frac{\frac{1}{2} k_2 T \omega A''^2}{\frac{1}{2} k_1 T \omega A'^2} = \frac{4k_1 k_2}{(k_1 + k_2)^2} \]  \hspace{1cm} (5.12)

Therefore,

\[ R_r + R_t = \frac{(k_1^2 + k_2^2 - 2k_1 k_2) + (4k_1 k_2)}{(k_1 + k_2)^2} = 1, \]  \hspace{1cm} (5.13)

as expected, there is no power loss in the system.

\section*{5.3 Impedence}

Impedence is a general term in physics that describes the opposition of a material to a time varying current (in an electrical circuit) or indeed any wave-carrying system.

A general definition that it is a measure of resistance to an alternating effect, and is equivalent to, the ratio of a \textit{push variable} (i.e. voltage or pressure) to a \textit{flow variable} (i.e. current or particle velocity).

\subsection*{5.3.1 Impedence along a stretched string}

One of the key assumptions that we made in the previous sections was that the tension in the string is uniform throughout the string. What happens if we relax this condition? What does this actually mean anyway?
First of all, let us consider how we might be able to alter the tension either side of

\( x = 0 \), given that this implies that the nearly massless atom within the string at \( x = 0 \),

would experience \( \sim \infty \) acceleration!

We can get around this by joining the two halves of the string via a massless ring, which

encircles a fixed frictionless pole (Fig. 5.3.1). The pole that sits at the boundary now

balances the horizontal components of the tensions, so that the net horizontal force on the

ring is zero. This obviously has to be the case as the ring must remain on the pole and can

only move vertically.

However, in this case the net vertical force on the ring must also be zero, otherwise

it would have infinite acceleration (as it is massless). This zero vertical component of the

force means that \( T_1 \sin \theta_1 = T_2 \sin \theta_2 \). This can be written as,

\[
T_1 \frac{\partial y_1(x = 0, t)}{\partial x} = T_2 \frac{\partial y_2(x = 0, t)}{\partial x} \tag{5.14}
\]

Figure 5.1: (left) Set up to imitate a system with non-uniform tension across the \( x = 0 \)

boundary. (right) Forces acting on the ring.

In the previous examples, the vertical component of \( T_1 \) was equal to the vertical

component of \( T_2 \), but now these vertical components to the tensions can be different. So

if we have the same form as Eq. 5.4, but now with the additional tension terms which

no longer cancel out. Therefore, implementing the same form of the wave solution, i.e.

\( y(x, t) = A \sin(\omega t - kx) \), for incident, reflected and transmitted waves, we arrive at a very

similar result, but with the tension in the string on either side of the massless ring also

included, i.e.
\[ A \sin \omega t + A' \sin \omega t = A'' \sin \omega t \]
\[ \Rightarrow (A + A') = A'' \]  
(5.15)

and differentiating with respect to \(x\),
\[ -k_1 T_1 A \cos \omega t + k_1 T_1 A' \cos \omega t = -k_2 T_2 A'' \cos \omega t \]
\[ \Rightarrow k_1 T_1 (A - A') = k_2 T_2 A''. \]  
(5.16)

Therefore, the new coefficients of reflection and transmission becomes slightly modified,
\[
\begin{align*}
    r &= \frac{A'}{A} = \frac{k_1 T_1 - k_2 T_2}{k_1 T_1 + k_2 T_2} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \\
    t &= \frac{A''}{A} = \frac{2k_1 T_1}{k_1 T_1 + k_2 T_2} = \frac{2Z_1}{Z_1 + Z_2}
\end{align*}
\]  
(5.17)

where \(Z_1 = k_1 T_1\) and \(Z_2 = k_2 T_2\), or more correctly,
\[
\begin{align*}
    Z_1 &= T_1/v_1 \\
    Z_2 &= T_2/v_2
\end{align*}
\]  
(5.18)

where \(v_{1,2}\) are the wave velocities. Note that replacing \(k_{1,2}\) with \(1/v_{1,2}\) just means that we have assumed \(\omega_1 = \omega_2\), which it must across the massless ring (remembering \(v = \omega/k\)).

### 5.3.2 Physical meaning of impedence

Although we can describe impedence in the way we have above, what does it actually mean for this system?

