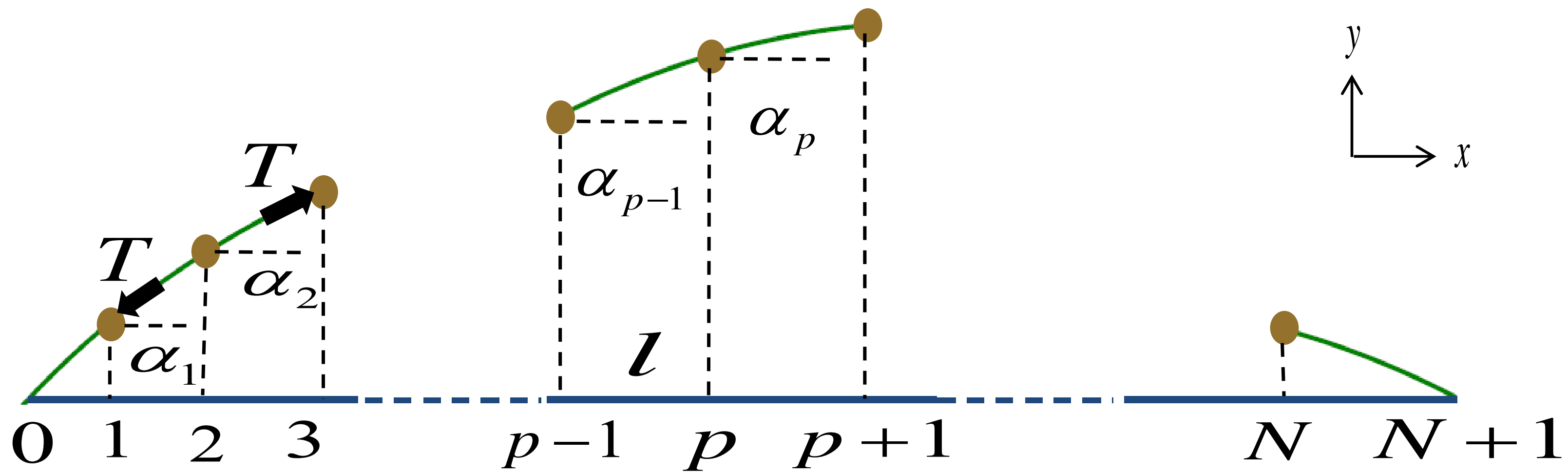
Waves

Waves 1

1. N coupled oscillators – towards the continuous limit
2. Stretched string and the wave equation
3. The d'Alembert solution
4. Sinusoidal waves, wave characteristics and notation

N coupled oscillators

Consider flexible elastic string to which are attached *N* particles of mass *m*, each a distance *l* apart. The string is fixed at each end. Small transverse displacements are applied → transverse oscillations



All angles small, *i.e.* $\sin\alpha_i \approx \tan\alpha_i \approx \alpha_i$ and $\cos\alpha_i \approx 1-\alpha_i^2/2 \approx 1$

No significant horizontal force, since $-T\cos\alpha_{p-1} + T\cos\alpha_p \approx 0$. But vertically:

$$F_p = m\ddot{y}_p = -T\sin\alpha_{p-1} + T\sin\alpha_p$$
$$= -\frac{T}{l}(y_p - y_{p-1}) + \frac{T}{l}(y_{p+1} - y_p)$$

$$\Rightarrow \ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = 0$$

with $\omega_0^2 = T/ml$

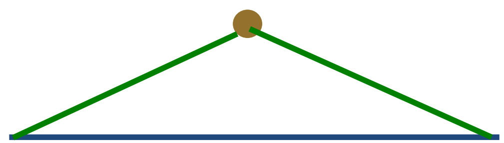
N coupled oscillators: special cases

First consider the special cases *N=1* and *N=2*

$$\omega_0^{\,2} = T/ml$$

N=1

$$\ddot{y}_1 + 2\omega_0^{\,2} y_1 = 0 \quad \Rightarrow \quad \boxed{\omega = \sqrt{2}\omega_0}$$

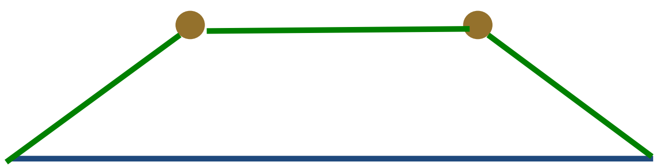


N=2

$$\ddot{y}_1 + 2\omega_0^{\,2} y_1 - \omega_0^{\,2} y_2 = 0$$
$$\ddot{y}_2 + 2\omega_0^{\,2} y_2 - \omega_0^{\,2} y_1 = 0$$

