Keble First Year Mathematics: Xmas Vacation 2014-15

Goals:

- Learn to estimate the difficulty of problems without solving them.
- Gain confidence in your ability to solve different types of problems.
- Gain further experience and practice in problem solving.
- Start to learn how to acquire mathematical skills in self-study for the examples of Lagrange multipliers and multiple integrals.

Tasks:

a) Carefully read through questions 1 – 23 below.
b) Rank the questions according to your level of confidence that you will be able to solve them. Distinguish between questions for which you do and do not expect to have to read up on mathematical concepts before solving them. Hand in this ordered list.
c) Select at least ten questions including the five questions you consider most difficult to solve. Solve these questions and hand in the solutions.
d) For each of the solved questions reflect on how your expectation on being able to solve them has agreed or disagreed with your actual experience in solving them. Also, consider whether expectations about the remaining questions have changed.
e) Read through all questions which you have not solved yet on the lecturers' problem sets. For each of these questions reflect on whether you expect to run into any problem when attempting to solve them. Solve them (no need to hand in the solutions).
f) Identify and write down the most important mathematics concepts of which you have gained a deeper understanding through studying CP3 and CP4 in MT.
g) Write down any questions you might still have about the topics covered in CP3 and CP4 during MT and hand them in.
h) Hand in solutions to Keble lodge by 5pm Thursday 0^{th} week HT.

Miscellaneous Mathematics Problems

(from past papers, by WWMA, revised by DJ and TJ)

Question 1.

Solve the differential equations:

(i) $x^2 \dfrac{dy}{dx} + xy = x^n$, where n is a constant. Distinguish between the cases n=0 and n≠0.

(ii) $\dfrac{dy}{dx} + y\ln x = \exp(-x\ln x)$

(iii) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = e^x \sin 2x$.

Question 2.

Solve the following differential equations:

$(i) \quad x\dfrac{dy}{dx} - 2y = 2x^2, \qquad y(1) = 1$

$(ii) \quad \dfrac{d^2y}{dx^2} + 9y = 6\cos 3x, \qquad y(0) = 1, \qquad \dfrac{dy}{dx}(0) = 3.$

Question 3.

Solve the following equations:

(a) $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}, \qquad y = 1$ when $x = 0$

by using the substitution $y = vx$ or otherwise,

(b) $x \ln x \dfrac{dy}{dx} + y = \ln x, \qquad y = 0$ when $x =$ e.

Question 4. [Hint: Ensure that it is clear which variables are held constant in the definition of each differential coefficient.]

When $x = r\cos\theta, \quad y = r\sin\theta$ and $V = f(x,y) = g(r,\theta) = h(u,v)$, prove that

$$\frac{\partial V}{\partial x} + i\frac{\partial V}{\partial y} = e^{i\theta}\left(\frac{\partial V}{\partial r} + \frac{i}{r}\frac{\partial V}{\partial \theta}\right)$$

and that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

The equation $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$ is satisfied by $V = f(x,y)$. Prove that this occurs iff $\dfrac{\partial^2 V}{\partial u^2} + \dfrac{\partial^2 V}{\partial v^2} = 0$, with $u = \tfrac{1}{2}\ln(x^2 + y^2)$ or $r^2 = e^{2u}$ and $v = \tan^{-1}(y/x)$ or $v = \theta$.

Question 5. [NB ensure that it is clear which variables are held constant in the definition of each differential coefficient.]

Variables u,x and y are related so that each may be regarded as a function of the other two. Prove that

$$\left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_u \left(\frac{\partial x}{\partial u}\right)_y = -1.$$

Hence, or otherwise, show that for the case of four related variables u,v,x and y

$$\left(\frac{\partial u}{\partial y}\right)_{x,v}\left(\frac{\partial y}{\partial x}\right)_{u,v}\left(\frac{\partial x}{\partial v}\right)_{u,y}\left(\frac{\partial v}{\partial u}\right)_{x,y}=1.$$

Verify this result by calculating each of the four derivatives for the case where

$$uv=\sinh x\sinh y.$$

Question 6. [NB ensure that it is clear which variables are held constant in the definition of each differential coefficient.]

Four variables u,t,p,v are such that any one can be expressed as a function of any two others. Prove that

$i)$ $\left(\frac{\partial u}{\partial t}\right)_{p}=\left(\frac{\partial u}{\partial t}\right)_{v}+\left(\frac{\partial u}{\partial v}\right)_{t}\left(\frac{\partial v}{\partial t}\right)_{p}$,

$ii)$ $\left(\frac{\partial u}{\partial p}\right)_{t}=\left(\frac{\partial v}{\partial p}\right)_{t}\left(\frac{\partial u}{\partial v}\right)_{t}$,

$iii)$ $\left(\frac{\partial u}{\partial p}\right)_{v}=-\left(\frac{\partial v}{\partial p}\right)_{u}\left(\frac{\partial u}{\partial v}\right)_{p}$.

Question 7.

