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Keble College - Michaelmas 2014

CP3&4: Mathematical methods I&II

Tutorial 7 - Advanced problems

Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.
Leave these at Keble lodge by 5pm on Monday of 7th week.
Look at the 'class problems' in preparation for the tutorial session.
Suggested reading: RHB, all lecturers' problem sets, and past examination papers.

Goals

- Learn to apply various mathematical methods studied during term to more advanced problems.

Problems

While previous problem sets have attempted to walk you through a single topic, here we aim foremost to practice the art of problem solving — how to spot what the question is after without aid, how to spot short-cuts and simplify, how to avoid dead ends, and how to work confidently yet carefully in the face of multiple unknowns. Practice good exam technique, answer the questions directly, concisely, yet fully. It should also show you how far you have come since the beginning of term, serve as a bit of revision, and be a little bit fun. Good luck.

1. Prove that

$$\sum_{r=1}^{n} \binom{n}{r} \sin(2r\theta) = 2^n \sin(n\theta)\cos^n(\theta)\,.$$

[Hint: express the left side as $\Im\left(\sum \binom{n}{r} \mathrm{e}^{\mathrm{i}2r\theta}\right)$ and use the binomial theorem.]

2. Show that the complex polynomial equation with the four roots $z = (\pm\sqrt{3}\pm\mathrm{i})/2$ is $z^4 - z^2 + 1 = 0$.

3. Find the roots of the equation $(z-1)^n + (z+1)^n = 0$. Hence or otherwise solve the equation $x^3 + 15x^2 + 15x + 1 = 0$.

4. Find all the roots, real and complex, of the equation $z^3 - 1 = 0$. If ω is any one of the complex roots, prove that $1 + \omega + \omega^2 = 0$. Find the sums of the following series:

$$S_1 = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots\,; \qquad S_2 = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots\,; \qquad S_3 = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots$$

[Hint: Note that $S_1 + S_2 + S_3 = \mathrm{e}^x$ and calculate $\mathrm{e}^{\omega x}$ and $e^{\omega^2 x}$.]

5. The relativistic expression for the energy of a particle of mass m is

$$E = \frac{mc^2}{(1 - v^2/c^2)^{1/2}}$$

where v is the particle velocity and c the speed of light. Expand this $O(v^4/c^4)$ and identify the terms you obtain.

[1] These problems were compiled by Prof. D. Jaksch based on problem sets by Prof. G.G. Ross, Prof. J. Yeomans and Prof. N. Harnew, and past Oxford Prelims exam questions.

6. Find the stationary points of $z(x, y) = x^2 - y^2$ and determine the shapes of the contour lines close to these points. Sketch $z(x, y)$ in 3-dimensions and draw a contour map.

7. Find the eigenvalues and normalised eigenvectors of the Hermitian matrix

$$\mathbf{H} = \begin{pmatrix} 10 & 3\mathrm{i} \\ -3\mathrm{i} & 2 \end{pmatrix} .$$

Hence construct a unitary matrix $\mathbf{U}$ such that $\mathbf{U}^\dagger\mathbf{H}\mathbf{U} = \mathbf{D}$ where $\mathbf{D}$ is the real diagonal matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix} ,$$

that has the eigenvalues of $\mathbf{H}$ as its diagonal elements.

[Hint: Avoid a common mistake. Remember that the normalisation condition for a vector $\mathbf{v}$ that has complex elements is $\mathbf{v}^\dagger\mathbf{v} = 1$ (as opposed to $\mathbf{v}^T\mathbf{v} = 1$).]

8. Show that the quadratic surface $5x^2 + 11y^2 + 5z^2 - 10yz + 2xz - 10xy = 4$ is an ellipsoid with semi-axes of lengths 2, 1 and 1/2. Find the direction of its longest axis.

[Hint: You might find this question quite tricky, but nevertheless it is very instructive. First write the left hand side of the equation in matrix form $\mathbf{r}^T\mathbf{O}\mathbf{r}$, with $\mathbf{r} = (x, y)^T$, and calculate the eigenvectors and eigenvalues of symmetric matrix $\mathbf{O}$. This will then allow a change of basis (importantly to a rotated and/or reflected basis) from which the answer just drops out. If you need a good reference for this question, see Riley, Hobson and Bence.]

9. [From Prelims 1999] Leibnitz theorem and McLaurin series.

Show that the function $y(x) = \cos(a \arccos(x))$ fulfils the ODE

$$(1 - x^2)y''(x) - xy'(x) + a^2 y(x) = 0 ,$$

where a is a constant. Use Leibnitz' theorem to differentiate this ODE n times and then put $x = 0$ to show that for $n \geq 0$

$$y^{(n+2)}(0) = (n^2 - a^2)y^{(n)}(0) ,$$

where $y^{(n)}(0)$ is the n^{th} derivative of $y(x)$ evaluated at $x = 0$.

Use this result to obtain a terminating power series expansion for $y(x) = \cos(3 \arccos(x))$ in terms of x. Verify that your solution solves the above ODE.