The force acting on the right-hand side of the massless ring is just given by,
\[ F_y = T_2 \frac{\partial y_2(x = 0, t)}{\partial x} \]  
(5.19)

Substituting in a solution of the form \(y_2(x, t) = A \sin[\omega(t-x/v_2)]\) (which is equivalent to \(y_2(x, t) = A \sin(\omega t - kx)\), using the normal relation between \(k, \omega\) and \(v\)).
The partial derivatives with respect to $t$ and $x$,

\[
\frac{\partial y_2}{\partial x} = -\frac{A\omega}{v_2} \cos[\omega(t - x/v_2)]
\]

\[
\frac{\partial y_2}{\partial t} = A\omega \cos[\omega(t - x/v_2)]
\]

therefore,

\[
\frac{\partial y_2}{\partial x} = -\frac{1}{v_2} \frac{\partial y_2}{\partial t}.
\]

Then we find,

\[
F_y = -\frac{T_2}{v_2} \frac{\partial y_2(x = 0, t)}{\partial t} = -\frac{T_2}{v_2} v_y \equiv -\gamma v_y,
\]

where $v_y$ is the transverse velocity of the ring at $x = 0$, and $\gamma$ is defined as $T_2/v_2$. So we have a force that is proportional to the negative of the transverse velocity. Therefore, it acts exactly like a damping force! This means that from the perspective of the left string, the right string acts like a resistance that is being dragged against.

### 5.4 Reflection from a mass at the boundary

Now let us consider a slightly more complicated system, rather than having a massless ring around a frictionless pole at the boundary, we now have an object of mass $M$. Either side of this mass we have semi-infinite strings of linear density $\rho_1$ to the left and $\rho_2$ to the right, as shown in Fig. 5.4.

![Figure 5.2: System of two strings of density $\rho_1$ and $\rho_2$ attached to an object of mass $M$ at $x = 0$.](image)

We can solve this system as we did in the previous examples, but in this case the boundary conditions are different.
We have the usual boundary condition that the system in continuous and the at \( x = 0 \) the \( y \) displacement must be the same for the left and right side of the mass, i.e. \( y_1(0, t) = y_2(0, t) \).

However, if we consider the forces at the boundary, we now have to consider the transverse acceleration of the objects which has finite mass, i.e.

\[
-T \frac{\partial y_1(0, t)}{\partial x} + T \frac{\partial y_2(0, t)}{\partial x} = M \frac{\partial^2 y_1(0, t)}{\partial t^2} = M \frac{\partial^2 y_2(0, t)}{\partial t^2}. \tag{5.23}
\]

As we will have to consider second derivatives, let us express the wave in terms of exponentials rather than sines and cosines. Then for a wave travelling from left to right (Fig. 5.4), we have

\[
y_1(x, t) = \Re \left\{ A e^{i(\omega t - k_1 x)} + A' e^{i(\omega t + k_1 x)} \right\} \tag{5.24}
\]

and

\[
y_2(x, t) = \Re \left\{ A'' e^{i(\omega t - k_2 x)} \right\} \tag{5.25}
\]

As before, with a continuous system we have \( A + A' = A'' \), but from Eq. 5.23 we have

\[
\begin{align*}
&ik_1 TA - ik_1 TA' - ik_2 TA'' = -\omega^2 M (A + A') = -\omega^2 MA'' \\
\Rightarrow & ik_1 (A - A') = \left( ik_2 - \frac{\omega^2 M}{T} \right) A'' \tag{5.26}
\end{align*}
\]

From these we find,

\[
\begin{align*}
\frac{A'}{A} & = \frac{(k_1 - k_2)T - i\omega^2 M}{(k_1 + k_2)T + i\omega^2 M} = Re^{i\theta} \tag{5.27} \\
\frac{A''}{A} & = \frac{2k_1 T}{(k_1 + k_2)T + i\omega^2 M} = Te^{i\phi} \tag{5.28}
\end{align*}
\]

where \( R \) and \( T \) are real numbers.