Find the values of x and y for which $f(x,y)=(ax^2+by^2)\exp(-cx^2-dy^2)$ has (i) a maximum, (ii) a minimum, and (iii) a saddle point; the constants a, b, c, d are all real and positive, and $ad < bc$.

Find the greatest value of f in the region $cx^2+dy^2\le \frac{1}{2}$

Question 8.

Assuming Taylor's theorem for a function of a single variab1e, show that as far as second order terms in h, k,

$$f(x+h,y+k)=f(x,y)+h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}+\frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right),$$

where the derivatives are evaluated at (x,y). Find the conditions for f to have a maximum, minimum or saddle point at (x,y).

Find the maximum and minimum values of

$$f(x,y)=\sin x\sin y\sin(x+y)$$

in $0<x<\pi, 0<y<\pi$, and draw a *rough* contour map of the surface $z=f(x,y)$.

Question 9. [NB ensure that it is clear which variables are held constant in the definition of each differential coefficient.]

(a) z is defined as a function of x and y by $x^2+y^2+z^2=a^2$. Show that:

$$\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=\frac{a^2}{z^4}.$$

(b) If *z* is a function of *u* and *v*, where $u = \ln(x^2 + y^2)$ and $v = \tan^{-1}(y/x)$, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\frac{\partial z}{\partial u}.$$

Hence, or otherwise, show that the genera1 so1ution to

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x^2 + y^2$$

is: $z = \frac{1}{2}\left(x^2 + y^2\right) + f\left(\tan^{-1}(y/x)\right)$ where *f*(.) is an arbitrary function.

Question 10.

Prove that the values of *x, y, z* for which the function $u = x^2 + y^2 + z^2$ is stationary when *x, y, z* are restricted by the conditions

$$ax^2 + by^2 + cz^2 = 1, \quad lx + my + nz = 0$$

satisfy the equation

$$(b-c)lyz + (c-a)mzx + (a-b)nxy = 0.$$

Hence, or otherwise, find the lengths of the semi-axes of the ellipse in which the surface $x^2 + y^2 + 2z^2 = 1,$ is cut by the plane $x + y + z = 0$.

Question 11.

Explain the use of Lagrange multipliers in determining the stationary values of the function $f(x_1, x_2)$ of two variables subject to a constraint $g(x_1, x_2) = 0.$
Use Lagrange multipliers to find the maximum and minimum distances from the origin to the curve

$$5x^2 + 8xy + 5y^2 - 9 = 0,$$

and interpret your results geometrically. Could you answer this question in a different way, using vectors and matrices?

Question 12. [NB ensure that it is clear which variables are held constant in the definition of each differential coefficient.]

The independent variables *x,y* are transformed into new variables *u,v* given by the equation $u = xy, \quad v = 1/y$. If a function *f(x,y)* is thus transformed into *g(u,v),* prove that

$$y\frac{\partial f}{\partial y}\left\{x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right\} = v\frac{\partial g}{\partial v}\left\{u\frac{\partial g}{\partial u} - v\frac{\partial g}{\partial v}\right\}$$

and

$$y\left\{x\frac{\partial^2 f}{\partial x \partial y} - y\frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial y}\right\} = v\left\{u\frac{\partial^2 g}{\partial u \partial v} - v\frac{\partial^2 g}{\partial v^2} - \frac{\partial g}{\partial v}\right\}.$$

Question 13.

Evaluate

$$I_1 = \int_A^B 3x^2 y \, dx, \quad I_2 = \int_A^B x^3 \, dy,$$

where *A* and *B* are the points *(a,0)* and *(0,a)* respectively, and the path of integration lies

(i) along the straight line joining A and B,

(ii) along part of the semi-circle $x^2 + y^2 = a^2, \quad y \geq 0.$

Find $I_1 + I_2$ in each case, and comment on your results.

Question 14.

The transformation from Cartesian coordinates *x,y,z* to cylindrical polar coordinates r, θ, z is defined by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

State how a volume integral of the form $\iiint f(x,y,z)\, dx\, dy\, dz$ may be transformed into one involving cylindrical coordinates.

The region V_1 is defined by $x^2 + y^2 \leq z, \quad x^2 + y^2 \leq 8 - z;$ the region V_2 is defined by. $x^2 + y^2 \leq z, \quad z \leq 4.$ Show that, for a function *g(x,y)* depending only on *x* and *y*,

$$\iiint_{V_1} g(x,y)\, dx\, dy\, dz = 2 \iiint_{V_2} g(x,y)\, dx\, dy\, dz.$$

Evaluate $\iiint_{V_1} \left(x^2 + y^2\right)^{\frac{1}{2}} dx\, dy\, dz.$

Question 15. [This simple looking question will require you to think very carefully!]

Find the mean distance of the points on the surface of a sphere of radius *a* from a point distance *b* from the centre of the sphere. Distinguish between the cases $a < b, \quad a > b.$

What is the mean distance of the points inside a sphere of radius *c* from a point within the sphere, distance *b* from the centre?

Question 16.

