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# Keble College - Michaelmas 2014

CP3&4: Mathematical methods I&II

Tutorial 6 - Ordinary differential equations

*Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.*
*Leave these at Keble lodge by 5pm on Monday of 5th week.*
*Look at the 'class problems' in preparation for the tutorial session.*
*Suggested reading: RHB 14 and 15, and the lecturer's problem set.*

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## Goals

- Learn how to recognise different types of ordinary differential equations (ODEs) and how to solve them.
- Extend your understanding of linear operators by considering linear differential operators, and appreciate the importance of linearity for ODEs.
- Apply methods for solving ODEs to equations appearing in physical problems.

## Problems

*An ODE relates a function $y(x)$ of one independent variable $x$ to its derivatives and terms containing $x$ only. It is linear if the sum of two solutions is also a solution and similarly for a multiple of a solution. An ODE is homogeneous if every term contains the function or its derivatives (but note that the notion of homogenous differential equation is also used differently with first order ODEs — see below). The order of the equation refers to the highest order derivative appearing in the ODE and is equal to the number of arbitrary constants appearing in its general solution and the number of boundary conditions needed to give a specific solution.*

*There are numerous different types of ODE, and we will consider some useful and commonly appearing types for which a solution is known. You will practice identifying the type of an ODE, or how to change variables and manipulate expressions to obtain an ODE of that type, and become familar with the solving procedure.*

*We begin with first order ODEs, which may be written*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y),$$

*and identify two main solvable types, seperable and exact ODEs. To begin, we note that if $f(x, y) = g(x)h(y)$ then the equation can be solved by integrating with respect to the two variables separately $\int \mathrm{d}y(h(y))^{-1} = \int \mathrm{d}x g(x)$. Such equations are called separable. If $f(x, y) = g(v)$ then the ODE is not separable but may, in some cases, be made separable by changing variables from $y$ to $v$. As a first example, if $v = y/x$, called a homogeneous equation, then changing variable to $v$ gives a separable equation $\frac{\mathrm{d}v}{\mathrm{d}x} = (g(v) - v)/x$. As a second example, consider the same thing with $v = Y/X$ with $Y = y + a$ and $X = x + b$, i.e. homogeneous but for constants. As a third example, referred to as almost separable, if $v = ax + by + c$ then changing variable to $v$ gives separable equation $\frac{\mathrm{d}v}{\mathrm{d}x} = a + bg(v)$.*

[1]These problems were compiled by Prof. D. Jaksch based on problem sets by Prof. G.G. Ross and past Oxford Prelims exam questions.

*Note that, as the solutions are given, Questions 1-5 will not be thoroughly marked. Please make an extra effort to highlight any problems.*

**1**. Solve the following differential equations using the method stated:

(a) Separable $\frac{\mathrm{d}x}{\mathrm{d}t} = (2tx^2 + t)/(t^2x - x)$.

(b) Homogeneous $2\frac{\mathrm{d}y}{\mathrm{d}x} = (xy + y^2)/x^2$.

(c) Homogeneous but for constants $\frac{\mathrm{d}y}{\mathrm{d}x} = (x + y - 1)/(x - y - 2)$.

(d) Almost separable $\frac{\mathrm{d}y}{\mathrm{d}x} = 2(2x + y)^2$.

**Solution**: $\ln(2x^2 + 1) = 2\ln(t^2 - 1) + \mathcal{C}$, (b) $(y - x)/y = \mathcal{C}\sqrt{x}$, (c) $\ln(x - 3/2) = \arctan(u) - \ln(1 + u^2)/2 + \mathcal{C}$ with $u = (y + 1/2)/(x - 3/2)$, (d) $2x + y = \tan(2x + \mathcal{C})$.

*The second main type of solvable ODE is an exact ODE. This is an ODE where $f(x, y) = -A(x, y)/B(x, y)$, $A(x, y) = \partial_x F(x, y)$ and $B(x, y) = \partial_y F(x, y)$. The last two equations are possible iff $\partial_y A(x, y) = \partial_x B(x, y)$ (assuming analycity). An exact ODE is solved by using $A(x, y) = \partial_x F(x, y)$ and $B(x, y) = \partial_y F(x, y)$ to deduce $F(x, y)$, and noting that integrating the ODE reduces to solving $\int \mathrm{d}F(x, y) = \mathcal{C}$. As with separable equations, there are a number of ways to transform inexact equations into exact equations. Given some decomposition into a quotient $f(x, y) = -A(x, y)/B(x, y)$ for arbitrary $A(x, y)$ and $B(x, y)$, one may seek another decomposition $f(x, y) = -A'(x, y)/B'(x, y)$, related to the first by $A'(x, y) = \mu A(x, y)$ and $B'(x, y) = \mu B(x, y)$, such that $\partial_y A'(x, y) = \partial_x B'(x, y)$. The so-called integrating factor $\mu$ can be found by solving the equation $\partial_y A'(x, y) = \partial_x B'(x, y)$. One way to do this is by inspection or trial and error. It can be done systematically in the case $\mu = \mu(x)$ or $\mu = \mu(y)$, leading to $\mu(x) = \exp\left\{\int \mathrm{d}x(A_y - B_x)/B\right\}$ or $\mu(y) = \exp\left\{\int \mathrm{d}y(B_x - A_y)/A\right\}$, respectively (it is easily verified* post factum *whether the assumption $\mu = \mu(x)$ or $\mu = \mu(y)$ is valid). A particular example is a generic linear first order equation $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$, where we might choose to write $A(x, y) = P(x)y - Q(x)$, $B(x, y) = 1$ and thus $\mu(x) = \exp\left\{\int \mathrm{d}x P(x)\right\}$. Instead of finding $F(x, y)$ one usually proceeds via the faster route and notes $\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mu(x)P(x)y = \frac{\mathrm{d}(\mu(x)y)}{\mathrm{d}x} = \mu(x)Q(x)$, which gives $y = (\mu(x))^{-1}(\int \mathrm{d}x\mu(x)Q(x))$. It is equally valid to deduce $F(x, y) = \mu(x)y - \int \mathrm{d}x\mu(x)Q(x)$, leading to the same thing.*