\( \theta \) is the phase shift of the reflected wave and \( \phi \) is the phase shift of the transmitted wave with respect to the incident wave. Therefore, combining Eq. 5.24 with Eq. 5.31 and Eq. 5.24 with Eq. 5.32, we obtain
\[
y_1(x, t) = A \cos(\omega t - k_1 x) + RA \cos(\omega t + k_1 x + \theta)
\]
and
\[
y_2(x, t) = TA \cos(\omega t - k_2 x + \phi)
\]
where,
\[
R = \left[ \frac{(k_1 - k_2)^2 T^2 + \omega^4 M^2}{(k_1 + k_2)^2 T^2 + \omega^4 M^2} \right]^{\frac{1}{2}} \quad \text{and} \quad \theta = \tan^{-1} \left[ \frac{-\omega^2 M}{(k_1 - k_2) T} \right] - \tan^{-1} \left[ \frac{\omega^2 M}{(k_1 + k_2) T} \right]
\]
\[
T = \left[ \frac{4k_1^2 T^2}{(k_1 + k_2)^2 T^2 + \omega^4 M^2} \right]^{\frac{1}{2}} \quad \text{and} \quad \phi = -\tan^{-1} \left[ \frac{\omega^2 M}{(k_1 + k_2) T} \right]
\]
Checking that energy is conserved,
\[
|r|^2 + \frac{k_2}{k_1}|t|^2 = R^2 + \frac{k_2}{k_1} T^2 = 1,
\]
so it is.

5.5 Standing Waves

5.5.1 Infinite string with a fixed end
Consider a leftward-moving single sinusoidal wave that is incident on a brick wall at its left end, located at \( x = 0 \). The most general form of a leftward-moving sinusoidal wave is given by
\[
y_i(x, t) = A \cos(kx - \omega t + \phi)
\]
where \( \omega/k = v = \sqrt{T/\rho} \), \( \phi \) is arbitrary and depends only on where the wave is at \( t = 0 \). The brick wall is equivalent to a system with infinite impedance, i.e. \( Z_2 = \infty \), and the reflection coefficient \( r = -1 \), which gives rise to a reflected wave with amplitude of the same magnitude as the incident wave but with the opposite sign and travelling in the opposite direction, i.e.
\[
y_r(x, t) = -A \cos(kx + \omega t + \phi).
\]
If we were to observe this system, we would see the summation of these two waves,

\[ y(x, t) = A \cos(kx - \omega t + \phi) - A \cos(kx + \omega t + \phi) \]

which, using trig identities, can be expressed as

\[ y(x, t) = -2A \sin(\omega t + \phi) \sin kx \quad (5.36) \]

or,

\[ y(x, t) = 2A \sin \left( \frac{2\pi x}{\lambda} \right) \sin \left( \frac{2\pi t}{T} + \phi \right). \quad (5.37) \]

Thus we have a wave that is factorised in space- and time-dependent parts, where every point on the string is moving with a certain time dependence, but the amplitude of the oscillation is dependent on the displacement along the string.

It is also important that we have a sine function rather than cosine in the \( x \)-dependent part, as the cosine would not satisfy the boundary condition of \( y(0, t) = 0 \) for all values of \( t \). However, it wouldn’t matter if we had sine or cosine for the time dependent part as we can always turn one into the other with a phase shift \( \phi \).

We therefore obtain stationary points along the \( x \)-direction, these are the nodes with \( y = 0 \) and they occur every \( \lambda/2 \) wavelengths. Between these nodes, i.e. the peaks, are the anti-nodes. All points on the string have the same phase, or are multiples of \( \pi \), in terms of how the oscillations move in time. For example, all the points are at rest at the same time, when the string is at a maximum displacement from the equilibrium position, and they all pass through the origin or equilibrium position at the same time. These waves are therefore called standing waves, as opposed to travelling waves.