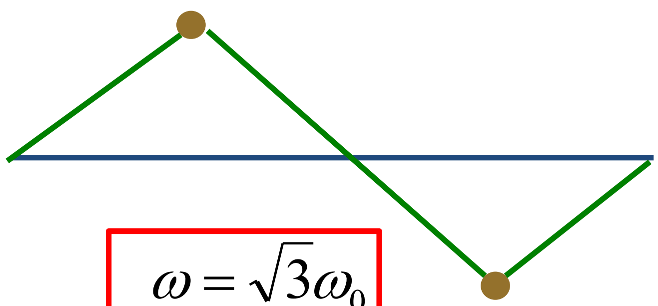
Try $q_1 = \frac{1}{\sqrt{2}}(y_1 + y_2)$ and $q_2 = \frac{1}{\sqrt{2}}(y_1 - y_2)$

$$\Rightarrow \quad \ddot{q}_1 + \omega_0^{\,2} q_1 = 0$$



$$\boxed{\omega = \omega_0}$$

$$\Rightarrow \quad \ddot{q}_2 + 3\omega_0^{\,2} q_2 = 0$$



$$\boxed{\omega = \sqrt{3}\omega_0}$$

N coupled oscillators: general case

We have

$$\ddot{y}_p + 2\omega_0^{\ 2} y_p - \omega_0^{\ 2}(y_{p+1} + y_{p-1}) = 0$$

Consider trial solution $y_p = A_p \cos \omega t$

(For simplicity we're not allowing for additional phase offset , *e.g.* $y_p = A_p \cos(\omega t + \phi_p)$. Equivalent to imposing all masses start at rest)

Substituting in trial solution gives *N* equations:

$$(-\omega^2 + 2\omega_0^{\ 2})A_p - \omega_0^{\ 2}(A_{p+1} + A_{p-1}) = 0 \qquad (p = 1,2,\ldots,N)$$

$$\Rightarrow \frac{A_{p-1} + A_{p+1}}{A_p} = \frac{-\omega^2 + 2\omega_0^{\ 2}}{\omega_0^{\ 2}}$$

(For *p=0* and *p=N+1* we know A_p*=0*)

Since ω is same for all masses we see RHS can't depend on *p* & so nor can LHS

N coupled oscillators: general case

Look for form for A_p that

1. Leaves $\dfrac{A_{p-1} + A_{p+1}}{A_p}$ independent of p
2. Satisfies $A_p=0$ for $p=0$ and $N+1$

Try $A_p = C \sin p\theta$ so

$$A_{p-1} + A_{p+1} = C\left[\sin(p-1)\theta + \sin(p+1)\theta\right] = 2C \sin p\theta \cos\theta$$

$$\Rightarrow \frac{A_{p-1} + A_{p+1}}{A_p} = 2\cos\theta$$ which satisfies 1.

and requiring $(N+1)\theta = n\pi \quad (n = 1,2,3,\ldots)$ satisfies 2.

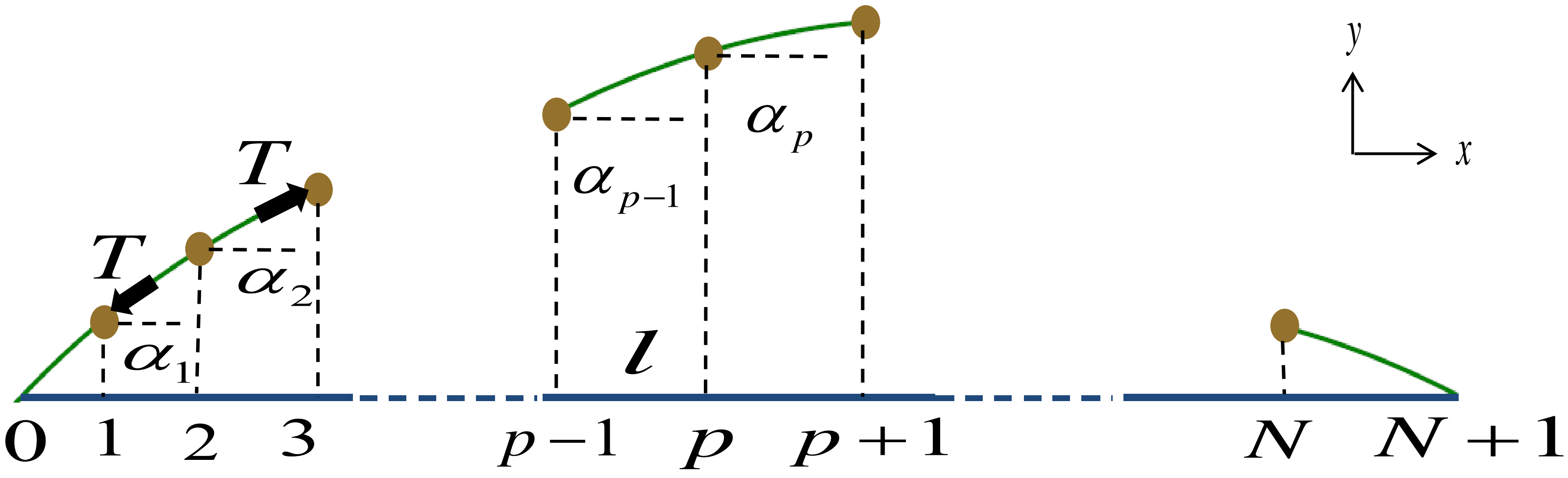
$$\boxed{A_p = C \sin\left(\frac{pn\pi}{N+1}\right)}$$

$$\frac{A_{p-1} + A_{p+1}}{A_p} = \frac{-\omega^2 + 2\omega_0^{\,2}}{\omega_0^{\,2}} = 2\cos\left(\frac{n\pi}{N+1}\right)$$

$$\Rightarrow \omega^2 = 2\omega_0^{\,2}\left[1 - \cos\left(\frac{n\pi}{N+1}\right)\right]$$

$$\boxed{\Rightarrow \omega = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right]}$$

