Waves 3

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   - Longitudinal waves in a bar
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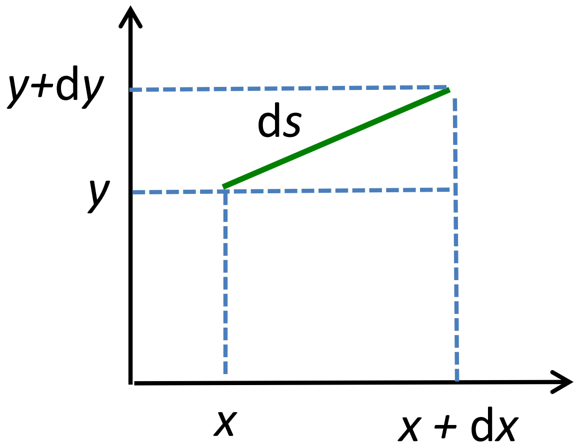
Energy stored in a mechanical wave

A vibrating string must carry energy – but how much? Lets calculate the kinetic energy density (i.e. KE / unit length) and potential energy density



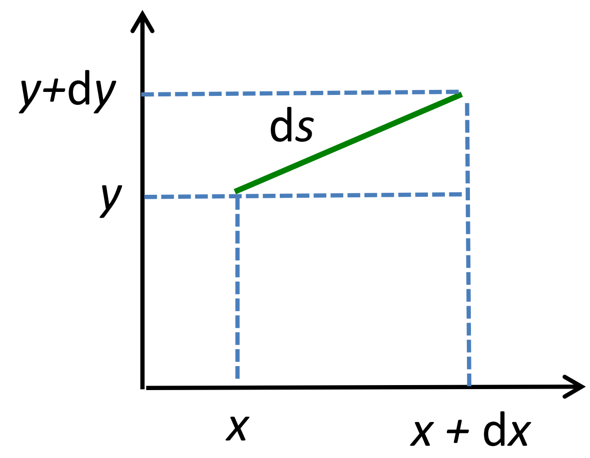
Consider small segment of string of linear density ρ between x and x+dx displaced in the y direction. As usual, assume displacement is small

Kinetic energy, $K = \frac{1}{2}(\rho\, dx) u_y^2 = \frac{1}{2}\rho\, dx \left(\frac{\partial y}{\partial t}\right)^2$

$$\text{KE density} \equiv \frac{dK}{dx} = \frac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2$$

Energy stored in a mechanical wave

Potential energy, *U*, is work done by deformation



$$U = T(\mathrm{d}s - \mathrm{d}x)$$

$$\mathrm{d}s = \mathrm{d}x\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2}$$

$$\approx dx\left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 + \dots\right]$$

$$U = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x$$

$$\text{PE density} \equiv \frac{\mathrm{d}U}{\mathrm{d}x} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

Energy stored in a mechanical wave

We have shown

$$\text{KE density} \equiv \frac{\mathrm{d}K}{\mathrm{d}x} = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 \qquad \text{PE density} \equiv \frac{\mathrm{d}U}{\mathrm{d}x} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

and we know solutions take form $y(x,t) = f(x \pm vt) = f(z)$, say

therefore

$$\frac{\partial y}{\partial x} = f'(z) \qquad \frac{\partial y}{\partial t} = \pm v f'(z)$$

and so

$$\frac{\mathrm{d}K}{\mathrm{d}x} = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\rho v^2[f'(z)]^2 \qquad \frac{\partial U}{\partial x} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}T[f'(z)]^2$$

These are *equal* since $v = \sqrt{(T/\rho)}$ - one manifestation of the *Virial Theorem*.

Energy stored in a mechanical wave

Let's explicitly calculate KE and PE for a sinusoidal wave $y = A\sin(kx - \omega t)$

Evaluate these quantities over an integer number of wavelengths

$$K = \frac{1}{2}\rho \int_{x}^{x+n\lambda} A^2\omega^2 \cos^2(kx-\omega t)\,\mathrm{d}x = \frac{1}{2}\rho A^2\omega^2 \int_{x}^{x+n\lambda} \frac{1}{2}\left(1+\cos[2(kx-\omega t)]\right)\mathrm{d}x = \frac{1}{2}\rho A^2\omega^2 \frac{n\lambda}{2}$$

$$U = \frac{1}{2}T \int_{x}^{x+n\lambda} A^2k^2 \cos^2(kx-\omega t)\,\mathrm{d}x = \frac{1}{2}TA^2k^2 \int_{x}^{x+n\lambda} \frac{1}{2}\left(1+\cos[2(kx-\omega t)]\right)\mathrm{d}x = \frac{1}{2}TA^2k^2 \frac{n\lambda}{2}$$

Now $v = \sqrt{(T/\rho)} = \omega/k \Rightarrow \rho\omega^2 = Tk^2$ and so these expressions are equal

Thus total energy / unit length $\frac{1}{2}\rho A^2\omega^2$

Moreover, we can evaluate energy flow/unit time (= power to generate wave)

P = (energy / wavelength) x (distance travelled / time)

$$P = \frac{1}{2}\rho A^2\omega^2 v = \frac{1}{2}Tk^2A^2\frac{\omega}{k} = \frac{1}{2}T\omega k A^2$$

Wave equation revisited – solving by separation of variables

We have already solved the wave equation using the d’Alembert approach

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Lets attack it again, now looking for solutions which have the ‘separated’ form

$$y(x,t) = X(x)T(t)$$

i.e. that factorise into functions that are separate functions of x and t

$$T(t)\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} = \frac{1}{c^2} X(x)\frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2}$$

$$\frac{\ddot{X}}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T}$$

Wave equation revisited – solving by separation of variables

Separated solution to wave equation e.g. $y(x,t) = X(x)T(t)$ gives $\frac{\ddot{X}}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T}$

Can only be satisfied if both sides equal a constant:

$$\frac{\ddot{X}}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T} = C$$

← the separation constant

Lets consider first the case when C is negative. We write $C=-k^2$

$$\ddot{X} = -k^2 X \qquad \ddot{T} = -(ck)^2 T$$

These are SHM equations, so we can write

$$X(x) = A\cos kx + B\sin kx \quad T(t) = D\cos ckt + E\sin ckt$$

with *A,B,D* and *E* constants defined by initial conditions

so *e.g.* in case *B=0* and *D=0* we have

$$y(x,t) = F\cos kx \sin \omega t$$

with *ω=ck (and F=AE…)*

a form we have seen before, when manipulating d'Alembert solution

Wave equation revisited – solving by separation of variables

Wave equation with separated variables:

$$\frac{\ddot{X}}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T} = C$$

C negative already considered. Here are some other possibilities:

- *C* is positive = k^2

$$y(x,t) = (Ae^{kx} + Be^{-kx})(De^{kct} + Ee^{-kct})$$

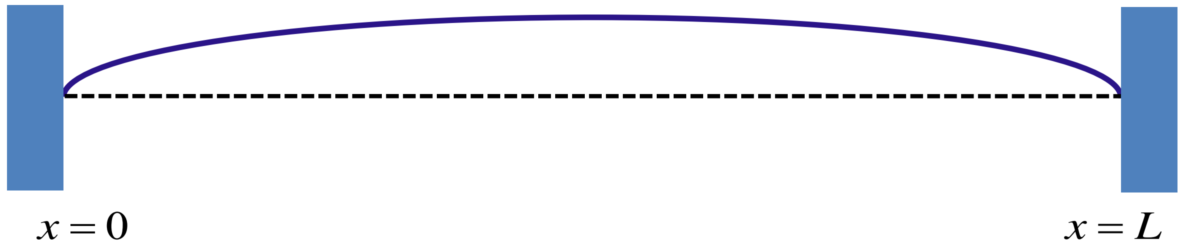
- *C = 0*

$$y(x,t) = (A + Bx)(D + Et)$$

Still others exist (e.g. *C* is complex). Which of solutions is relevant depends on the physical situation. Here we are usually interested in *C* is negative, since then we get oscillations! But even then many possibilities exists (=different values of *k*), and we will often get a linear superposition of these.

Wave on string with fixed ends

Consider a string with fixed ends, initially at rest, given an initial displacement and released. Describe its subsequent motion.



Know from separation of variables that a solution to wave equation is

$$y(x,t) = (A\cos kx + B\sin kx)(C\cos kct + D\sin kct)$$

and we also have four boundary conditions. Three of them are as follows:

1. String initially at rest, *i.e.* $\partial y/\partial t = 0$ for all $x \Rightarrow D = 0$
2. *y(0,t)=0* $\Rightarrow A = 0$
3. *y(L,t)=0* $\Rightarrow kL = n\pi$ where *n* any integer. This is the eigenvalue eqn. and discretises *k*. Each value of *n* corresponds to a normal mode. It is clear that for this continuous system *n* can go to infinity...

Wave on string with fixed ends

Therefore, by the principle of superposition, the solution is:

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}$$

i.e. sum of all possible solutions with coefficients F_n given by initial displacement, which is the boundary condition we have not yet invoked

$$y(x,0) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{L} = h(x)$$

where h(x) is pattern of initial displacement

Simplest case is when *h(x)* is just a normal mode, *e.g.*

$$h(x) = \sin\frac{5\pi x}{L} \qquad \Rightarrow F_5 = 1 \text{ and } F_n = 0 \text{ when } n \neq 5$$

$$\Rightarrow \quad y(x,t) = \sin\frac{5\pi x}{L} \cos\frac{5\pi ct}{L}$$

Wave on string with fixed ends

Consider a more complicated situation when *h(x)* is not a single normal mode

$$h(x) = \sin\frac{\pi x}{L} + \frac{1}{2}\sin\frac{2\pi x}{L}$$

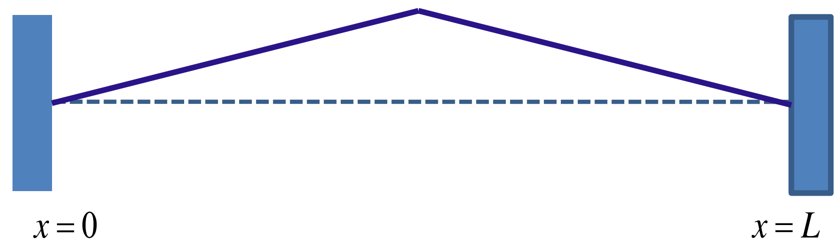
Then it is clear $F_1=1$, $F_2=0.5$ and $F_n=0$ when $n \neq 1$ or 2, so

$$y(x,t) = \sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} + \frac{1}{2}\sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L}$$

In contrast to case with just a single normal mode the subsequent motion is not equal to initial displacement x varying amplitude. Since the shorter waves are faster the shape of the wave varies during oscillation.

Even if the initial displacement looks like this (i.e. plucked string), it *can* be expressed as sum of normal modes!

→ Fourier series (year 2)



Wave on string with fixed ends: energies of normal modes

Normal mode *n* for our problem, with given boundary conditions:

$$y_n(x,t) = F_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

Calculate kinetic energy, K_n, and potential energy, U_n, for each mode

$$K_n = \int_0^L \frac{1}{2}\rho\left(\frac{\partial y_n}{\partial t}\right)^2 \mathrm{d}x \qquad U_n = \int_0^L \frac{1}{2}T\left(\frac{\partial y_n}{\partial x}\right)^2 \mathrm{d}x$$

$$K_n = \frac{1}{2}\rho F_n^{\,2}\left(\frac{n\pi c}{L}\right)^2 \sin^2\frac{n\pi ct}{L}\int_0^L \sin^2\frac{n\pi x}{L}\,\mathrm{d}x = \frac{\rho(F_n n\pi c)^2}{4L}\sin^2\frac{n\pi ct}{L}$$

$$U_n = \frac{1}{2}TF_n^{\,2}\left(\frac{n\pi}{L}\right)^2 \cos^2\frac{n\pi ct}{L}\int_0^L \cos^2\frac{n\pi x}{L}\,\mathrm{d}x = \frac{T(F_n n\pi)^2}{4L}\cos^2\frac{n\pi ct}{L}$$

Since $c = \sqrt{(T/\rho)}$ we have $E_n = K_n + U_n = \frac{\rho L F_n^{\,2}}{4}c^2\left(\frac{n\pi}{L}\right)^2 \Rightarrow \boxed{E_n = \frac{\rho L F_n^{\,2}\omega_n^{\,2}}{4}}$

Wave on string with fixed ends: energies of system

Now let's calculate energy of system, with arbitrary initial displacement

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

This is simple extension of exercise for individual normal modes, But with additional terms

$$E = \sum_{n=1}^{\infty} E_n + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \text{cross-terms}$$

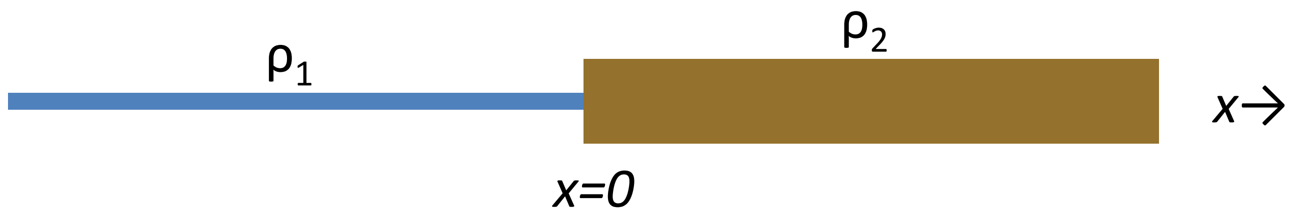
but cross-terms all include factors of the type $\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx \text{ with } n \neq m$ which are zero!