Rather than invoking the fact that \( r = -1 \) for a wall, we could always derive this result using the fact \( y = 0 \) at all \( t \) and start from the general solution to the wave equation, i.e.
Figure 5.3: Standing wave with two full wavelengths shown. The solid curves are for $t_0 = 0$ and $t = t_0 = \delta t$ and the dashed curved shows where the waves would be at a time $t = t_0 + \pi$ and $t = t_0 + \pi \pm \delta t$ later. For this standing wave the nodes are the stationary points where the wave crosses the $y = 0$ axis.

$$y(x, t) = A_1 \sin(kx - \omega t) + A_2 \cos(kx - \omega t) + A_3 \sin(kx + \omega t) + A_4 \cos(kx + \omega t)$$

$$\Rightarrow y(x, t) = B_1 \cos kx \cos \omega t + B_2 \sin kx \sin \omega t + B_3 \sin kx \cos \omega t + B_4 \cos kx \sin \omega t$$

at $y(0, t) = 0$ for all $t$, therefore we should only have the sin $kx$ terms, i.e.

$$y(x, t) = B_2 \sin kx \sin \omega t + B_3 \sin kx \cos \omega t$$

$$= (B_2 \sin \omega t + B_3 \cos \omega t) \sin kx$$

$$= B \sin(\omega t + \phi) \sin kx$$

where if $B_1 = B_2$ then $B = 2B_1 = 2B_2$. 
5.5.2 Standing waves with a free end

We can also consider a similar system as discussed in Sec. 5.3, where we fix one end to a massless ring which encircles a frictionless pole at \( x = 0 \). This ensures that the wave cannot move in the longitudinal direction, but is still free to move in the transverse direction. This is similar to assuming that the string beyond the pole has a density of zero. If we assume that the wave is travelling towards the pole from the left hand side (i.e. along negative \( x \)), then we can write

\[
y_i(x, t) = A \cos(\omega t - kx + \phi).
\]  

(5.38)

Since the massless ring has zero impedence (remember it was the string on the other side of the ring that provided the impedence in Sec. 5.3), then the reflection coefficient \( r = +1 \) as \( k_2 = 0 \). Therefore, we find for the reflected wave we have,

\[
y_r(x, t) = r y_i(x, t) = A \cos(\omega t + kx + \phi),
\]  

(5.39)

and therefore the wave we would observe is the summation of the incident and reflected waves,

\[
y(x, t) = y_i(x, t) + y_r(x, t) = A \cos(\omega t - kx + \phi) + A \cos(\omega t + kx + \phi) \]

\[= 2A \cos(\omega t + \phi) \cos kx
\]  

(5.40)

As in the case considered before, you can also apply the usual boundary conditions to the general solution to the wave equation and reach the same result. In both of these cases, \( \omega \) and \( k \) can and number and are not necessarily discrete, unlike the case which we will look at next, where we find that only discrete values are allowed.

5.6 Waves on a finite string

Up until now we have considered only infinite strings which are either free or fixed at one end. In this section we will look at a finite string with both fixed and free ends. We consider a string on length \( L \) and with the two ends assigned the values of \( x = 0 \) and \( x = L \). We can think of what the general boundary conditions for such a system are. At a fixed end we know that the displacement in the \( y \)--direction must be zero at all times, and that the displacement at any free end must result in \( \partial y / \partial x = 0 \), because the slope must be zero,
Figure 5.4: Standing wave for a system with a free end, with two full wavelengths shown. As in Fig. 5.5.1, the solid curves are for \( t_0 = 0 \) and \( t = t_0 + \delta t \) and the dashed curved shows where the waves would be at a time \( t = t_0 + \pi \) and \( t = t_0 + \pi \pm \delta t \) later. For this standing wave the nodes are the stationary points where the wave crosses the \( y = 0 \) axis. In this case the end of the string are anti-nodes.

otherwise we would have a vertical force on a massless end, which in turn would result in infinite acceleration.

5.6.1 Two fixed ends

First, let us consider a system in which the string is fixed at both ends, i.e. at \( x = 0 \) and \( x = L \). Then we have similar boundary conditions to that considered for the infinite string fixed at one end, i.e. the boundary conditions that resulted in Eq. 5.36, but we require not only that \( y(0, t) = 0 \), but also \( y(L, t) = 0 \). Therefore, the only way to have \( y(L, t) = 0 \) for
all $t$ is to ensure that $\sin kL = 0$. This implies that $kL$ must be an integer number of $\pi$, i.e.

$$k_n = \frac{n\pi}{L},$$

where $n$ is an integer and defines which mode is excited in the string.