N coupled oscillators: the solution



Displacement for mass *p* when oscillating in mode *n* and angular frequency:

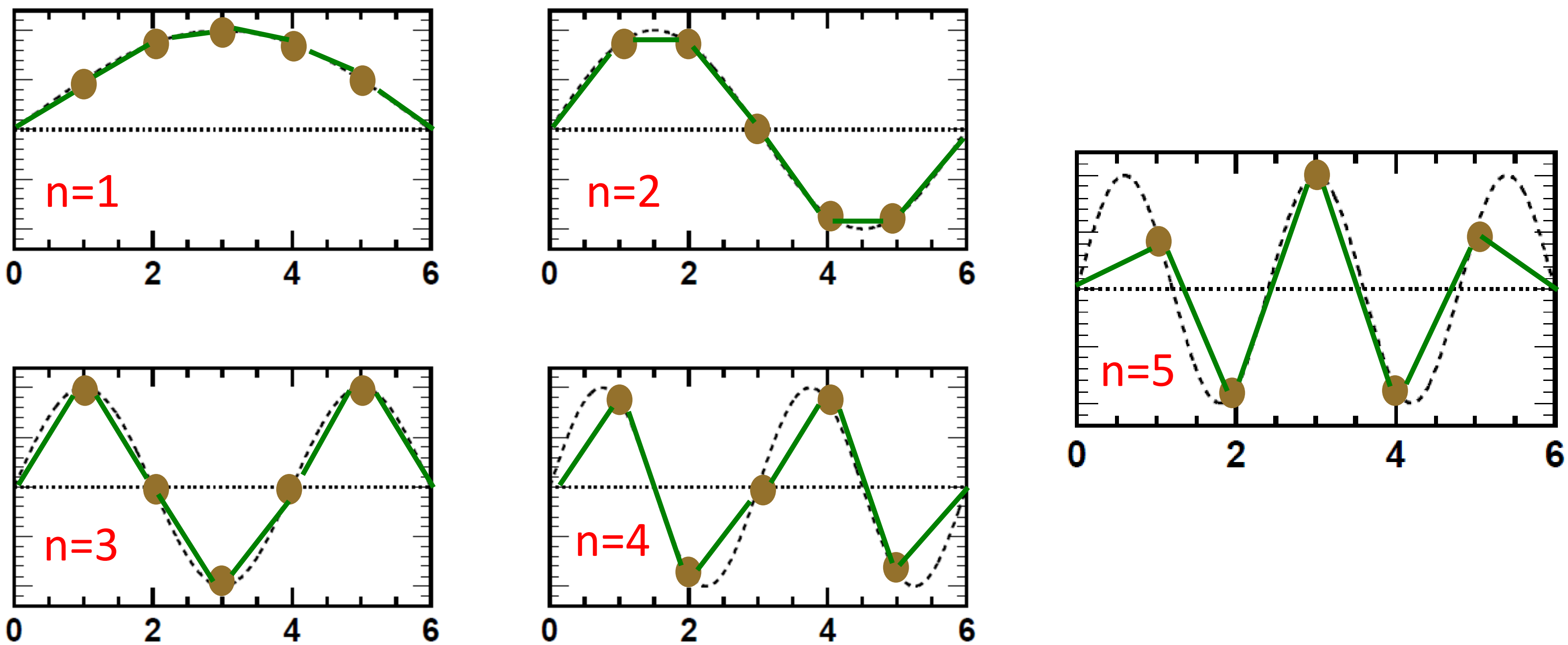
$$y_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right)\cos(\omega_n t + \phi_n) \qquad \omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \qquad \omega_0 = \sqrt{T/ml}$$

(note we have smuggled back in the phase offset φ_n)

Although the value of *n* can go beyond *N*, this just generates duplicate solutions, *i.e.* there are *N* normal modes in total.

N coupled oscillators: modes for N=5

Look at each mode for *N=5*, with snapshot taken at *t=0*



Note how the displacement of every particle falls on a sine curve!

N coupled oscillators: N very large

Let's consider string-mass system of fixed length *L* and mass *M*, *i.e.*

$$L = (N+1)l \qquad M = Nm$$

and define linear density $\rho \equiv m/l$

We'll focus on case *N* very large, which is starting to approximate to real string

***n* is small**

Look first at mode numbers which are low in value compared with *N*

$$\omega_n = 2\sqrt{\frac{T}{ml}}\sin\left[\frac{n\pi}{2(N+1)}\right] \qquad \Rightarrow \omega_n \approx 2\sqrt{\frac{T}{m/l}}\frac{n\pi}{2(N+1)l}$$

$$\Rightarrow \omega_n \approx n\frac{\pi}{L}\sqrt{\frac{T}{\rho}}$$

so normal frequencies are integer multiples of a lowest frequency ω_1

$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{T}{\rho}}$$

N coupled oscillators: N very large

***n* is small**

We have $\omega_n \approx n\frac{\pi}{L}\sqrt{\frac{T}{\rho}}$, what about displacements?

