Waves 2

1. Standing waves
2. Transverse waves in nature: electromagnetic radiation
3. Polarisation
4. Dispersion
5. Information transfer and wave packets
6. Group velocity

Standing waves

Consider a string with 2 waves of equal amplitude moving in opposite directions

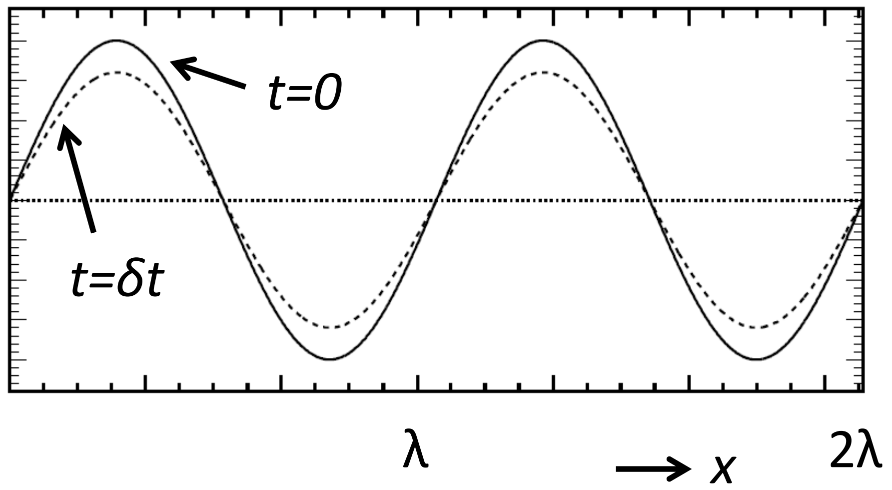
$$\begin{aligned} y(x,t) &= A\sin(kx-\omega t)+A\sin(kx+\omega t) \\ &= 2A\sin kx\cos \omega t \end{aligned}$$

or, if you prefer

$$y(x,t) = 2A\sin\left(\frac{2\pi x}{\lambda}\right)\cos\left(\frac{2\pi t}{T}\right)$$

i.e. has factorised into space and time-dependent parts. This means every point on string is moving with a certain time-dependence (*cosωt*), but the amplitude of the motion is a function of the distance from the end of the string

An example – a string on two which two wavelengths are excited



Stationary points are the nodes – occur every λ/2. Between these are the antinodes.

Standing waves

Boundary condition that each end of a fixed string must be a node...

$$y(x,t) = 2A\sin\left(\frac{2\pi x}{\lambda}\right)\cos\left(\frac{2\pi t}{T}\right) \quad \text{with} \quad y(0,t) = y(L,t) = 0$$

...means that only certain discrete frequencies – the modes – are available. These modes are multiples of the basic mode, which is the *fundamental* .

Wavelengths must 'fit', hence wavelength of mode *n*

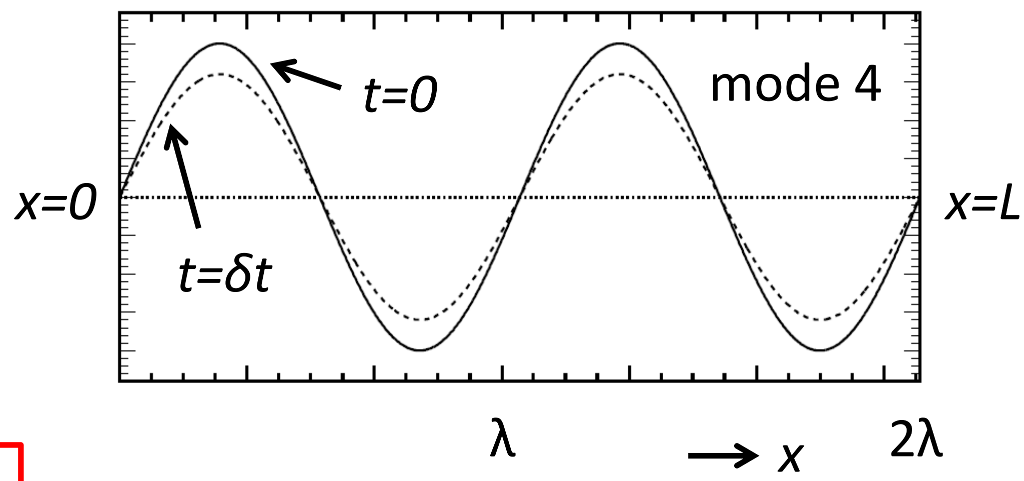
$$\lambda_n = \frac{2L}{n}$$

Recalling

$$c = \sqrt{T/\rho}$$

gives angular frequency of mode *n*

$$\omega_n = n\frac{\pi}{L}\sqrt{\frac{T}{\rho}}$$



We already obtained this result when discussing 'lumpy string' with *N* large!

Standing waves – violin string

E string of a violin is to be tuned to a frequency of 640 Hz. Its length and mass (from bridge to end) are 33 cm and 0.125 g respectively.

What tension is required?

Fundamental given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

$$\Rightarrow T = (2Lf)^2 \rho$$

Above parameters give 68 N

Transverse waves in nature: EM radiation

The most important example of waves in nature is electromagnetic radiation, *i.e.* light etc. This will be properly covered in EM lectures. Here is just a taster.

James Clerk Maxwell
1831-1879

Maxwell's equations in free space for electric field **E**, and magnetic inductance **B**

$$\nabla \cdot \mathbf{E} = 0 \qquad (1) \qquad \nabla . \mathbf{B} = 0 \qquad (3)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (2) \qquad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad (4)$$

ε_0 = permittivity of free space = 8.854 x 10^{-12} F/m
μ_0 = permeability of free space = 4π x 10^{-7} Hm^{-1}

Vector calculus identity plus (3):

$$\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B} + \nabla(\nabla \cdot \mathbf{B}) = -\nabla^2 \mathbf{B}$$

Making use of (4) and (2):

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial (\nabla \times \mathbf{E})}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Transverse waves in nature: EM radiation

Maxwell's equations in free space yield:

$$\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

(equivalent expression is obtainable for **E**)

which is the wave equation with $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.997 \times 10^8 \ \mathrm{ms^{-1}}$ → the speed of light!