The fact that each end must be a node implies that we can only have wavelengths which are related to the length of the string by $n$, i.e.

$$\lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n}. \tag{5.41}$$

Therefore, we now have a solution of the form

$$y(x, t) = -2A\sin(\omega t + \phi)\sin\left(\frac{n\pi}{L}\right) = -2A\sin(\omega t + \phi)\sin\left(\frac{n\pi}{L}\right) \tag{5.42}$$

So the allowed wavelengths on the string are all integer divisors of twice the length of the string. This can easily be seen if you consider what the $n = 1$ mode actually is based on previous lectures, i.e. the lowest mode is one in which there are two nodes and a single anti-node halfway between the ends. This unavoidably has half of a full wavelength, where this half wavelength is the length of the string. You can obviously have an $n = 0$ mode as well, but this just means that $\sin(0) = 0$ and the string is just at rest in its equilibrium position.

Now looking at the angular frequency $\omega$, we know that it is related to the velocity of the wave through $\omega/k = \sqrt{T/\rho} = v$, so that $\omega_n = vk_n$, i.e. the frequency of oscillation also has a dependence on $n$. The frequency is therefore given by,

$$\omega_n = k_nv = \frac{n\pi}{L}v. \tag{5.43}$$

Therefore, the frequency of the oscillations of the string are all integer multiples of the fundamental frequency, $\omega_1 = v/2L$.

Combining Eqs. 5.41 and 5.43, we find that $v = \lambda_n/2\pi\omega_n$ as you would expect.

Since the wave equation in Eq. 4.7 is linear, the most general motion of a string with two fixed ends is a linear combination of the solution given in Eq. 5.36, where $k$ can only take a form $k_n = n\pi/L$ and $\omega/k = v$. Therefore the general expression for $y(x, t)$ is the summation over all $n$, i.e.
This is the sum of all possible solutions with the coefficients $F_n$ given by the initial displacement, which is the boundary condition we have yet to invoke. Note that the sine function for the time dependent term could be replaced by a cosine, with the phase difference $\phi_n$ adjusted accordingly, but this cannot be done for the $x$–dependent sine term.

### 5.6.2 One fixed end

Now we will look at what happens if one end of a finite string is left completely free. If we take the fixed end to be at $x = 0$ then the boundary conditions are $y(0, t) = 0$ and $\frac{\partial y}{\partial x}|_{x=L} = 0$ for all $t$. From Eq. 5.36 we find that the slope ($\frac{\partial y}{\partial x}$) is proportional to $\cos kx$. Therefore, for this to be zero at $x = L$, we require that $kL = n\pi + \pi/2$ for any integer $n$. Therefore,

$$k_n = \frac{(n + 1/2)\pi}{L}. \quad (5.46)$$

The first thing to note here is that now with $n = 0$ we have an excited wave, as $k_0 = \pi/2L$. As $\lambda_n = 2\pi/k$, then $\lambda_0 = 4L$, i.e. the $n = 0$ mode produces a quarter of a wavelength, where the string has length $L$. This is straightforward to visualise: with one free we have an anti-node, whereas at the fixed end there is a node. In this case, the general solution is again the summation over all possible modes, $n$, and is given by,

$$y(x, t) = \sum_{n=0}^{\infty} F_n \sin(\omega_n t + \phi_n) \cos k_n x \quad (5.47)$$
or

\[ y(x, t) = \sum_{n=0}^{\infty} F_n \sin \left( \frac{n \pi v}{L} t + \phi_n \right) \cos \left( \frac{(n + 1/2) \pi}{L} x \right) \]  \hspace{1cm} (5.48)

5.6.3 Two free ends

Finally, we will look at the case where we have two free ends. In terms of the boundary conditions, we now do not require that \( y(x, t) = 0 \) at any end of the string, and only require that the gradient of the string \( \partial y/\partial x = 0 \) at both \( x = 0 \) and \( x = L \), for all \( t \). Therefore Eq. 5.40 provides us with the most general solution for this system, therefore the slope \( \partial y/\partial x \) is proportional to \( \sin kx \). To ensure that this is zero at \( x = L \) and \( x = 0 \), we require \( kL = n\pi \) for any integer \( n \). In this case, we have

\[ k_n = \frac{n\pi}{L}, \]  \hspace{1cm} (5.49)

which is the same as we found for the case with two fixed ends, and again the possible wavelengths are all integral divisors of \( 2L \), similarly \( \omega_n = n\pi v/L \). So writing down the general solution as the superposition of all the \( n \) modes, we find

\[ y(x, t) = \sum_{n=0}^{\infty} F_n \cos(\omega_n t + \phi_n) \cos k_n x \]  \hspace{1cm} (5.50)

or

\[ y(x, t) = \sum_{n=0}^{\infty} F_n \cos \left( \frac{n \pi v}{L} t + \phi_n \right) \cos \left( \frac{n \pi}{L} x \right). \]  \hspace{1cm} (5.51)