General result $y_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right)\cos(\omega_n t)$. Now separation between particles becomes smaller & we approach a continuous variable $x = pl$

$$y_n(x,t) = C_n \sin\left(\frac{xn\pi}{L}\right)\cos(\omega_n t)$$

i.e. string gets closer and closer to lying on a sine curve

N coupled oscillators: N very large

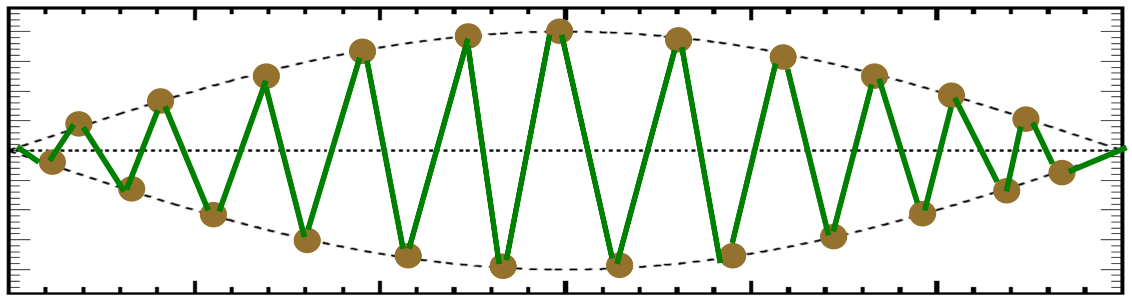
n* =*N

Look first at frequency of highest mode → $\omega_N = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right]$

$$\approx 2\omega_0$$

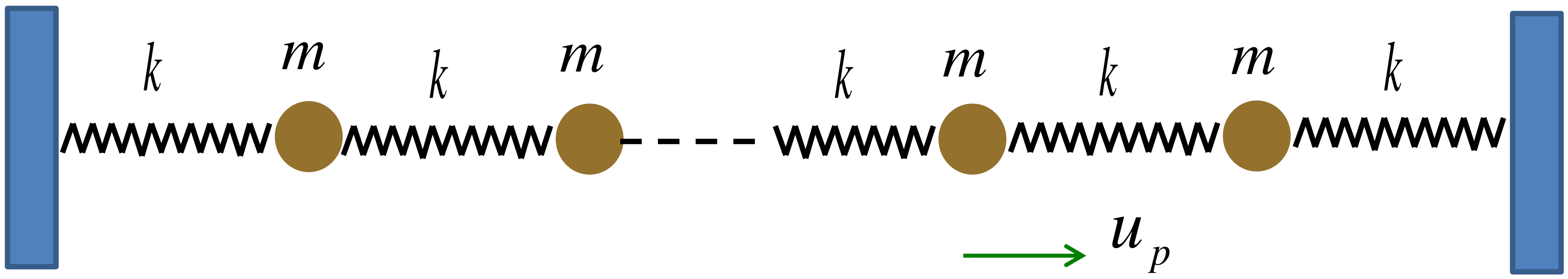
Now consider displacement of successive masses → $\frac{A_p}{A_{p+1}} = \frac{\sin\left(\frac{pN\pi}{N+1}\right)}{\sin\left(\frac{(p+1)N\pi}{N+1}\right)} \approx \frac{\sin(p\pi)}{\sin(p\pi+\pi)} \approx -1$

So we have something like this →



Realising this, we see eqn of motion for a given mass is $m\ddot{y} = -2T\frac{2y}{l}$ from which we can recover the result $\omega_N \approx 2\omega_0$

System of springs and N masses: longitudinal oscillations



Let u_p be displacement from equilibrium position of mass *p*

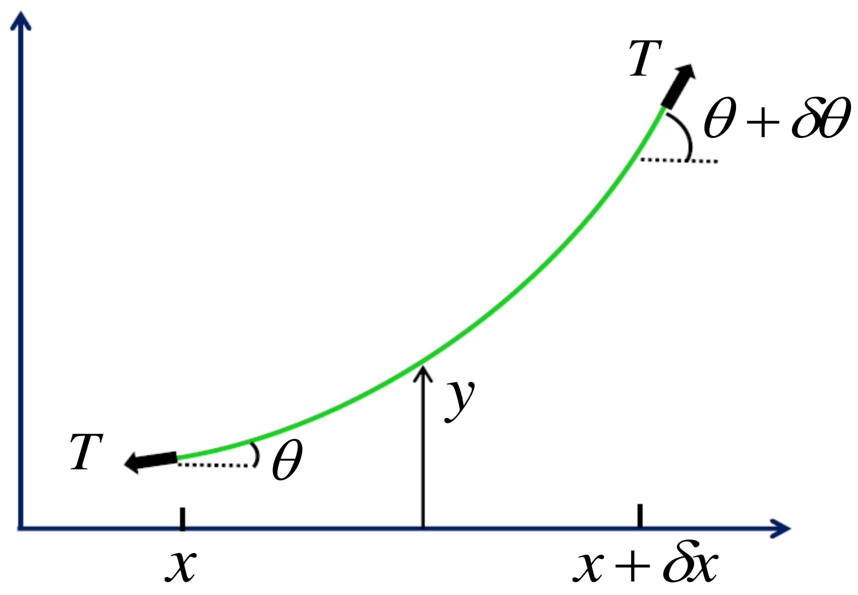
$$m\ddot{u}_p = k(u_{p+1} - u_p) + k(u_{p-1} - u_p)$$

$$\Rightarrow \ddot{u}_p + 2\omega_0^2 u_p - \omega_0^2 (u_{p+1} + u_{p-1}) = 0 \qquad \text{with} \quad \omega_0^2 = k/m$$

Equations of motion have same form as for masses on string → same solutions!