Let's then consider a wave travelling in z-direction (using complex notation):

$$\begin{aligned} \mathbf{B} &= \mathbf{B_0} \exp[i(kz - \omega t)] \\ &= (B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}) \exp[i(kz - \omega t)] \end{aligned}$$

(that we take real part is implicit)

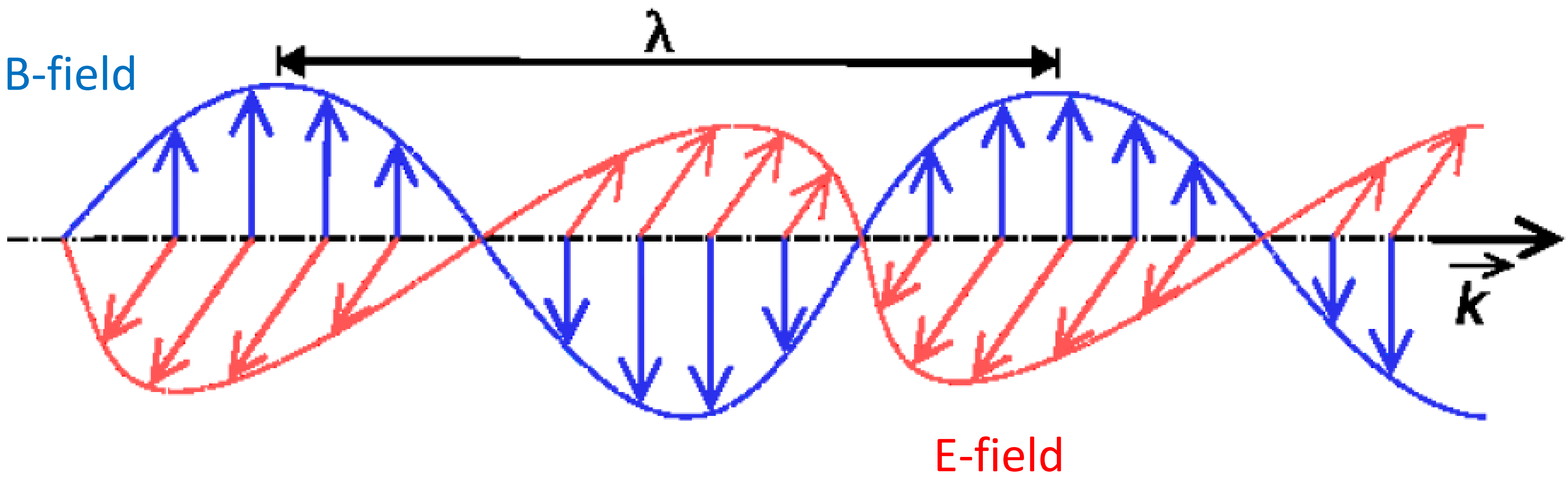
From Maxwell $\nabla \cdot \mathbf{B} = 0$, that is $\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$ and so $kB_z = 0$

B field has no component in direction of propagation - any oscillation is in transverse plane. Same argument follows for E field → wave is transverse!

Maxwell's equations also imply that E and B vectors are transverse to each other (not shown here – exercise for the student!)

Transverse waves in nature: EM radiation

EM waves in vacuum: both E and B vectors oscillate transverse to the direction of propagation and, in phase, transverse to each other



Transverse vs longitudinal waves

For coupled oscillators we considered both transverse and longitudinal excitations. The same is true here – can certainly have longitudinal waves

Some systems support only transverse waves, some only longitudinal, some both

- Transverse only: stretched string, EM waves in vacuum...
- Longitudinal only: sound waves in air – this because air has no elastic resistance to change in shape, only to change in density
- Both: stretched spring, crystal...

Transverse waves have an important attribute not available to longitudinal waves:

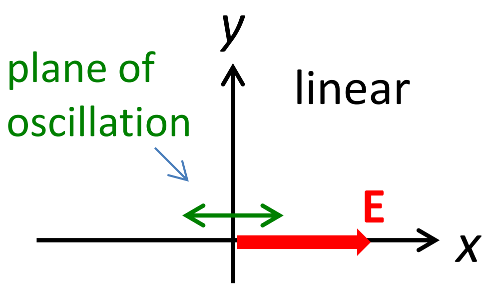
POLARISATION

Polarisation

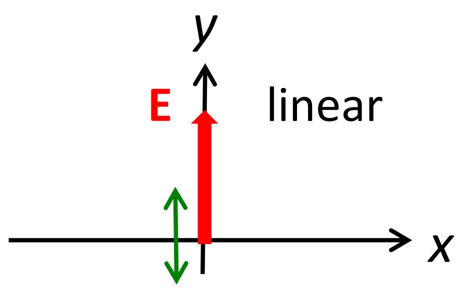
Transverse vibrations can be in one of two directions (or both) orthogonal to the direction of wave propagation. We talk of two different directions of *polarisation*.

(It can even be that wave velocities are different for the two polarisation states, due to *e.g.* the different interatomic spacings in a crystal.)