$$\Rightarrow \quad \boxed{E = \sum_{n=1}^{\infty} E_n}$$

i.e. total energy is sum of energies of normal modes

Waves at boundaries

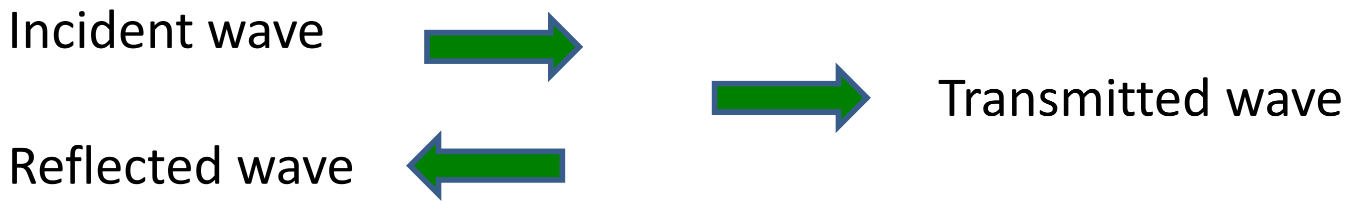
What happens for a wave propagating on a string when it encounters a boundary across which the string characteristics change, *e.g.* change of linear density $\rho_{1,2}$?



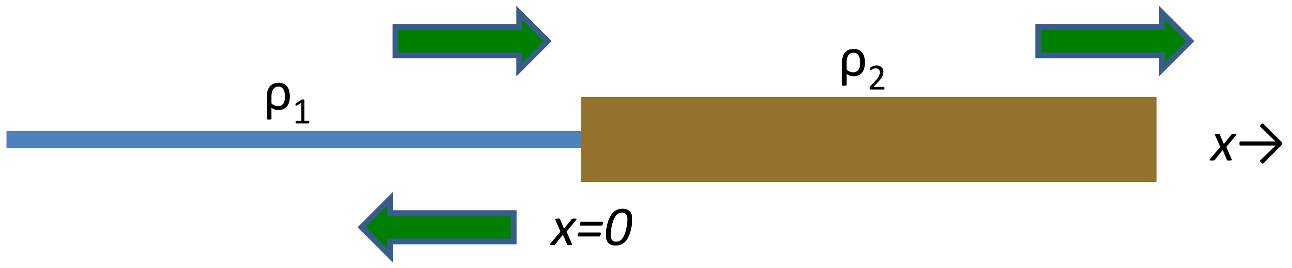
Since the tension, *T*, is constant across boundary, it follows phase velocities change

$$c_{1,2} = \sqrt{T / \rho_{1,2}}$$

We must allow for the possibility of three waves:



Waves at boundaries



Let's write down the three waves

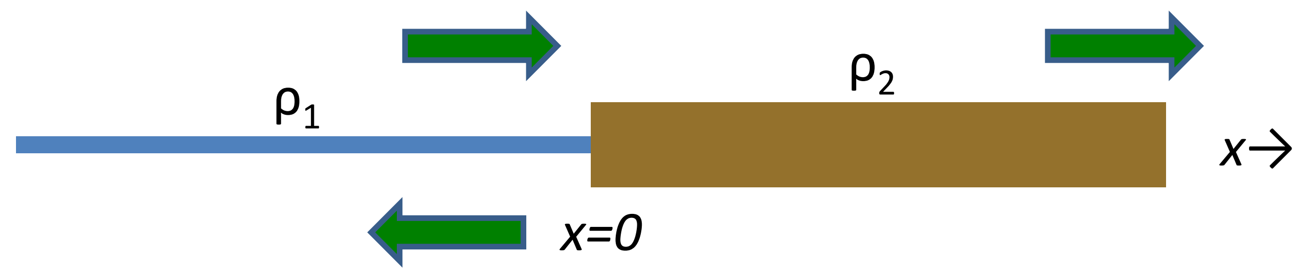
Incident	$A\sin(\omega t - k_1 x)$
Reflected	$A'\sin(\omega t + k_1 x)$
Transmitted	$A''\sin(\omega t - k_2 x)$

← Observe the + sign in argument for left-going wave

(we have switched to writing sin(ωt-kx) rather than sin(kx-ωt), as this notation is usual for these problems. But this does not affect anything.)

Note:
- We assume same angular frequency, ω, on both strings. This is required by boundary conditions (see later). but wavelength, and hence wave number, medium-dependent
- The relative values of A, A' and A'' are to be determined

Boundary conditions



Two boundary conditions:

- String is continuous $$y_1(0,t) = y_2(0,t)$$

- Vertical component of force on left of boundary must be balanced by vertical component on right $$\frac{\partial y_1}{\partial x}(0,t) = \frac{\partial y_2}{\partial x}(0,t)$$

Here:

$$y_1(x,t) = A\sin(\omega t - k_1 x) + A'\sin(\omega t + k_1 x)$$ net wave to left of boundary

$$y_2(x,t) = A''\sin(\omega t - k_2 x)$$ wave to right of boundary

Applying boundary conditions

Applying $y_1(0,t) = y_2(0,t) \quad (1)$

$$\frac{\partial y_1}{\partial x}(0,t) = \frac{\partial y_2}{\partial x}(0,t) \quad (2)$$

with

$$y_1(x,t) = A\sin(\omega t - k_1 x) + A'\sin(\omega t + k_1 x)$$

$$y_2(x,t) = A''\sin(\omega t - k_2 x)$$

String continuous (1) →

$$A\sin\omega t + A'\sin\omega t = A''\sin\omega t$$
$$\Rightarrow A + A' = A''$$

Balanced vertical tension (2) →

$$-k_1 A\cos\omega t + k_1 A'\cos\omega t = -k_2 A''\cos\omega t$$
$$\Rightarrow k_1(A - A') = k_2 A''$$

Solve to obtain *amplitude reflection* and *transmission coefficients*

$$r \equiv \frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$

$$t \equiv \frac{A''}{A} = \frac{2k_1}{k_1 + k_2}$$

Applying boundary conditions

We have $r \equiv \frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2}$ and $t \equiv \frac{A''}{A} = \frac{2k_1}{k_1 + k_2}$

Lets consider some specific cases:

- $k_1 = k_2$ then $r=0$, $t=1$ – no reflection and full transmission
- $k_1 < k_2$ then A' is negative and can write reflected wave

  $$-|A'|\sin(\omega t + k_1 x) = |A'|\sin(\omega t + k_1 x + \pi)$$

  i.e. there is a phase change at a rare-dense boundary
  (since $c = \omega / k = \sqrt{T/\rho}, k_1 < k_2 \Rightarrow \rho_1 < \rho_2$)
- $k_1 > k_2$ then A' is positive
- $\rho_2 \to \infty$ $k_2 \to \infty$ hence $r = \frac{A'}{A} \to -1$

  Full reflection (with phase change) and no wave in second string

Power flow at a boundary

We have $r \equiv \frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2}$ and $t \equiv \frac{A''}{A} = \frac{2k_1}{k_1 + k_2}$

and earlier we showed that $P = \frac{1}{2} T \omega k A^2$

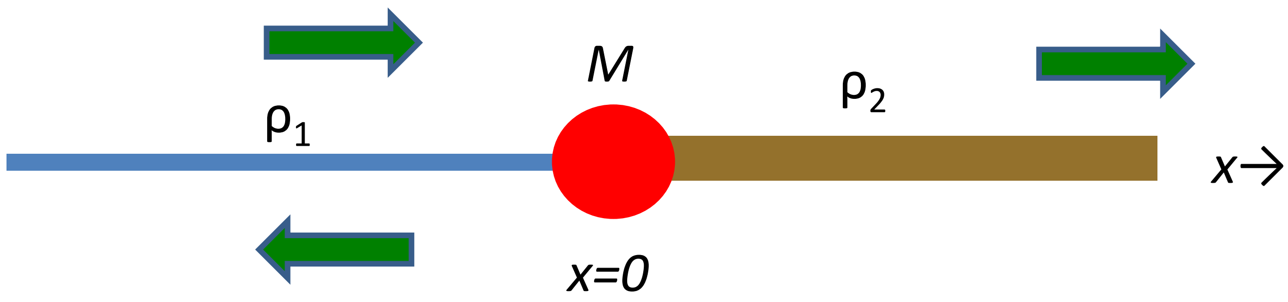
So ratios of reflected to incident power, R_r, and transmitted to incident power, R_t, are given by

$$R_r = \frac{\frac{1}{2} k_1 T \omega A'^2}{\frac{1}{2} k_1 T \omega A^2} = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 \qquad R_t = \frac{\frac{1}{2} k_2 T \omega A''^2}{\frac{1}{2} k_1 T \omega A^2} = \frac{4k_1 k_2}{(k_1 + k_2)^2}$$

and so $R_r + R_t = \frac{(k_1{}^2 + k_2{}^2 - 2k_1 k_2) + (4k_1 k_2)}{(k_1 + k_2)^2} = 1$, as expected

Reflection from a mass at the boundary

Consider situation where a finite point mass, *M*, is fixed at the boundary between two semi-infinite strings of density ρ_1 and ρ_2

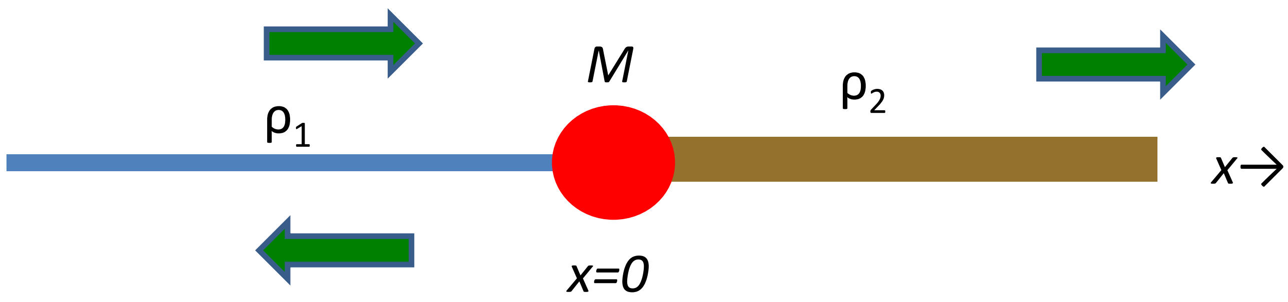


Solve as before, but in this case one of the boundary conditions changes

Sum of forces at boundary act on mass and generates transverse acceleration $-T\frac{\partial y_1}{\partial x}(0,t)+T\frac{\partial y_2}{\partial x}(0,t)=M\frac{\partial^2 y_1}{\partial t^2}(0,t)=M\frac{\partial^2 y_2}{\partial t^2}(0,t)$

Other boundary condition (system continuous, so $y_1(0,t)=y_2(0,t)$) unchanged

Reflection from a mass at the boundary



Here, with 2nd derivatives involved, it's convenient to use exponential notation

$$y_1(x,t) = \mathrm{Re}\{A\exp[i(\omega t - k_1 x)] + A'\exp[i(\omega t + k_1 x)]\}$$

$$y_2(x,t) = \mathrm{Re}\{A''\exp[i(\omega t - k_2 x)]\}$$

System continuous → $A + A' = A''$, as before

New condition → $ik_1 TA - ik_1 TA' - ik_2 TA'' = -\omega^2 M(A + A') = -\omega^2 MA''$

$$\Rightarrow \quad ik_1(A - A') = (ik_2 - \omega^2 M/T)A''$$

Reflection from a mass at the boundary

We have $A + A' = A''$ and $ik_1(A - A') = (ik_2 - \omega^2 M / T)A''$

From these can show

$$r \equiv \frac{A'}{A} = \frac{(k_1 - k_2)T - i\omega^2 M}{(k_1 + k_2)T + i\omega^2 M} = R\exp(i\theta)$$

$$t \equiv \frac{A''}{A} = \frac{2k_1 T}{(k_1 + k_2)T + i\omega^2 M} = T\exp(i\phi)$$

Here *R* and *T* are real numbers

Here *ϑ (φ)* is phase shift of reflected (transmitted) wave w.r.t. incident wave.

$$y_1(x,t) = \mathrm{Re}\{A\exp[i(\omega t - k_1 x)] + A'\exp[i(\omega t + k_1 x)]\}$$
$$\Rightarrow y_1(x,t) = A\cos(\omega t - k_1 x) + RA\cos(\omega t + k_1 x + \theta)$$

$$y_2(x,t) = \mathrm{Re}\{A''\exp[i(\omega t - k_2 x)]\}$$
$$\Rightarrow y_2(x,t) = TA\cos(\omega t - k_2 x + \phi)$$