Stretched string

Consider a segment of string of linear density ρ stretched under tension T



$\theta, \delta\theta$ small

$$F_y = T\sin(\theta+\delta\theta) - T\sin\theta$$
$$F_x = T\cos(\theta+\delta\theta) - T\cos\theta$$

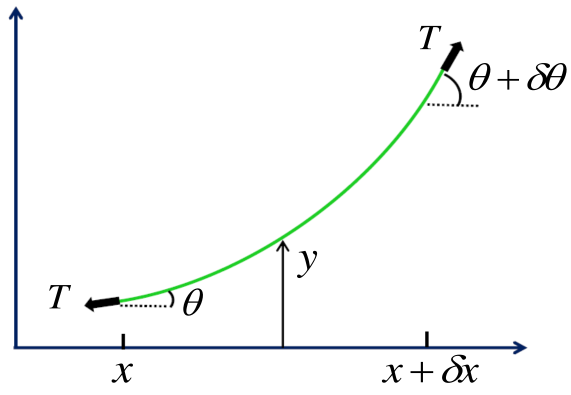
since $\delta\theta$ small

$$F_y \approx T\delta\theta$$
$$F_x \approx 0$$

and so $(\rho\,\delta x)\,a_y = T\,\delta\theta$

Stretched string

$$(\rho\,\delta x)\,a_y = T\,\delta\theta \quad \textit{i.e.} \quad (\rho\,\delta x)\frac{\partial^2 y}{\partial t^2} = T\,\delta\theta$$



Note the partial double derivative, rather than $\ddot{y}$ which implies $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}$. *y* depends on both *x* and *t*.
This and much of what follows will concern partial differential equations.

Now $\quad \tan\theta = \frac{\partial y}{\partial x} \quad$ so $\quad \sec^2\theta \frac{\partial \theta}{\partial x} = \frac{\partial^2 y}{\partial x^2}$

and since $\theta, \delta\theta$ small $\quad \sec\theta \approx 1 \quad$ and $\quad \sec^2\theta \frac{\partial \theta}{\partial x} \approx \frac{\delta\theta}{\delta x}$

Therefore $\quad \frac{\delta\theta}{\delta x} \approx \frac{\partial^2 y}{\partial x^2} \quad$ and so

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \left(\frac{\rho}{T}\right)\frac{\partial^2 y}{\partial t^2}}$$

Stretched String and the wave equation

We have performed a similar analysis to the N oscillating masses on a string.

- Then we had N coordinates, $y_p(t)$, $p=1...N$.
- Now we have $y(x,t)$, x which are continuous variables

Have obtained

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\rho}{T}\right)\frac{\partial^2 y}{\partial t^2}$$

Now (ρ/T) has dimensions of 1/speed^2 and indeed we will see that these parameters do define the velocity of travelling waves on the string.

This is the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

$$c = \sqrt{\frac{T}{\rho}}$$

Jean-Baptiste le Rond d’Alembert

- 1717-1783
- Lived in Paris
- Mathematician and physicist
- Also a music theorist and co-editor with Diderot of a famous encyclopaedia

d'Alembert solution of wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

y is a function of *x* and *t*.
Define new variables so that
y is now a function of *u* and *v*

$$u = x - ct$$
$$v = x + ct$$

Chain rule to get first derivatives

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u}\cdot 1 + \frac{\partial y}{\partial v}\cdot 1$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial y}{\partial u}\cdot(-c) + \frac{\partial y}{\partial v}\cdot c$$

and then second derivatives

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t}\right) = \left(\frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v}\right)\left(-c\frac{\partial y}{\partial u} + c\frac{\partial y}{\partial v}\right) = c^2\left[\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right]$$

d'Alembert solution of wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

y is a function of *x* and *t*.
Define new variables so that
y is now a function of *u* and *v*

$$u = x - ct$$
$$v = x + ct$$

$$\frac{\partial^2 y}{\partial u \partial v} = 0$$

$$\Rightarrow y(u,v) = f(u) + g(v)$$

So general solution of wave equation is

$$y(x,t) = f(x-ct) + g(x+ct)$$

Here *f* and *g* are any functions of *(x-ct)* and *(x+ct)*.
They are determined by initial conditions.

Interpretation of d'Alembert solution

$$y(x,t) = f(x-ct) + g(x+ct)$$ is general solution of $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Lets focus on *f(x-ct)* part of solution $$y(x,t) = f(x-ct)$$

At time $t=t_1$:

$$y(x,t_1) = f(x-ct_1)$$

At time $t=t_2$:

$$\begin{aligned} y(x,t_2) &= f(x-ct_2) \\ &= f(x+ct_1-ct_2-ct_1) \\ &= f([x-c(t_2-t_1)]-ct_1) \end{aligned}$$