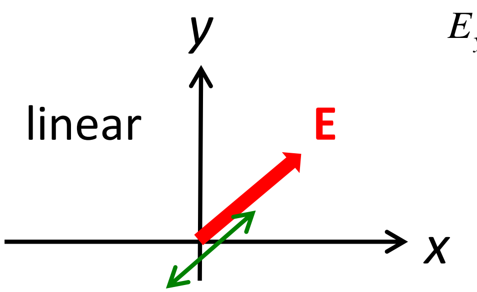
Some possibilites for polarisation of E vector in EM wave travelling in *z*-direction:



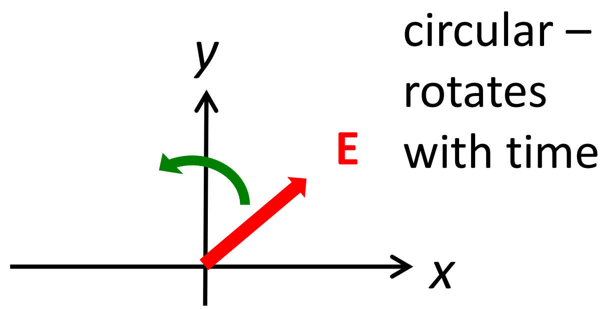
$$E_x = E_0 \cos(kz - \omega t)$$
$$E_y = 0$$



$$E_x = 0$$
$$E_y = E_0 \cos(kz - \omega t)$$



$$E_x = \left(E_0 / \sqrt{2}\right)\cos(kz - \omega t)$$
$$E_y = \left(E_0 / \sqrt{2}\right)\cos(kz - \omega t)$$



$$E_x = E_0 \cos(kz - \omega t)$$
$$E_y = E_0 \sin(kz - \omega t)$$

Dispersion

For our stretched string we found that the wave velocity is, $c = \sqrt{T / \rho}$ *i.e*. depends only on properties of string and has no dependence on frequency (or wavelength) of wave. But this is an idealised system!

For most systems the velocity of a wave *does* have a dependence on ω and λ

→DISPERSION

One well known example is light in a prism. Light in a medium *m* with refractive index *n* Has a velocity c_m, where $c_m = c / n$.

But the refractive index, and hence wave velocity, varies with wavelength. Hence light is bent at different angles by prism according to wavelength.

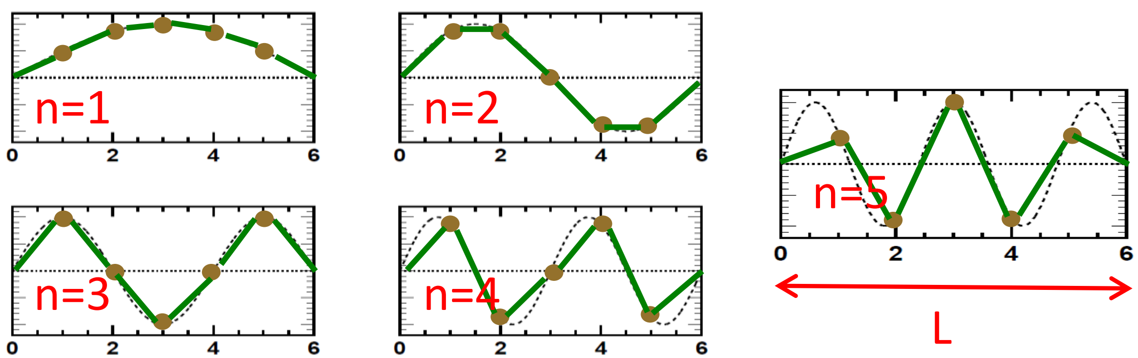
Dispersion – lumpy string revisited

The stretched string has an idealised mass / unit length.
But earlier we analysed normal modes of the lumpy string. We found:

$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \quad \text{with} \quad \omega_0 = \sqrt{T/mL}$$

and $\lambda_n = 2L/n$; also we have $k_n \equiv 2\pi/\lambda_n = n\pi/L$

Recall normal modes for *N=5*:

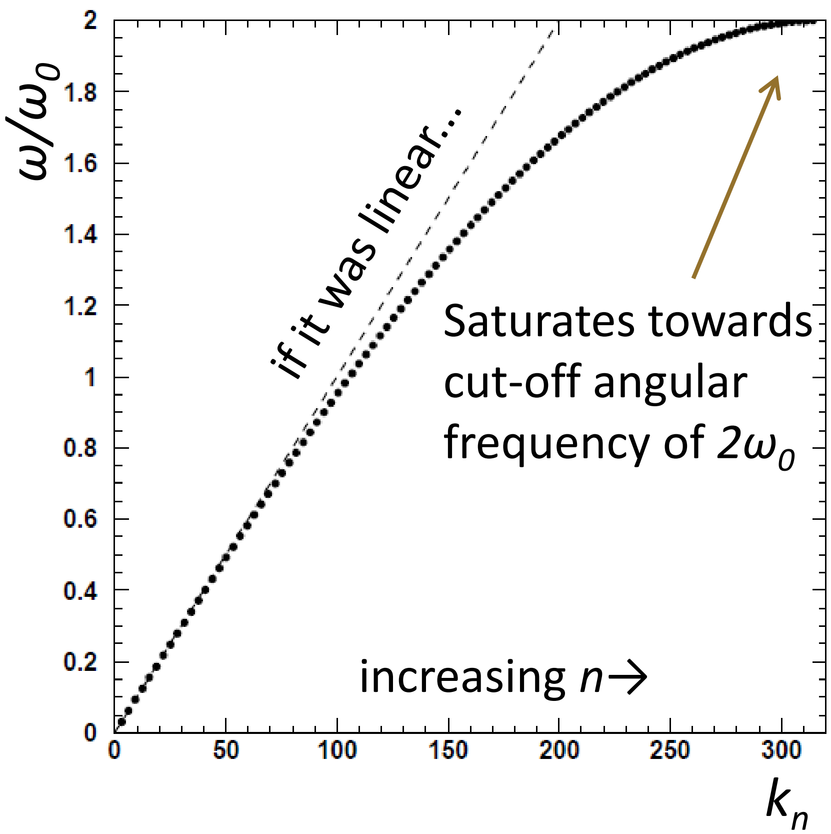


Look at behaviour of ω_n vs *k* (for *n=1...N*), recalling that wave speed=*ω/k*

Dispersion curve for lumpy string

For a lumpy string with N=100 masses (other properties arbitrary) calculate ω and k for each normal mode