Reflection from a mass at the boundary

For completeness – we have:

$$r \equiv \frac{A'}{A} = \frac{(k_1 - k_2)T - i\omega^2 M}{(k_1 + k_2)T + i\omega^2 M} = R\exp(i\theta) \qquad t \equiv \frac{A''}{A} = \frac{2k_1 T}{(k_1 + k_2)T + i\omega^2 M} = T\exp(i\phi)$$

and so

$$R = \left[\frac{(k_1 - k_2)^2 T^2 + \omega^4 M^2}{(k_1 + k_2)^2 T^2 + \omega^4 M^2}\right]^{1/2} \qquad \theta = \tan^{-1}\left[\frac{-\omega^2 M}{(k_1 - k_2)T}\right] - \tan^{-1}\left[\frac{\omega^2 M}{(k_1 + k_2)T}\right]$$

$$T = \left[\frac{4k_1^{\,2} T^2}{(k_1 + k_2)^2 T^2 + \omega^4 M^2}\right]^{1/2} \qquad \varphi = -\tan^{-1}\left[\frac{\omega^2 M}{(k_1 + k_2)T}\right]$$

once more note $|r|^2 + \frac{k_2}{k_1}|t|^2 = R^2 + \frac{k_2}{k_1}T^2 = 1$, *i.e.* energy conserved

Reflection from a mass at the boundary

Consider special case where second string has zero density: $\rho_2=0$.
Then $k_2=0$ and it as if we are just terminating the first string with the mass.

$$R = \left[\frac{k_1{}^2 T^2 + \omega^4 M^2}{k_1{}^2 T^2 + \omega^4 M^2} \right]^{1/2} = 1$$

Total reflection

$$\theta = \tan^{-1}\left[\frac{-\omega^2 M}{k_1 T} \right] - \tan^{-1}\left[\frac{\omega^2 M}{k_1 T} \right]$$

$$= 2\tan^{-1}\left[\frac{-\omega^2 M}{k_1 T} \right]$$

→0 if M is small

→π if M is large

Impedance

Familiar with concept of electrical impedance from circuit theory
= a measure of opposition to a time varying electric current

$$V = Z\,I$$

different components have different impedances, some frequency dependent

$$Z_R = R$$ resistor

$$Z_C = -i/\omega C$$ capacitor

$$Z_L = i\omega L$$ inductor

For structures along which electromagnetic waves propagate,
i.e. transmission lines, even free space, talk of *characteristic impedance*

But concept of impedance and characteristic impedance can be used in other wave-carrying systems. Here is an (admittedly) woolly definition:

> Impedance is ratio of 'push variable' (*i.e.* voltage or pressure) to a 'flow variable' (*i.e.* current or particle velocity)

Impedance & waves on string

For transverse waves on a string:

The characteristic impedance *Z* is defined as the applied driving force acting in the *y*-direction divided by the velocity of the string in the *y*-direction

$$Z = \frac{F_y}{v_y} = \frac{-T\frac{\partial y}{\partial x}}{\frac{\partial y}{\partial t}}$$

so with

$$y(x,t) = A\sin(kx - \omega t)$$

$$\Rightarrow Z = \frac{Tk}{\omega} = \frac{T}{c} = \sqrt{(T\rho)}$$

Also, can take reflection and transmission coefficients from ‘mass at a boundary’ problem and write these in terms of impedances

$$r \equiv \frac{A'}{A} = \frac{(k_1 - k_2)T - i\omega^2 M}{(k_1 + k_2)T + i\omega^2 M} = \frac{Z_1 - (Z_2 + Z_M)}{Z_1 + (Z_2 + Z_M)}$$

$$t \equiv \frac{A''}{A} = \frac{2k_1 T}{(k_1 + k_2)T + i\omega^2 M} = \frac{2Z_1}{Z_1 + (Z_2 + Z_M)}$$

Here we have the characteristic impedances $Z_{1,2} = Tk_{1,2} / \omega$

which are ‘resistive’, and an impedance for the mass itself of $Z_M = i\omega M$

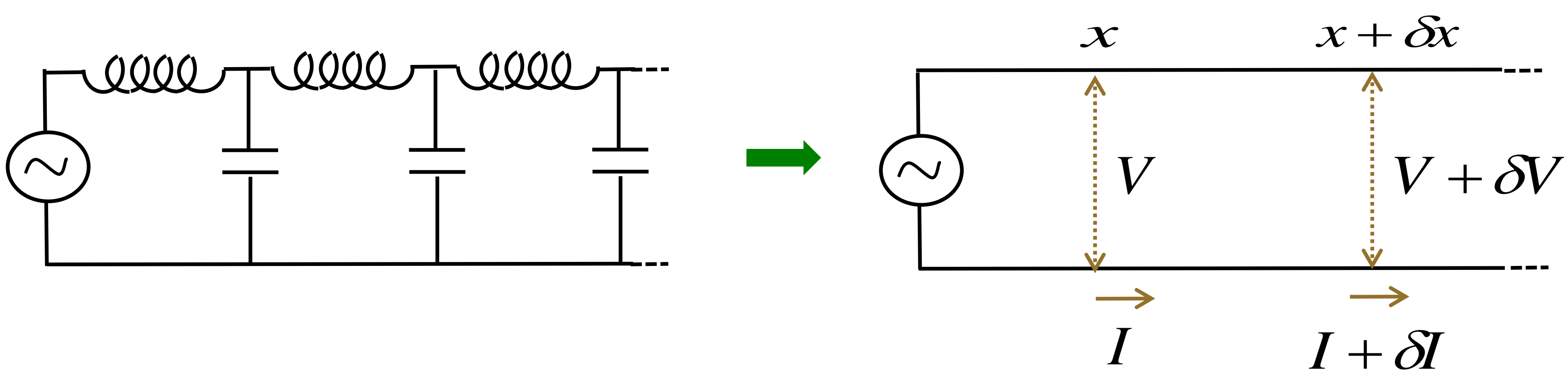
which is ‘inductive’

Other important examples

- Transmission lines
- Longitudinal elastic waves
  - Oscillations in a solid bar
  - Acoustic waves in gas

Lossless transmission lines

Consider a system made of inductors and capacitors, and then let it become continuous so that we now speak of inductance / unit length L’ & capacitance / unit length C’ (*e.g*. coaxial cable). Let it have zero resistance (‘lossless’).