10. Find a continuous solution with continuous first derivative of the system

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \sin(x) + f(x) ,$$

subject to $y(-\pi/2) = y(\pi) = 0$, where

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & x > 0 \end{cases} .$$

[Hint: Calculate a general solution for each of the cases $x \leq 0$ and $x > 0$ and then obtain relations between your four arbitrary constants by making the solutions agree at $x = 0$.]

11. [From Prelims 2010] Let

$$L = a_0 + \sum_{n=1}^{N} a_n \frac{\mathrm{d}^n}{\mathrm{d}t^n} ,$$

be a differential operator with constant coefficients.

(a) Apply L to the function e^{rt} to obtain the characteristic polynomial $p_L(r)$. Assume that $p_L(r) = 0$ has N distinct roots r_k. Show that the equation $L(y) = F_0 e^{\alpha t}$, where F_0 and α are constants and α is not a root of p_L, has the general solution

$$y(t) = y_0(t) + y_1(t)$$

where

$$y_0(t) = \sum_{k=1}^{N} c_k e^{r_k t}$$

is the solution to the homogeneous equation $L(y) = 0$ (the c_k are constants), and

$$y_1(t) = \frac{F_0 e^{\alpha t}}{p_L(\alpha)}$$

is a particular solution to the inhomogeneous equation $L(y) = F_0 e^{\alpha t}$.

(b) Write the general solution to the differential equation representing a damped, driven oscillator

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\gamma \frac{\mathrm{d}y}{\mathrm{d}t} + \omega_0^2 y = F_0 e^{(-\beta + i\omega)t}$$

where γ, ω_0, F_0, β, and ω are real constants, $\omega_0^2 > \gamma^2$, and $-\beta + i\omega$ is not a root of the equation's characteristic polynomial. Give an example of initial conditions that would uniquely determine the solution.

(c) Calculate the magnitude and phase shift, relative to the driving term, of the particular solution $y_1(t)$ of (b). In what way does this particular solution fail if $-\beta + i\omega$ is in fact a root of the characteristic polynomial?

(d) Find an expression for the magnitude and phase shift when $-\beta + i\omega$ is a root of the characteristic polynomial. What new behaviour is exhibited which was not present when $-\beta + i\omega$ was not a root?

Class problems

12. Evaluating derivatives numerically: Use Taylor's theorem to show that when h is small

(a) $f'(a) = (f(a+h) - f(a-h))/(2h)$ with an error $O(h^2 f'''(a))$.

(b) $f''(a) = (f(a+h) - 2f(a) + f(a-h))/h^2$ with an error $O(h^2 f''''(a))$.

Taking $f(x) = \sin(x)$, $a = \pi/6$, and $h = \pi/180$ find from (a) and (b) the approximate values of $f'(a)$ and $f''(a)$ and compare them to exact values.

These finite-difference formulae are often used to calculate derivatives numerically. How would you construct a more precise finite-difference approximation to $f'(a)$?

13. Surfaces: (a) Find the stationary points of $z(x, y)$ and determine the shapes of the contour lines close to these points. (b) Sketch $z(x, y)$ in 3-dimensions and (c) draw a contour map of the surface, where

(i) $z = (4 - x^2 - y^2)^{1/2}$,

(ii) $z = 1 - 2(x^2 + y^2)$,

(iii) $z = xy$,

14. Prove using vector methods that:

(i) The diagonals of a parallelogram bisect each other.

(ii) The diagonals of a rhombus are perpendicular to each other.

(iii) Two lines, drawn from the end-points of the line of diameter of a circle to a common end-point on the circumference, intersect at a right angle.

15. Find the eigenvalues and a set of normalized eigenvectors of the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Verify that its eigenvectors are mutually orthogonal. Would you have expected this?

16. Consider N identical atoms of mass m whose motion is restricted to one dimension, arranged in a line with nearest-neighbours couple via a spring with sping constant κ.

(a) Show that the Newtonian equations of motion may be written in matrix form

$$\ddot{\mathbf{x}} = -\mathbf{M}\mathbf{x},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_N)^{\mathrm{T}}$ is a vector giving the displacements of the atoms from their equilibrium positions, and the dynamical matrix is

$$\mathbf{M} = \frac{\kappa}{m}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

(b) Now (as it is symmetric) consider some unitary matrix $\mathbf{U}$, which diagonalises $\mathbf{M}$, such that $\mathbf{U}^{\dagger}\mathbf{M}\mathbf{U} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix with non-negative diagonal elements ω_n^2. Show that the equations of motion may then be rewritten in terms of N independent harmonic oscillators as

$$\ddot{\mathbf{x}}' = -\mathbf{D}\mathbf{x}',$$

where $\mathbf{x}' = \mathbf{U}^{\dagger}\mathbf{x}$ and $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_N)^{\mathrm{T}}$.

17. Solve the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(3x^2 + 2xy + y^2)\sin(x) - (6x + 2y)\cos(x)}{(2x + 2y)\cos(x)}.$$

18. Find the general solution $x(t)$ to

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = F_0\cos(\omega t + \phi),$$

for all possible values of γ, ω_0, ω and F_0. Interpret your solution, including the special cases.