So in this case the \( \cos(k_n x) \) term ensures that we have an anti-node at either end of the string for all \( n \). One things to note about this system is that the equilibrium position of the string does not have to lie at \( y = 0 \).
Fig. 5.6.3 shows the possible oscillations for these three different set-ups for a finite string.

![Fig. 5.6.3](image)

Figure 5.5: Wave pattern for a finite string with two fixed ends (left panels), one fixed end (central panels) and two free ends (right panels). In this case the length of the string is fixed at twice the wavelength for the \( n = 1 \) mode and the excited modes, defined by \( n \), are shown in each panel. All panels show the wave pattern at \( t = \pi \).

### 5.7 Superposition of modes

Let us now look at a specific example of a string in which the initial conditions mean that more than one mode is excited.

If \( h(x) \) describes a pattern for the initial displacement of a finite string then,

\[
y(x, 0) = \sum_{n=0}^{\infty} F_n \sin \frac{n\pi x}{L} = h(x). \tag{5.52}
\]
When \( h(x) \) is just a single \( n = 5 \) normal mode then we find

\[
h(x) = \sin \frac{5\pi x}{L} \quad \text{for } F_5 = 1 \text{ and } F_n = 0 \text{ when } n \neq 5
\]  

(5.53)

therefore,

\[
y(x, t) = \sin \frac{5\pi x}{L} \cos \frac{5\pi vt}{L}.
\]  

(5.54)

However, when there is more than one mode active, i.e. when \( F_1 = 1, F_2 = 0.5 \) and \( F_n = 0 \) when \( n \neq 1 \) or 2, we obtain

\[
y(x, t) = \sin \frac{\pi x}{L} \cos \frac{\pi vt}{L} + \frac{1}{2} \sin \frac{2\pi x}{L} \cos \frac{2\pi vt}{L}.
\]  

(5.55)

In contrast to the case with just a single normal mode, the subsequent motion of the case with \( > 1 \) mode active is not equal to the initial displacement multiplied by a time-dependent amplitude. This is because the shorter waves move faster, resulting in the shape of the wave varying with time.

Note: Even if the initial displacement takes the most simple form (i.e. a plucked string at the centre), it can be expressed as a sum of normal modes. You will see more of this in Year 2 when considering Fourier Series.

### 5.8 Energies of normal modes

Finally let us consider the energy associated with each normal mode for the finite string solution discussed in the previous sections.

The general solution for the motion of a string fixed at both ends, is given by

\[
y(x, t) = \sum_{n=0}^{\infty} F_n \sin \frac{n\pi x}{L} \cos \frac{n\pi vt}{L}
\]  

(5.56)

So as before, we can calculate the kinetic energy in the fixed string for the \( n \)th normal mode.:
\[ K_n = \int_0^L \frac{1}{2} \rho \left( \frac{\partial y_n}{\partial t} \right)^2 \cdot dx \]
\[ = \frac{1}{2} \rho F_n^2 \left( \frac{n\pi v}{L} \right)^2 \sin^2 \frac{n\pi vt}{L} \int_0^L \sin^2 \frac{n\pi x}{L} \cdot dx \]  
\[ = \frac{\rho (F_n n\pi v)^2}{4L} \sin^2 \frac{n\pi vt}{L} \]  

and the potential energy in the fixed string:
\[ U_n = \int_0^L \frac{1}{2} T \left( \frac{\partial y_n}{\partial x} \right)^2 \cdot dx \]
\[ = \frac{1}{2} TF_n^2 \left( \frac{n\pi}{L} \right)^2 \cos^2 \frac{n\pi vt}{L} \int_0^L \cos^2 \frac{n\pi x}{L} \cdot dx \]  
\[ = \frac{T (F_n n\pi)^2}{4L} \cos^2 \frac{n\pi vt}{L} \]  