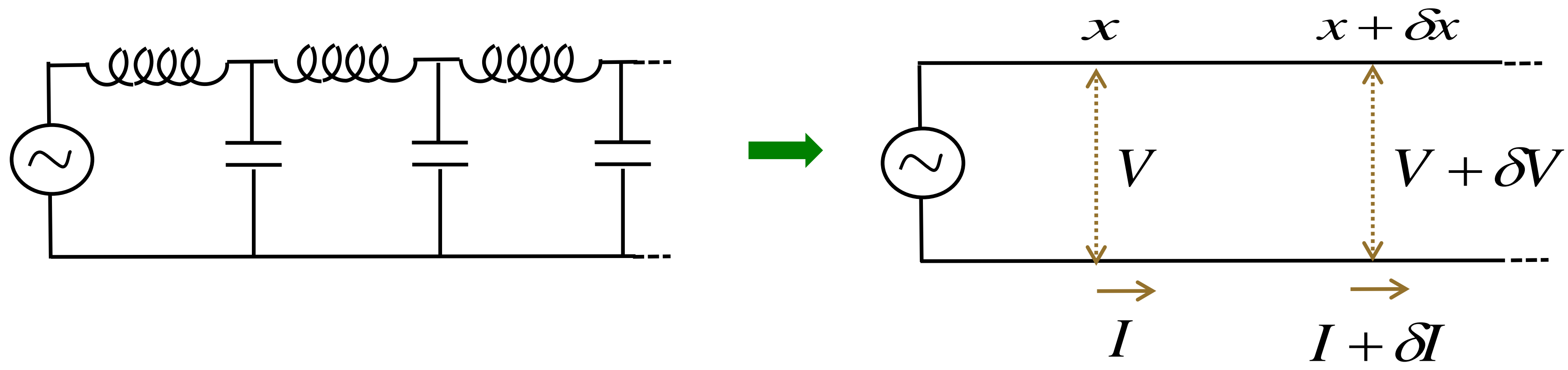
Self-inductance of δx = L’δx → voltage change $\delta V = -(L'\delta x)\frac{\partial I}{\partial t}$

$$\Rightarrow \frac{\partial V}{\partial x} = -L'\frac{\partial I}{\partial t}$$

Capacitance of δx = C’δx → voltage change $\delta V = -\delta Q/(C'\delta x)$

$$\Rightarrow C'\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x}$$

Lossless transmission lines



We have shown $\frac{\partial V}{\partial x} = -L'\frac{\partial I}{\partial t}$ (1) $C'\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x}$ (2) these are the *telegraph equations*

$$\frac{\partial}{\partial t}(1) \text{ and } \frac{\partial}{\partial x}(2) \Rightarrow \frac{\partial^2 I}{\partial x^2} = L'C'\frac{\partial^2 I}{\partial t^2}$$

$$\frac{\partial}{\partial t}(2) \text{ and } \frac{\partial}{\partial x}(1) \Rightarrow \frac{\partial^2 V}{\partial x^2} = L'C'\frac{\partial^2 V}{\partial t^2}$$

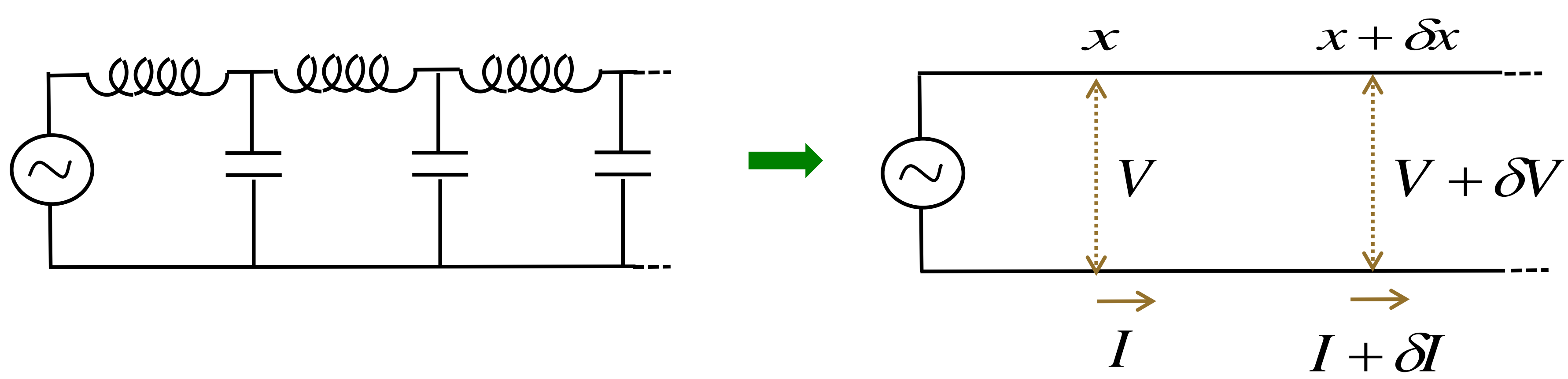
Wave equation!

So waves of form *e.g.*

$$V = V_0 \sin(\omega t \mp kx)$$
$$I = I_0 \sin(\omega t \mp kx)$$

can travel down line with $c = 1/\sqrt{C'L'}$

Characteristic impedance of transmission line



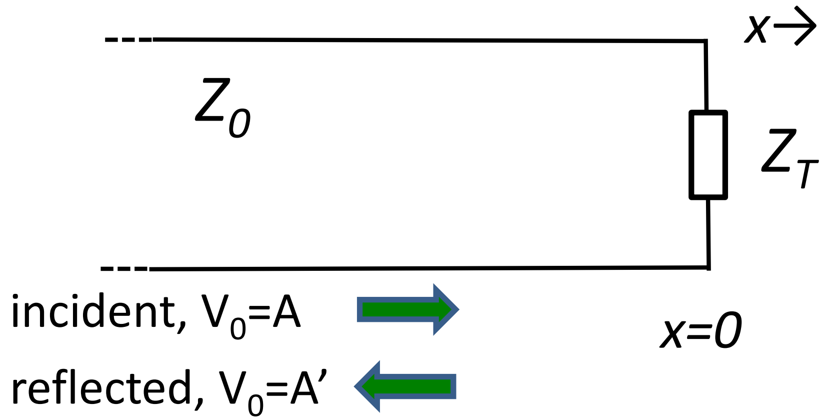
We have $\frac{\partial V}{\partial x} = -L'\frac{\partial I}{\partial t}$ and $V = V_0 \sin(\omega t \mp kx)$, $I = I_0 \sin(\omega t \mp kx)$

$$\Rightarrow \mp k V_0 \cos(\omega t \mp kx) = -L' I_0 \omega \cos(\omega t \mp kx)$$

Characteristic impedance $$Z \equiv \frac{V_0}{I_0} = \pm\frac{\omega}{k}L' = \pm\frac{1}{\sqrt{C'L'}}L' = \pm\sqrt{\frac{L'}{C'}}$$ (+ve for forward travelling wave)

Reflection at a terminated line

Consider how wave reflects for transmission line of characteristic impedance Z_0 terminated at $x=0$ by an impedance of Z_T