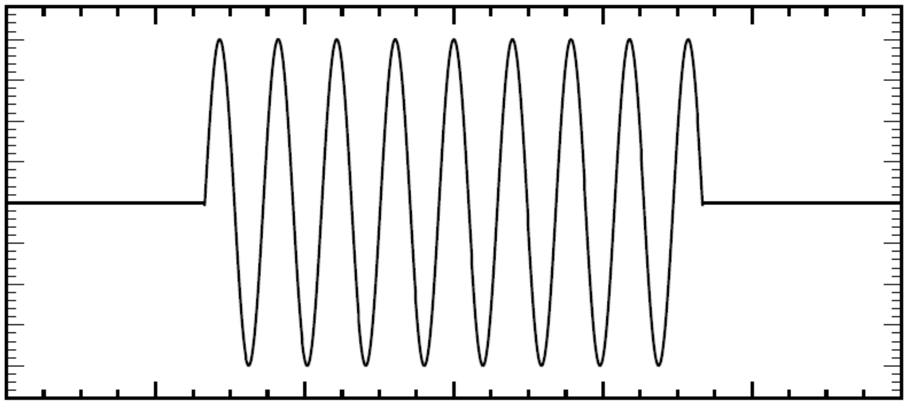
This is not linear! Velocity of wave corresponding to each mode depends on ω (or k). This is dispersion.



Note also that there is a 'cut-off' frequency – a maximum frequency above which it is not possible to excite system/transmit waves – this is a property often found in a dispersive system.

Information transfer & wave packets

To transmit information it is necessary to *modulate* a wave.
Consider the simplest case of turning a wave on and then off:



For a certain range of *(kx-ωt)* this signal has displacement *y=A*sin*(kx-ωt)*, outside this range the displacement *y=0*. *This is not a single wave*, for which *y=A*sin*(kx-ωt)* would apply for *all (kx-ωt)*! It is in fact a wave packet.

$$y(x,t)=\sum_{n=1}^{N} D_n \cos(k_n x-\omega_n t)$$

A wave packet can be formed by summing together (a possibly infinite number of) waves of different frequencies – this is a Fourier series (2nd year topic)

Modulation

A pure sine wave carries no information – to encode information for radio transmission need to modulate the wave.   General principle as follows:

Signal, typically characterised by low frequency variation (e.g. voice: a few 100 Hz -1kHz)

Carrier wave
High frequency
(e.g. ~ MHz)

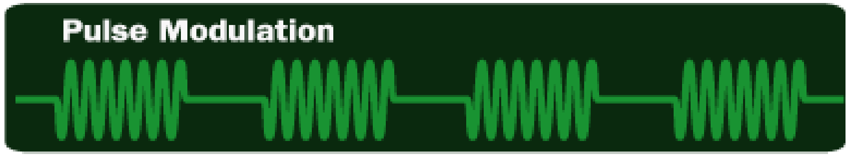
Carrier signal is modulated

Modulated signal, which is transmitted, received and then de-modulated

Various options exist for the modulation strategy

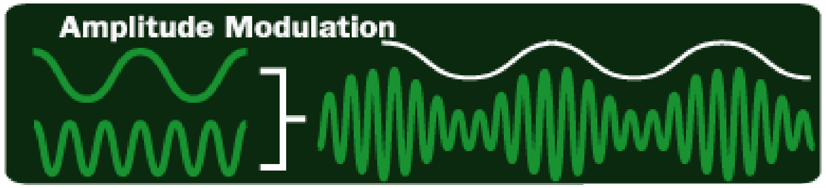
Modulation strategies

Pulse modulation



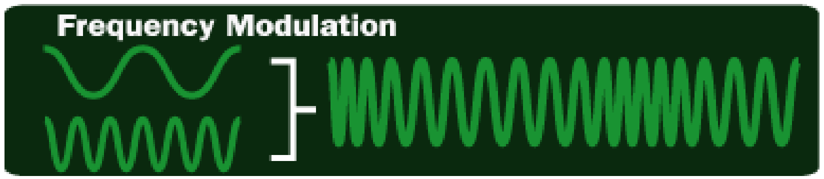
Simply turn sine wave off and on, e.g. morse code

Amplitude modulation



Modulate amplitude, e.g. $(\text{Offset} + \text{signal}(t)) \times \sin[2\pi f_{carrier}\, t]$

Frequency modulation



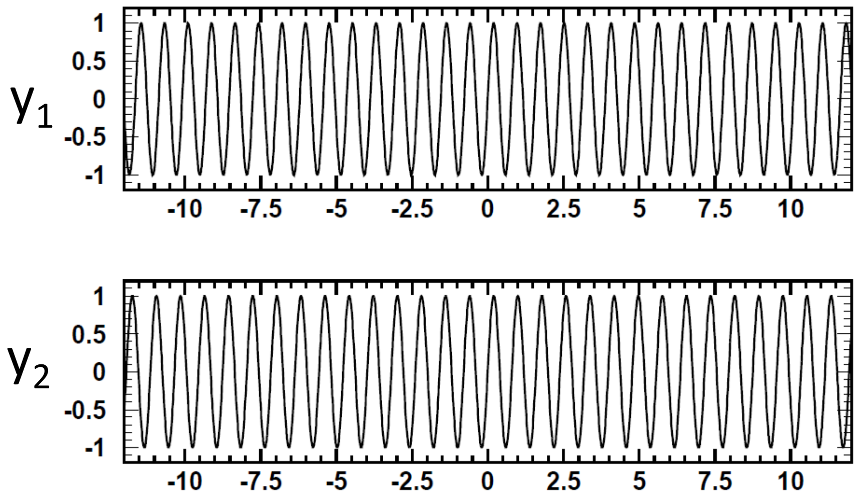
Encode information in modulation of frequency (also phase modulation)