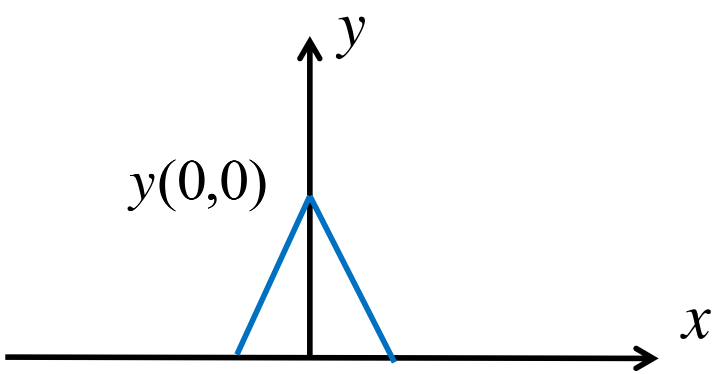
i.e. displacement at time t_2 and position x =
displacement at time t_1 a distance $c(t_2-t_1)$ to the left of x

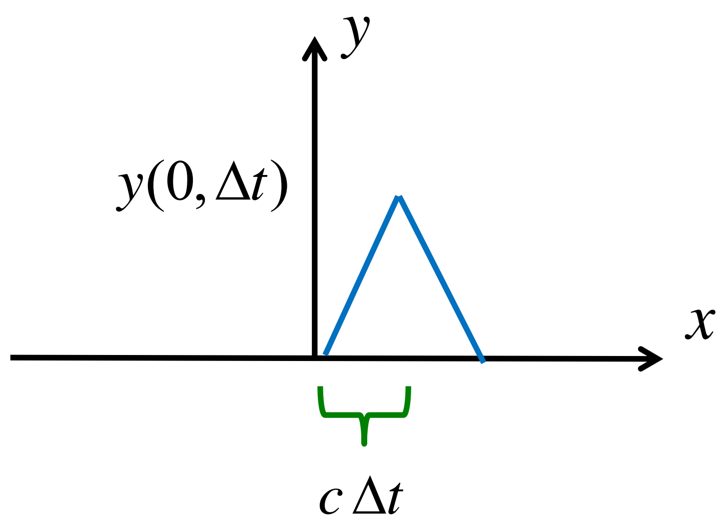
⟹ solution describes wave travelling to right with speed c

Interpretation of d'Alembert solution

$$y(x,t) = f(x - ct)$$

Focus on x=0 and consider situations at *t=0* and *t=Δt*



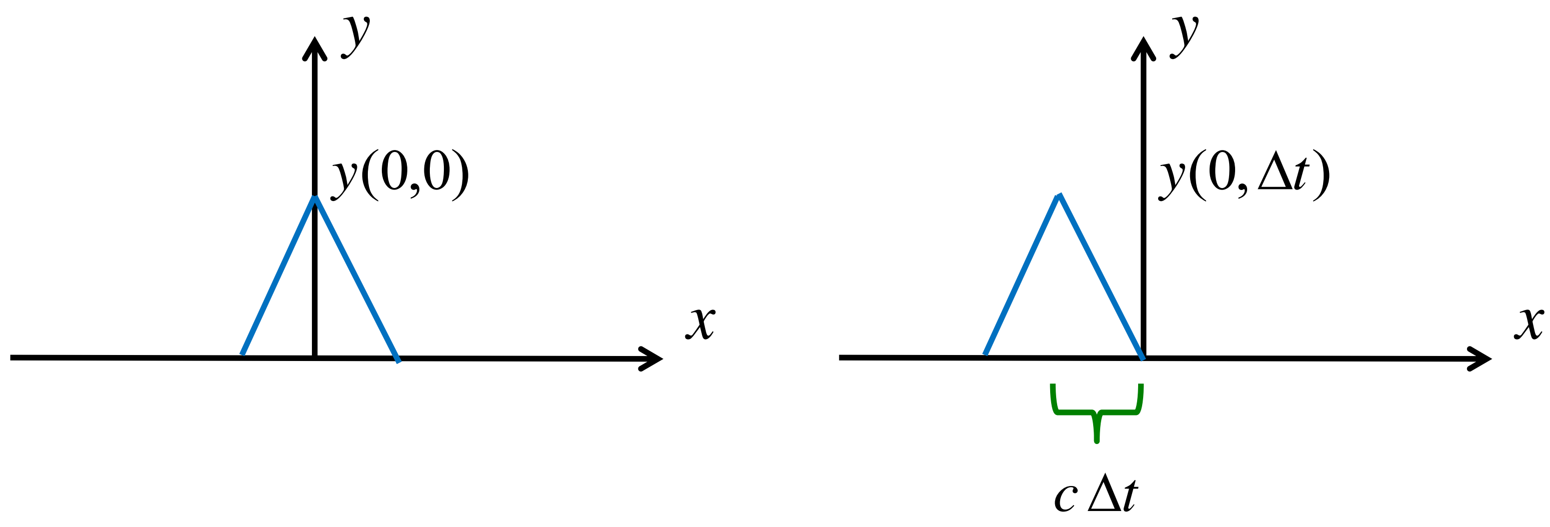


Wave moves to right with speed c

Interpretation of d'Alembert solution

$$y(x,t) = g(x+ct)$$

Focus on x=0 and consider situations at *t=0* and *t=Δt*



Wave moves to left with speed c

d'Alembert's solution with boundary conditions

$$y(x,t) = f(x-ct) + g(x+ct)$$

functions f and g are determined by initial conditions

Suppose at time $t=0$, the wave has initial displacement $U(x)$ and velocity $V(x)$

$$y(x,0) = U(x) = f(x) + g(x) \qquad (1)$$

$$\frac{\partial y(x,0)}{\partial t} = V(x) = \frac{\partial(x-ct)}{\partial t}\frac{\mathrm{d}f}{\mathrm{d}(x-ct)} + \frac{\partial(x+ct)}{\partial t}\frac{\mathrm{d}g}{\mathrm{d}(x+ct)}$$
$$= -cf'(x) + cg'(x) \qquad (2)$$