We have:

$$V(x,t) = A\exp(i[\omega t - kx]) + A'\exp(i[\omega t + kx])$$
$$Z_0 I(x,t) = A\exp(i[\omega t - kx]) - A'\exp(i[\omega t + kx])$$

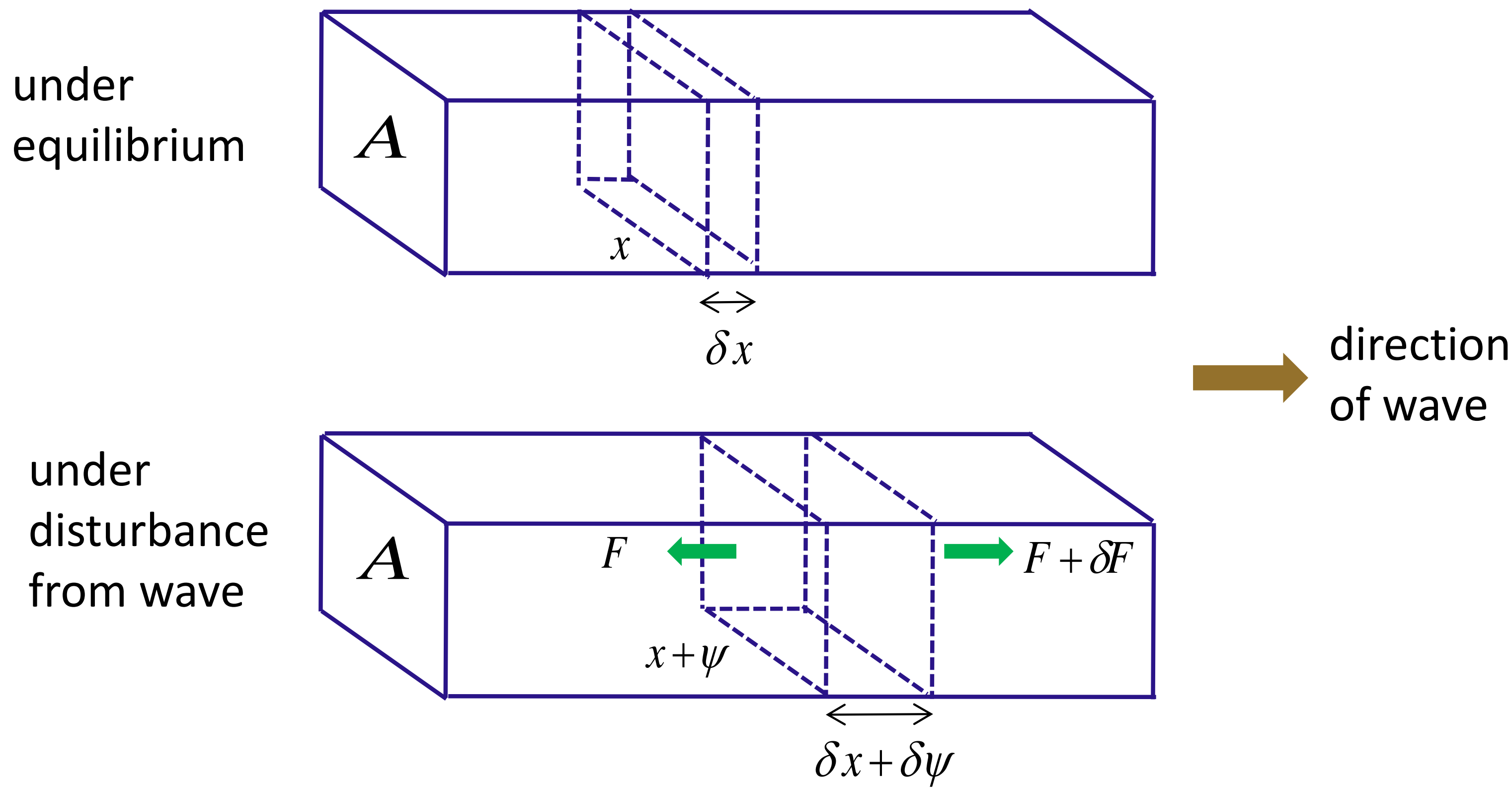
Now at $x=0$ the ratio V/I must = the terminating impedance! $\dfrac{V(0,t)}{I(0,t)} = Z_T$

Hence, $\dfrac{Z_T}{Z_0} = \dfrac{A + A'}{A - A'}$ so reflection coefficient $$r = \frac{A'}{A} = \frac{Z_T - Z_0}{Z_T + Z_0}$$

- $Z_T \to 0$ then $r \to -1$: full reflection with phase shift
- $Z_T = Z_0$ then $r \to 0$: no reflection – *matched impedance*; all power transmitted to terminating load
- $Z_T \to \infty$ then $r \to 1$: full reflection

Longitudinal waves in a solid bar

Consider a solid bar, initially in equilibrium, in which a disturbance perturbs the position and thickness of a slice of material



We denote by F the magnitude of the new stress force stretching the material and δF the excess force accelerating the segment to the right

Longitudinal waves in a solid bar

Force per unit area on slice is given by Hooke's law and Young's modulus of the material, *Y*

$$\frac{F}{A} = Y\left(\frac{\delta\psi}{\delta x}\right)$$

infinitely thin slice → $F = AY\frac{\partial\psi}{\partial x}$

So excess force is given by $\delta F = AY\frac{\partial^2\psi}{\partial x^2}\delta x$

Mass of slice (with density ρ) is given by $A\rho\delta x$ and acceleration is $\frac{\partial^2\psi}{\partial t^2}$