The function *f(x)* satisfies the differential equation

$$\frac{d^2 f}{dx^2} - 2x\frac{df}{dx} + 6f = 0$$

and the initial conditions $f(0) = 0, \quad \left(df/dx\right)_{x=0} = -2.$ If $f^{(n)}(x)$ denotes the *n*th derivative of *f*, show that

$$f^{(n+2)}(0) = (2n-6) f^{(n)}(0)$$

Hence obtain the Taylor Series for *f(x)*.

Verify that this series is the coefficient of s^3 in the expansion of $\exp(-s^2 + 2sx).$

Question 17.

(i) Prove that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\cos(\theta x),$$

and that

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} e^{-\theta^2 x^2},$$

where $0 < \theta < 1$.

(ii) Expand $\sqrt{x}$ about $x = 1$ using Taylor's series, and show that this expansion is valid only for $0 \le x \le 2$.

Question 18.

(i) Find the points of discontinuity of the following functions

(a) $\frac{x^3+4x+6}{x^2-6x+8}$ (b) sec x (c) $\frac{\sin x}{\sqrt{x}}$

(ii) Differentiate (Hint: where necessary take logs first.)

(a) $\log_e \cos\frac{1}{x}$ (b) e^{3x^2}

(c) $\sin^{-1}\frac{x}{x+1}$ (d) $e^{\sin 2x}$ (e) $\sin(\cos x)$

(f) $\log_e \cos(\frac{\pi}{4} - x^2)$ (g) $x^{\cos x}$ (h) x^x

(i) a^x $(a > 0)$ (j) $\log_e(\sin^{-1} x^2)$

Question 19.

Using vector methods:

(a) Show that the line of intersection of the planes $x + 2y + 3z = 0$ and $3x + 2y + z = 0$ is equally inclined to the x- and z-axes and makes an angle $\cos^{-1}(\frac{-2}{\sqrt{6}})$ with the y-axis.

(b) Find the perpendicular distance between one corner of a unit cube and the major diagonal not passing through it.

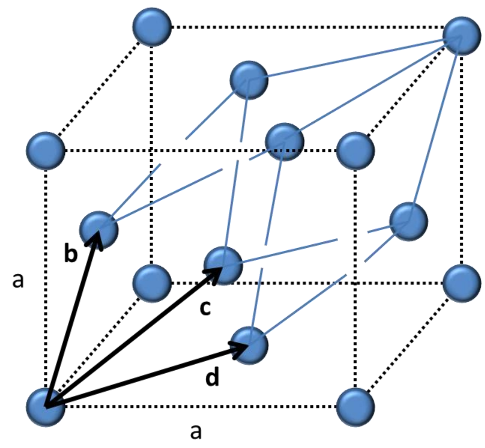
Question 20.

In a crystal with a face-centred cubic structure, the basic cell can be taken as a cube of edge a with its centre at the origin of coordinates and its edges parallel to the Cartesian coordinate axes; atoms are sited at the eight corners and at the centre of each face. However, other basic cells are possible. One is the rhomboid shown in the figure below, which has the three vectors **b**, **c** and **d** as edges.

(a) Show that the volume of the rhomboid is one-quarter that of the cube.

(b) Show that the angles between pairs of edges of the rhomboid are 60° and that the corresponding angles between pairs of edges of the rhomboid defined by the reciprocal vectors to **b**, **c**, **d** are each 109.5°. (This rhomboid can be used as the basic cell of a body-centred cubic structure, more easily visualised as a cube with an atom at each corner and one in its centre.)

(c) In order to use the Bragg formula, $2d \sin\theta = n\lambda$, for the scattering of X-rays by the crystal, it is necessary to know the perpendicular distance d between successive planes of atoms; for a given crystal structure, d has a particular value for each set of planes considered. For the face-centred cubic structure find the distance between successive planes with normals in the **k**, **i** + **j** and **i** + **j** + **k** directions.



Question 21.

Using the properties of determinants, solve with a minimum of calculation the following equations for x:

(a) $\begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$ (b) $\begin{vmatrix} x+2 & x+4 & x-3 \\ x+3 & x & x+5 \\ x-2 & x-1 & x+1 \end{vmatrix} = 0$

Question 22.

Prove the following results involving Hermitian matrices:

(a) If A is Hermitian and U is unitary then $U^{-1}AU$ is Hermitian.

(b) If A is anti-Hermitian then iA is Hermitian.

(c) The product of two Hermitian matrices A and B is Hermitian if and only if A and B commute.

(d) If S is a real antisymmetric matrix then $A = (I - S)(I + S)^{-1}$ is orthogonal. If A is given by $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ then find the matrix S that is needed to express A in the above form.

(e) If K is skew-hermitian, i.e. $K^\dagger = -K$, then $V = (I + K)(I - K)^{-1}$ is unitary.

Question 23.

Solve the set of linear equations

$$u + 2x - y + 3z = 1$$

$$x - 3z + 2y = -2$$
$$z - 3x + u - 2y = 0$$
$$u - 3x + 2z + y = 3$$