Integrating (2) gives

$$f(x) - g(x) = -\frac{1}{c}\int_b^x V(x)\mathrm{d}x$$

and combining with (1) yields

$$g(x) = \frac{1}{2}U(x) + \frac{1}{2c}\int_b^x V(x)\mathrm{d}x \qquad f(x) = \frac{1}{2}U(x) - \frac{1}{2c}\int_b^x V(x)\mathrm{d}x$$

d'Alembert's solution with boundary conditions

$$y(x,t) = f(x-ct) + g(x+ct)$$

functions *f* and *g* are determined by initial conditions

Suppose at time *t=0*, the wave has initial displacement *U(x)* and velocity *V(x)*

$$g(x) = \frac{1}{2}U(x) + \frac{1}{2c}\int_b^x V(x)\mathrm{d}x \qquad f(x) = \frac{1}{2}U(x) - \frac{1}{2c}\int_b^x V(x)\mathrm{d}x$$

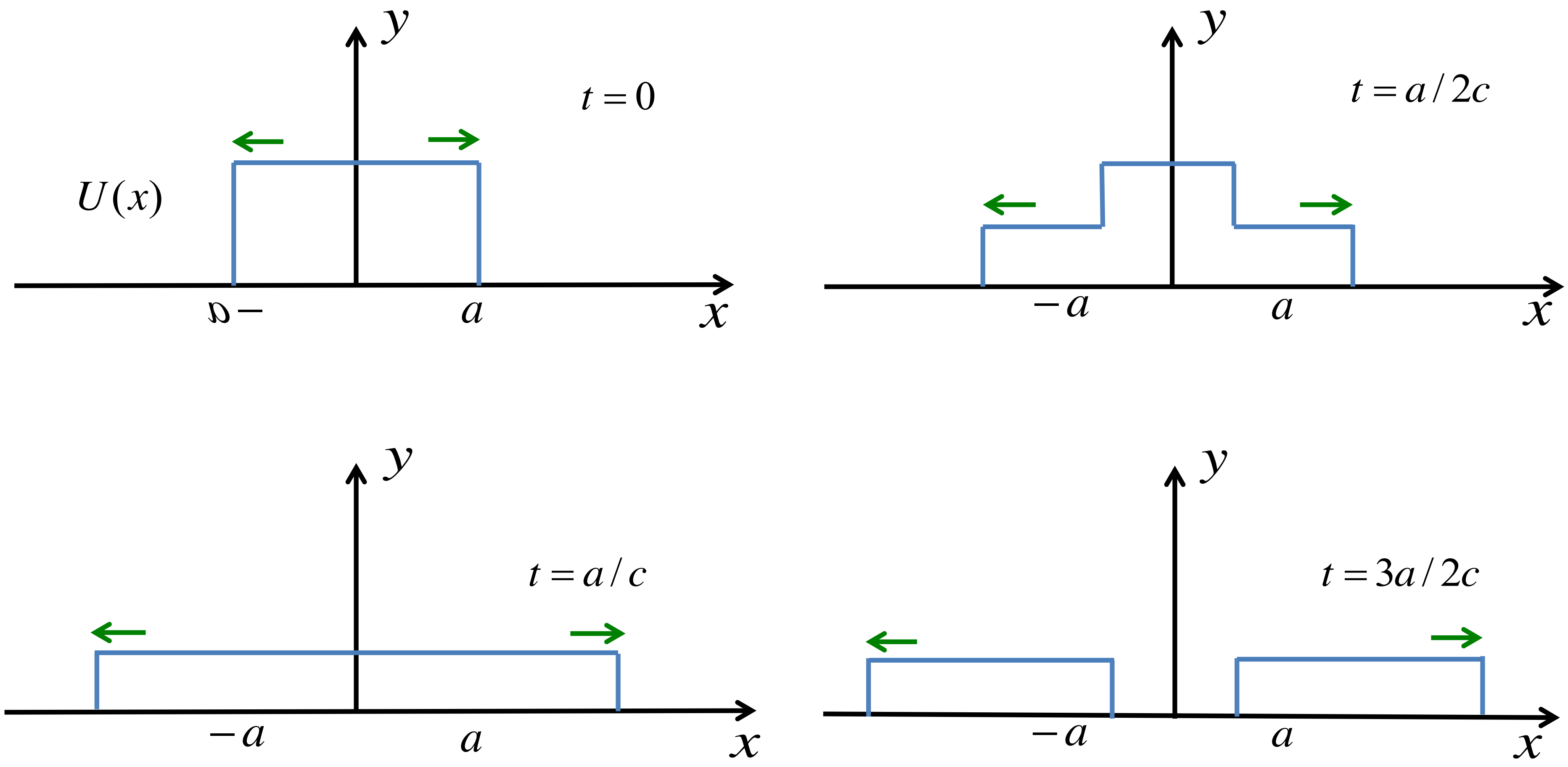
$$\Rightarrow y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\left[\int_b^{x+ct} V(x)\mathrm{d}x - \int_b^{x-ct} V(x)\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(x)\mathrm{d}x$$

d'Alembert's solution with boundary conditions

Example: rectangular wave of length 2a released from rest

$$\Rightarrow y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right]$$



Sinusoidal waves

A very common functional dependence for *f* and *g*...