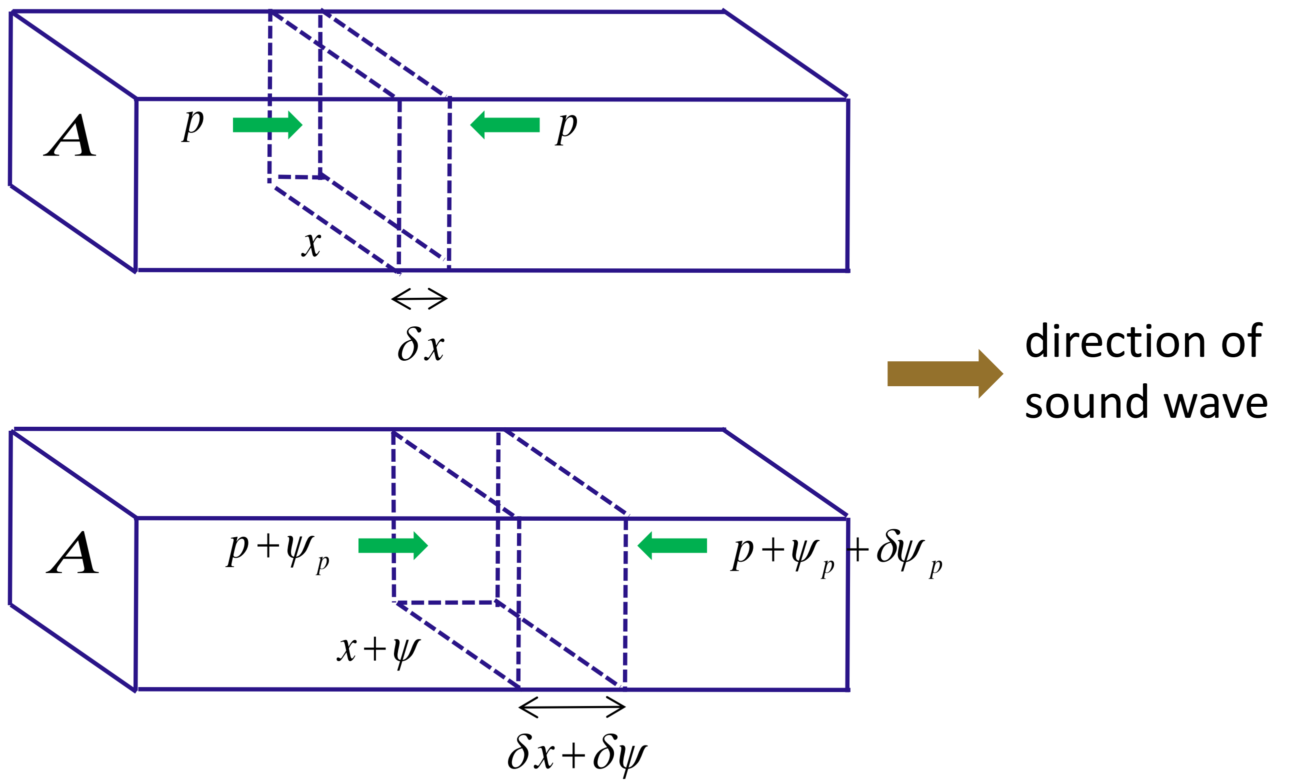
Newton II gives $(A\rho\delta x)\frac{\partial^2\psi}{\partial t^2} = AY\frac{\partial^2\psi}{\partial x^2}\delta x \quad \Rightarrow \quad \frac{\partial^2\psi}{\partial x^2} = \left(\frac{\rho}{Y}\right)\frac{\partial^2\psi}{\partial t^2}$

So velocity of waves is $v = \sqrt{\frac{Y}{\rho}}$

Steel has $Y=2 \times 10^{11}$ Nm^{-2} and $\rho= 8000$ kg m^{-3} $\Rightarrow v \approx 5\,\mathrm{km\,s^{-1}}$

Acoustic waves

Sound waves are longitudinal waves associated with compression of medium.



Consider a slice of gas, initially at equilibrium, in a tube of cross-sectional area A. Slice is between x and $x+\delta x$. A disturbance moves the slice to $x+\psi$ and changes its width to $\delta x+\delta\psi$. Pressure has changed from p to $p+\psi_p$ on LHS of slice, and to $p+\psi_p+\delta\psi_p$ on RHS.

Acoustic waves

Slice has had its volume changed by a fractional amount $(A.\delta\psi)/(A.\delta x)$, and this happens as a result of a pressure change ψ_p. The relationship is determined by the elasticity of the gas, the bulk modulus κ

$$\delta\psi / \delta x = (-1/\kappa)\psi_p$$

infinitely thin slice →

$$\psi_p = -\kappa \frac{\partial \psi}{\partial x}$$

From this we see $\delta\psi_p = -\kappa \frac{\partial^2 \psi}{\partial x^2} \delta x$ which will be useful in getting wave eqn.

- Mass of slice $A\rho\delta x$
- Force on slice in x-dir^n $-A\delta\psi_p$
- Acceleration of slice $\frac{\partial^2 \psi}{\partial t^2}$

These relations and Newton II yield:

$$A\kappa \frac{\partial^2 \psi}{\partial x^2} \delta x = A\rho \frac{\partial^2 \psi}{\partial t^2} \delta x$$

$$\Rightarrow \boxed{\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{\rho}{\kappa}\right) \frac{\partial^2 \psi}{\partial t^2}}$$

Acoustic waves

We have obtained result $\frac{\partial^2\psi}{\partial x^2} = \left(\frac{\rho}{\kappa}\right)\frac{\partial^2\psi}{\partial t^2}$, a wave equation describing motion of a slice of gas at position ψ. One might worry that a 'slice of gas' is a rather intangible experimental observable. Instead one can phrase problem in terms of the pressure variations, ψ_p, which are certainly measurable.

Since $\psi_p \propto \frac{\partial \psi}{\partial x}$ then ψ_p also satisfies wave equation, *i.e.* $\frac{\partial^2\psi_p}{\partial x^2} = \left(\frac{\rho}{\kappa}\right)\frac{\partial^2\psi_p}{\partial t^2}$

Either way, phase velocity of waves is $v = \sqrt{\kappa / \rho}$

Characteristic impedance can be defined by

$$Z = \frac{-\kappa \frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial t}}$$

so for forward-travelling wave

$$\psi(x,t) = A\sin(\omega t - kx)$$

$$Z = \frac{Kk}{\omega} = \frac{K}{v} = \sqrt{\rho K}$$

Acoustic waves – speed of sound

We have shown that $v = \sqrt{\kappa / \rho}$, so we can calculate v if we know κ (and ρ).

To calculate κ it is convenient to use this form of definition: $\kappa = -V \frac{\partial p}{\partial V}$

We also need to specify what else happens to system as p changes.

- **Isothermal compression** No temperature changes – justifiable if the pressure changes are slow enough to allow tube to exchange heat freely with surroundings.

  Ideal gas $PV = RT \quad \Rightarrow \frac{\partial P}{\partial V} = \frac{-RT}{V^2}$ so $\kappa = \frac{RT}{V} = p \quad \Rightarrow v = \sqrt{p / \rho}$

- **Adiabatic compression** Pressure changes occur so rapidly that heat cannot be exchanged from dense to less dense regions. Good approximation to reality.

  Adiabatic changes in an ideal gas $pV^\gamma = \text{constant}$ where $\gamma = C_p / C_V$
  i.e. the ratio of specific heats at constant p and constant V

  $\frac{\partial P}{\partial V} = \frac{-\gamma p}{V}$ so $\kappa = \gamma p \quad \Rightarrow \boxed{v = \sqrt{\gamma p / \rho} = \sqrt{\gamma (RT / M)}}$

  i.e. velocity independent of pressure for ideal gas
  Typical value for air at room temperature ~350 ms^{-1}