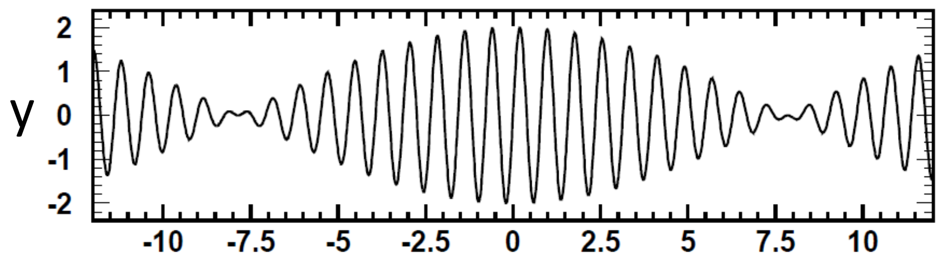
Wave packets – a toy example

Sum together two waves which differ by $2\delta\omega$ and $2\delta k$ in angular frequency and wave-number, respectively:

$$y_1 = A\sin[(k+\delta k)x-(\omega+\delta\omega)t]$$
$$y_2 = A\sin[(k-\delta k)x-(\omega-\delta\omega)t]$$

to give $y = y_1 + y_2 = 2A\cos(\delta k\, x - \delta\omega\, t)\sin(kx-\omega t)$





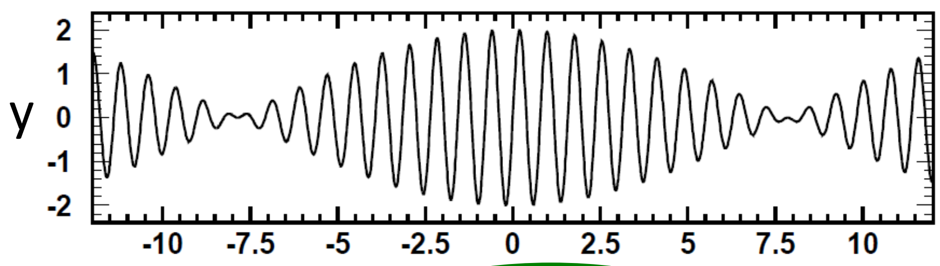
Not exactly a packet, more an infinite series of sausages – would need an infinite number of input waves to make a discrete wave packet

Group velocity

The velocity of the wave packet is known as the *group velocity*.
In almost all cases this is the velocity at which information is transmitted.

In a dispersive medium the group velocity is *not* the same as the velocity of the individual waves, which is known as *the phase velocity* (& in a dispersive medium the phase velocity, ω/k, varies with frequency & wavelength)

Consider our toy example:

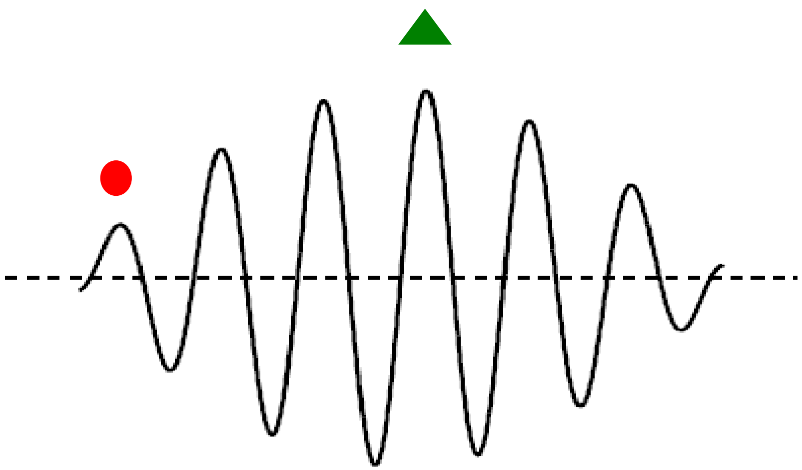


$$y = y_1 + y_2 = 2A\cos(\delta k\, x - \delta\omega\, t)\sin(kx - \omega t)$$

Describes envelope – so envelope moves with velocity $\frac{\delta\omega}{\delta k}$ and indeed

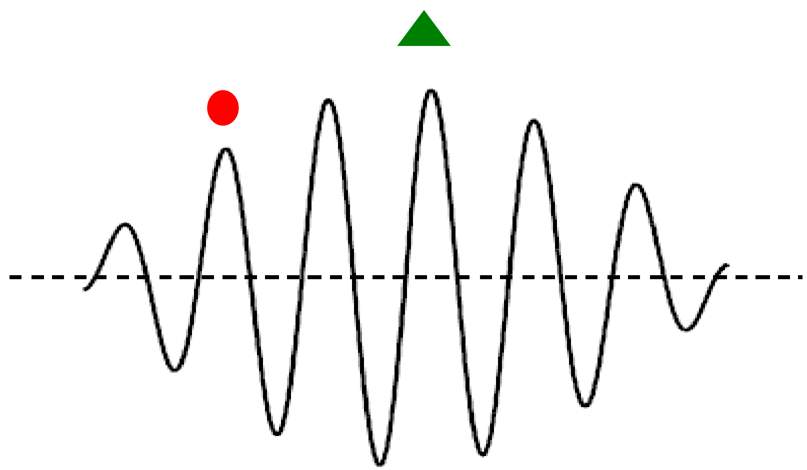
Group velocity $v_g = \frac{d\omega}{dk}$ while phase velocity $v_p = \frac{\omega}{k}$

Cartoon of wave packet (1/7)



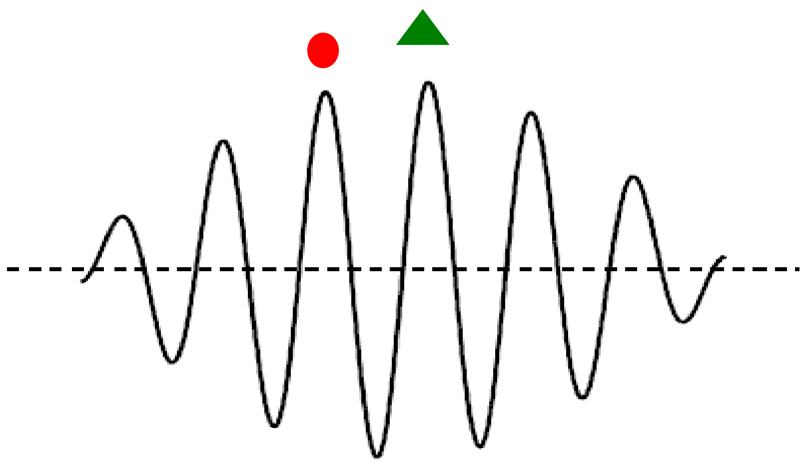
A travelling wave packet. The ▲ marks the top of the wave packet which moves with the group velocity. The ● indicates a component wave crest which enters the packet, moves through it, and leaves, with the phase velocity.