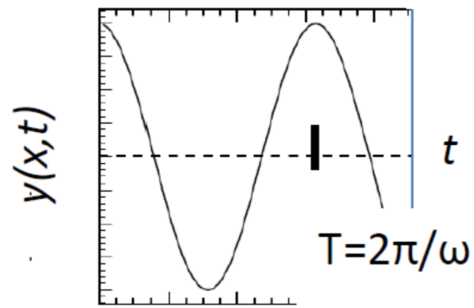
$$y(x,t) = f(x-ct) + g(x+ct)$$

...is sinusoidal. In this case it is usual to write:

$$y(x,t) = A\cos(kx-\omega t) + B\cos(kx+\omega t)$$

with *k* and *ω* (and *A* and *B*) constants

or *Asin(kx-ωt)* ... etc (choice doesn't matter, unless we are comparing one wave with another and then relative phases become important)

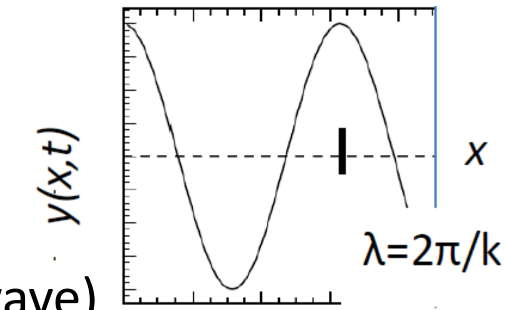


- **speed** of wave $c = \omega / k$
- **frequency** $f = 1/T = \omega / 2\pi$

  where ω is **angular frequency**
- **wavelength** $\lambda = 2\pi / k$

  where k is the **wave-number**

  (or **wave-vector** if also used to indicate direction of wave)



Notation choices

Sinusoidal solution $y(x,t) = A\cos(kx - \omega t)$ (writing here, for compactness, only the forward-going solution)

Using the relationships between *k,ω, λ* & *c* this can be expressed in many forms

$$y(x,t) = A\cos[k(x - ct)]$$

Also note that sometimes it is convenient to write $y(x,t) = A\cos(\omega t - kx)$

Changes nothing (for cosine, trivially so, & practically not even for sine function, as overall sign can be absorbed in constant) & still describes forward-going wave.

A very frequent approach is to use complex notation (we already made use of this when analysing normal modes, and you will have seen it in circuit analysis)

$$y(x,t) = \mathrm{Re}\big[A\exp[i(kx - \omega t)]\big]$$

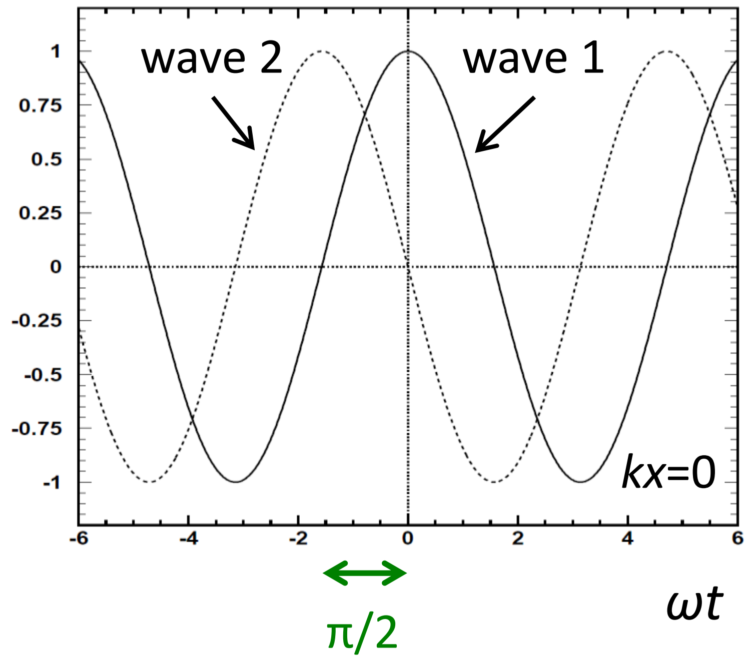
or $y(x,t) = \mathrm{Im}\big[A\exp[i(kx - \omega t)]\big]$ if its important to pick out sine function.
Note that often the 'Re' or 'Im' is implicit, and it gets omitted in discussion.

Phase differences

Often important to specify phase shifts. Only meaningful to do so when we are comparing one wave to another.

$$y_1(x,t) = A\cos(kx - \omega t)$$

$$y_2(x,t) = A\cos(kx - \omega t + \phi)$$



In this example wave 2 leads wave 1 by π/2, i.e. $\phi = -\pi/2$

Can be expressed with complex notation

$$y_2(x,t) = \mathrm{Re}\left[A\exp[i(kx - \omega t + \phi)]\right]$$

Nicer still to subsume phase into amplitude

$$y_2(x,t) = \mathrm{Re}\left[A\exp[i(kx - \omega t)]\right] \quad \text{with} \quad A = |A|\exp(i\phi)$$