The total energy in each normal mode is given by \( E_n = K_n + U_n \), and since, \( v = \sqrt{T/\rho} \), then

\[ E_n = K_n + U_n = \frac{\rho LF_n^2 v^2}{4} \left( \frac{n\pi}{L} \right)^2 \]
\[ = \frac{\rho LF_n^2 \omega_n^2}{4} \text{ as } \omega_n = \frac{n\pi v}{L} \]  

\[ (5.59) \]

### 5.9 Total energy in a fixed string

We can now determine the total energy in string fixed at both ends by just generalising the calculation in Sec. 5.8. So for a system with initial arbitrary displacement,
\[ y(x, t) = \sum_{n=0}^{\infty} F_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} \]
the partial derivative with respect to \( x \) and \( t \) have the form,
\[ \left( \frac{\partial y_n}{\partial t} \right)^2 = (\alpha c F_n)^2 \sin^2 \alpha x \sin^2 \alpha ct + (\beta c F_m)^2 \sin^2 \beta x \sin^2 \beta ct \]
\[ + 2\alpha \beta c^2 F_n F_m \sin \alpha x \sin \beta x \sin \alpha ct \sin \beta ct \]
\[ \left( \frac{\partial y_n}{\partial x} \right)^2 = (\alpha F_n)^2 \cos^2 \alpha x \cos^2 \alpha ct + (\beta F_m)^2 \cos^2 \beta x \cos^2 \beta ct \]  
\[ + 2\alpha \beta F_n F_m \cos \alpha x \cos \beta x \cos \alpha ct \cos \beta ct \]  

where \( \alpha = \frac{n\pi}{L} \) and \( \beta = \frac{m\pi}{L} \).
This is simple extension of exercise for individual normal modes, but with additional terms

\[ E = \sum_{n=1}^{\infty} E_n + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin \alpha x \sin \beta x \text{ or } \cos \alpha x \cos \beta x \text{ terms} \]  
(5.61)

Now if \( n \neq m \),

\[
\int_0^L \sin \alpha x \sin \beta x = \left. \left[ \frac{\sin((\alpha - \beta)x)}{2(\alpha - \beta)} - \frac{\sin((\alpha + \beta)x)}{2(\alpha + \beta)} \right] \right|_0^L = 0
\]
(5.62)

\[
\int_0^L \cos \alpha x \cos \beta x = \left. \left[ \frac{\sin((\alpha - \beta)x)}{2(\alpha - \beta)} + \frac{\sin((\alpha + \beta)x)}{2(\alpha + \beta)} \right] \right|_0^L = 0
\]

therefore the cross-terms all cancel, and we are left with,

\[ E_{\text{tot}} = \sum_{n=1}^{\infty} E_n, \]  
(5.63)

i.e. the total energy in the system is the sum of the energies in each normal mode, as we found in the first set of lectures for the coupled pendulum and spring-mass systems.

### 5.10 Power in a standing wave

We saw that for a travelling wave, that power is transmitted. A given point on the string does work (which may be positive or negative, depending on the direction of the waves velocity) on the part of the string to its right. And it does the opposite amount of work on the string to its left.

So is there any energy flowing in a standing wave? We know that there is an energy density as the string stretches and moves, this is what we saw in Sec. 5.9, but is any energy transferred along the string?

Given that a standing wave is just the superposition of two waves of equal amplitudes travelling in opposite directions then they should have equal and opposite energy flow. This would result in net energy flow of zero, on average.

The power flow in any wave is just given by the rate of work done, or the vertical force multiplied by the transverse velocity, i.e.
If our standing wave can be described by

$$y(x, t) = A \sin \omega t \sin kx,$$  \hspace{1cm} (5.65)

then

$$P(x, t) = -TA^2 (\sin kx \cos kx) (\sin \omega t \cos \omega t).$$  \hspace{1cm} (5.66)

This is non-zero for most values of $x$ and $t$, so energy does flow across a given point. However, at given value of $x$, the average power over a whole period, is zero. This is because the average of $\sin \omega t \cos \omega t$ over the period is zero. Therefore the average power is zero.