Cartoon of wave packet (2/7)



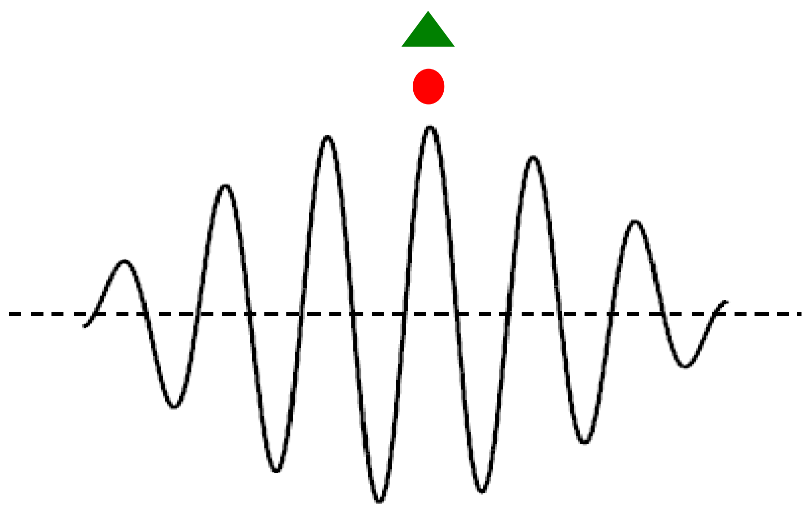
A travelling wave packet. The ▲ marks the top of the wave packet which moves with the group velocity. The ● indicates a component wave crest which enters the packet, moves through it, and leaves, with the phase velocity.

Cartoon of wave packet (3/7)



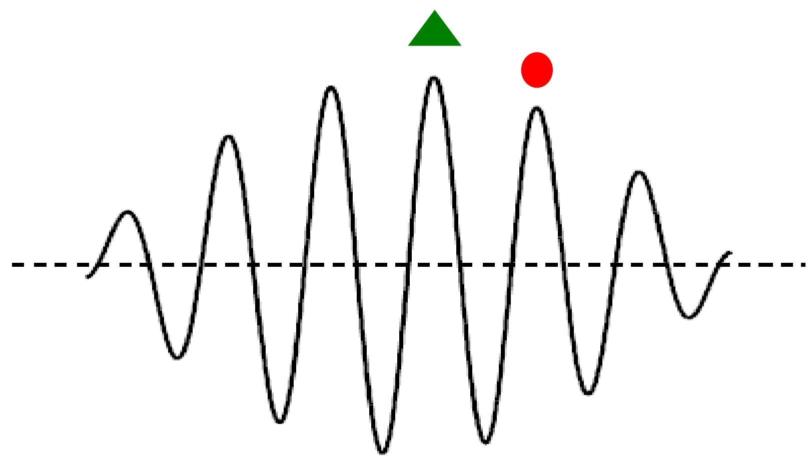
A travelling wave packet. The ▲ marks the top of the wave packet which moves with the group velocity. The ● indicates a component wave crest which enters the packet, moves through it, and leaves, with the phase velocity.

Cartoon of wave packet (4/7)



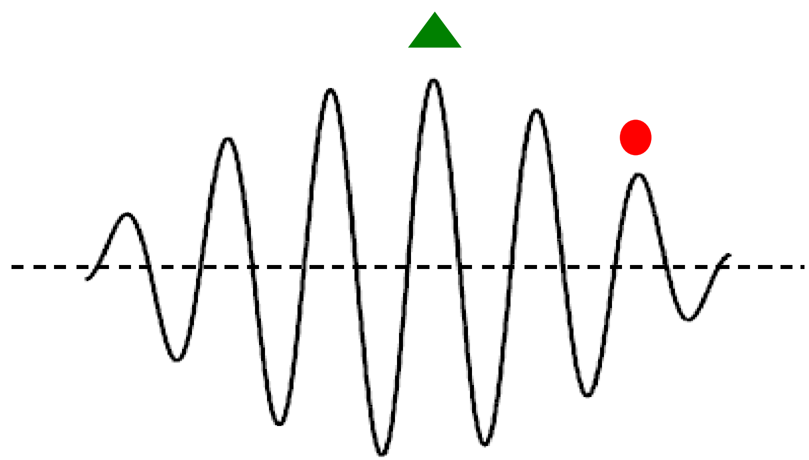
A travelling wave packet. The ▲ marks the top of the wave packet which moves with the group velocity. The ● indicates a component wave crest which enters the packet, moves through it, and leaves, with the phase velocity.

Cartoon of wave packet (5/7)



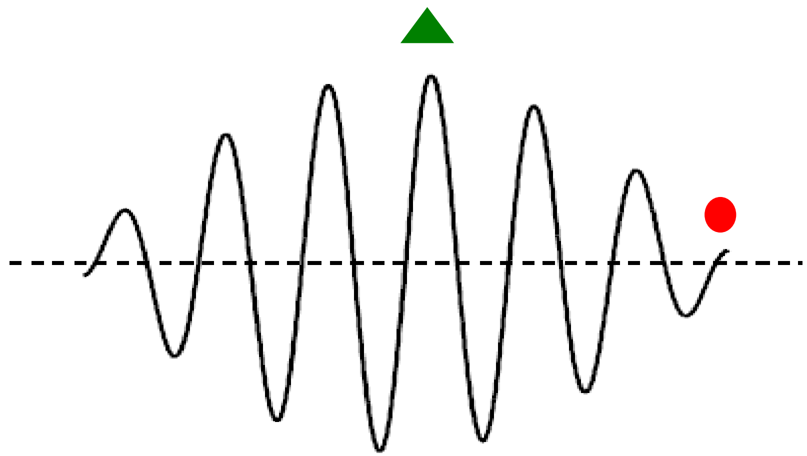
A travelling wave packet. The ▲ marks the top of the wave packet which moves with the group velocity. The ● indicates a component wave crest which enters the packet, moves through it, and leaves, with the phase velocity.

Cartoon of wave packet (6/7)



A travelling wave packet. The ▲ marks the top of the wave packet which moves with the group velocity. The ● indicates a component wave crest which enters the packet, moves through it, and leaves, with the phase velocity.

Cartoon of wave packet (7/7)



A travelling wave packet. The ▲ marks the top of the wave packet which moves with the group velocity. The ● indicates a component wave crest which enters the packet, moves through it, and leaves, with the phase velocity.

Different expressions for the group velocity

We have already stated $$v_g = \frac{d\omega}{dk}$$

but $\omega = v_p k$ so $$v_g = v_p + k\frac{dv_p}{dk}$$

also, since $k = 2\pi / \lambda$ $$v_g = v_p - \lambda\frac{dv_p}{d\lambda}$$

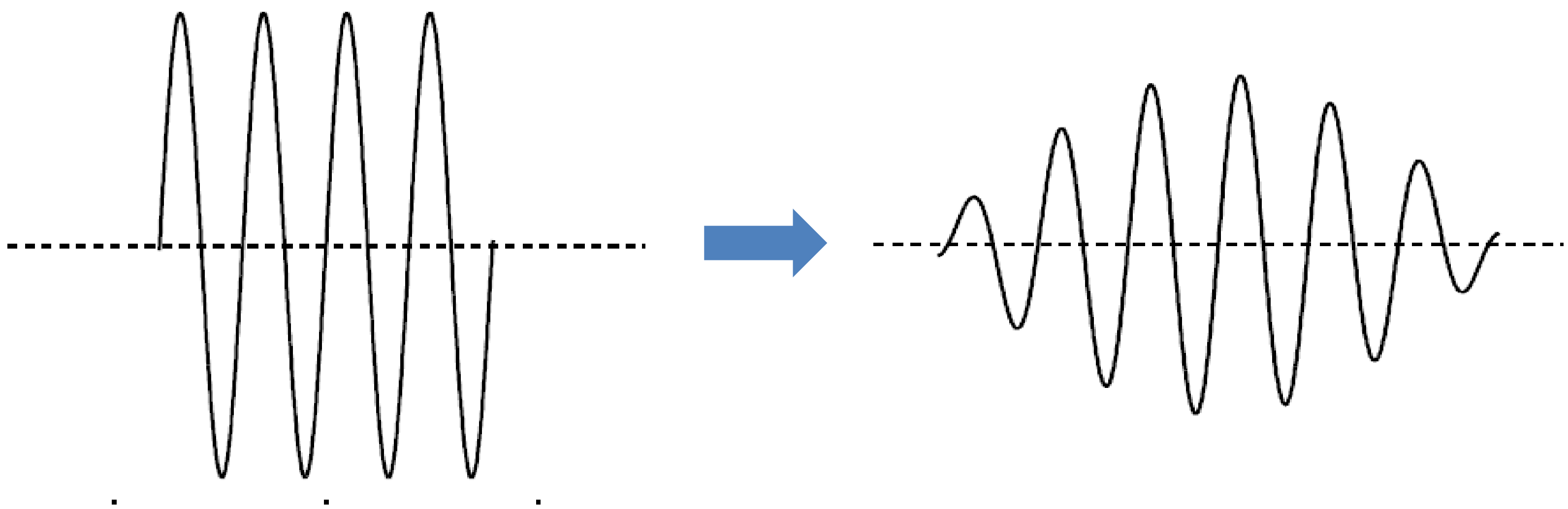
or if considering light, & a medium with refractive index n, we have $v_p = c / n$

$$v_g = \frac{c}{n}\left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right)$$

Observe that $v_g \neq c / n$!

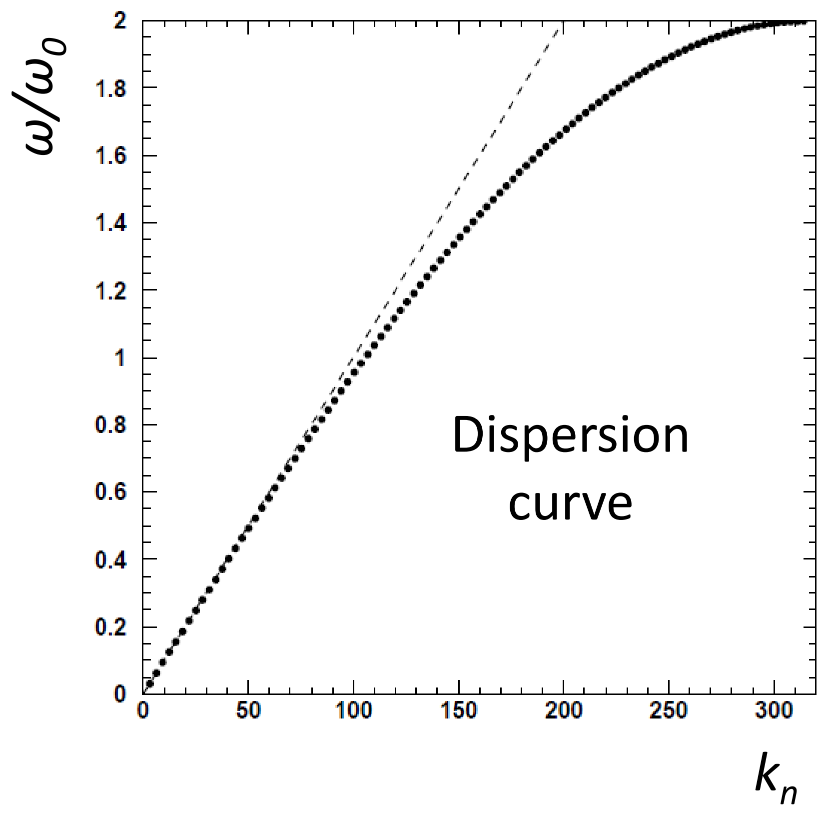
Dispersion and the spreading of the wave packet

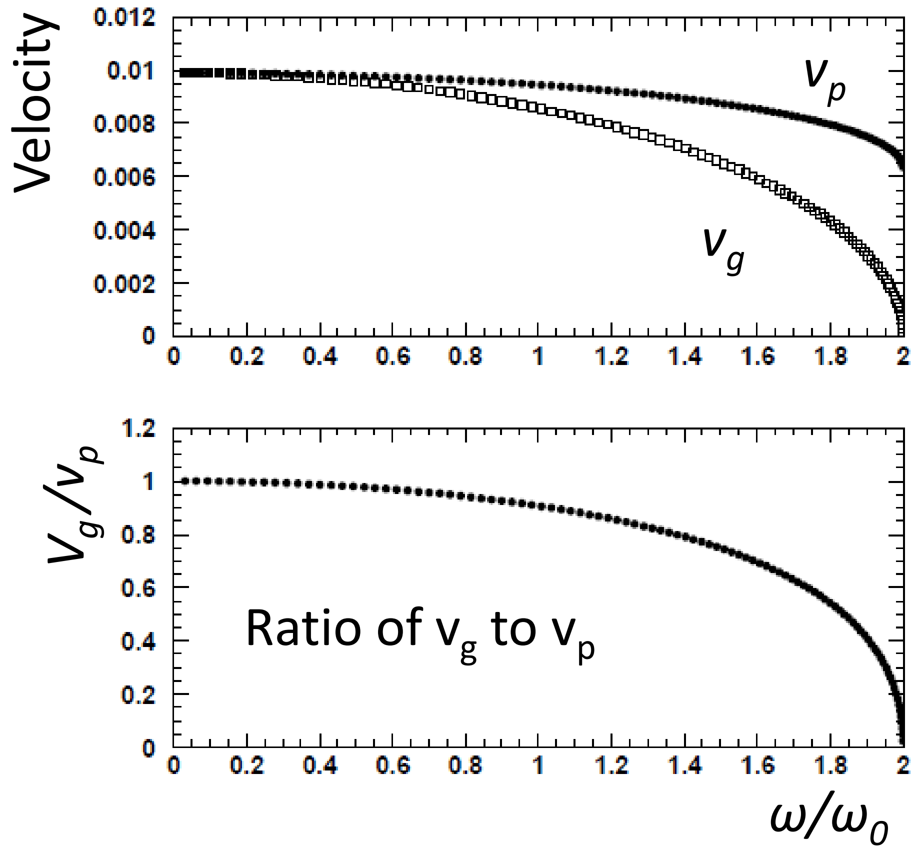
Another consequence of dispersion is that a wave-packet will not retain its shape perfectly, but will spread out. Can have consequences for signal detection



Group and phase velocities for lumpy string

Calculate phase and group velocity for the lumpy string with N=100





Phase and group velocity ~ the same at first, but $v_g \rightarrow 0$ as $\omega \rightarrow 2\omega_0$ (cut-off)

Waves in deep water

Waves in water with $\lambda > 2$ cm (below which surface tension effects are important), but still small compared to water depth, have a dispersion relation

$$\omega \approx \sqrt{(gk)}$$

i.e. driven by gravity, hence often called gravity waves - ocean swell

$$v_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\frac{\omega}{k} = \frac{1}{2}v_p$$

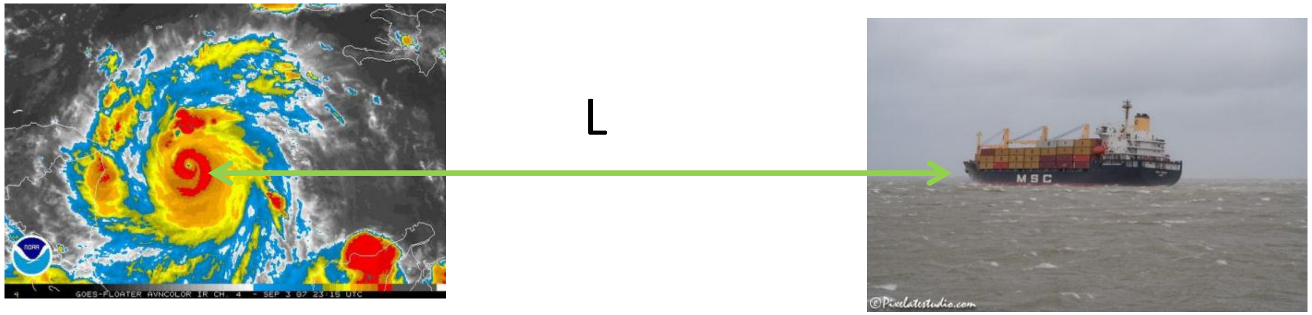
So group velocity is half that of phase velocity

Estimating how far away a sea storm is

Dispersion relation $\omega \approx \sqrt{(gk)}$ means that $v_p = \sqrt{(g\lambda)/(2\pi)}$ and so waves of longer wavelengths travel more quickly.

Storm

Measure interval between successive wave crests a long way (L) distant



Period of wave:

$$\tau = \frac{\lambda}{v_p} = (2\pi\lambda / g)^{1/2}$$

Time of arrival of wave crests:

$$t = t_0 + L / v_p = t_0 + L\tau / \lambda$$

Eliminate λ:

$$\tau = 2\pi L / g(t - t_0)$$

Hence

$$-\frac{d\tau}{dt} = 2\pi L / g(t - t_0)^2 = g\tau^2 / 2\pi L$$

So if measure period and rate of decrease in period can obtain L !

e.g. if L=1000 km and τ=10 s then -dτ/dt= 1.2 x 10^{-4} (0.6 s / hour)