**2**. Solve, using an integrating factor, the following differential equations (a) $\frac{\mathrm{d}y}{\mathrm{d}x} + y/x = 3$, where $x = 0$ at $y = 0$, and (b) $\frac{\mathrm{d}x}{\mathrm{d}t} + x\cos(t) = \sin(2t)$.

**Solution**: (a) $y = 3x/2$, (b) $x = 2\sin(t) - 2 + \mathcal{C}\mathrm{e}^{-\sin(t)}$.

*The final first order example is an equation of type named after Bernoulli $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$, which is solved by changing to the variable $v = y^{1-n}$ to obtain the linear equation $\frac{\mathrm{d}v}{\mathrm{d}x} + (1 - n)P(x)v = (1 - n)Q(x)$. This can be solved using an integrating factor to turn it into an exact ODE, as described above.*

**3**. The equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + ky = y^n \sin(x)\,,$$

where $k$ and $n$ are constants, is linear and homogeneous for $n = 1$. State a property of the solutions $y(x)$ to this equation for $n = 1$ that is not true for $n \neq 1$. Solve the equation for $n \neq 1$ by making the substitution $z = y^{1-n}$.

**Solution**: $y^{1-n} = \mathcal{C}\mathrm{e}^{(n-1)kx} + (n - 1)(\cos(x) + k(n - 1)\sin(x))/(k^2(n - 1)^2 + 1)$.

**4**. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} + y = xy^{2/3}$.

**Solution**: $y = \mathrm{e}^{-x}(-3\mathrm{e}^{x/3} + x\mathrm{e}^{x/3} + \mathcal{C})^3$.

*Eventually you'll have to become proficient in spotting the type of equation and its method of solution without hints. Try a few examples.*

**5**. Solve the following first order differential equations:

(a) $(3x + x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 5y - 8$.

(b) $xy\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = (x + y)^2 e^{-y/x}$.

(c) $x(x-1)\frac{\mathrm{d}y}{\mathrm{d}x} + y = x(x-1)^2$.

(d) $\frac{\mathrm{d}x}{\mathrm{d}t} = \cos(x + t)$, $x = \pi/2$ at $t = 0$.

(e) $\frac{\mathrm{d}y}{\mathrm{d}x} = (x - y)/(x - y + 1)$.

(f) $\frac{\mathrm{d}x}{\mathrm{d}y} = \cos(2y) - x\cot(y)$, $x = 1/2$ at $y = \pi/2$.

**Solution**: (a) $\ln(5y - 8)/5 = (\ln(x) - \ln(x + 3) + \mathcal{C})/3$, (b) $\ln x = \mathcal{C} + \mathrm{e}^{y/x}/(1 + y/x)$, (c) $y = (x^3/2 - 2x^2 + x\ln(x) + \mathcal{C}x)/(x - 1)$, (d) $t + 1 = \tan((x + t)/2) = -\cot(x + t) + \mathrm{cosec}(x + t) = \sin(x + t)/(1 + \cos(x + t))$, (e) $y^2/2 - xy - y + x^2/2 = \mathcal{C}$, (f) $x\sin(y) = -2\cos^3(y)/3 + \cos(y) + 1/2$.

*We could continue to higher order equations in the same manner but we quickly get lost in a zoo of equation types. Instead we focus on linear higher order equations with constant coefficients, which have a nice method for their solution and feature in important physical models (to some extent they are important because they are have a known solution).*

*An order $n$ linear equation with constant coefficients is*

$$Ly = a_0 + \sum_{i=1}^{n} a_n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} = g(x),$$

*but may be rewritten using*

$$L = a_n \left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_2\right)\cdots\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_n\right),$$

*where $\alpha_i$ are the $n$ roots of the auxiliary equation $\sum_{i=1}^{n} a_n \alpha^n = 0$.*

*We use linearity to divide the solving of the equation into two parts. First, we find the so-called complementary function $y = y_{\mathrm{CF}}$, the general solution to the so-called complementary equation $Ly = 0$, which is the homogeneous version of the inhomogeneous equation to be solved. Second, we find any[a] so-called particular integral $y = y_{\mathrm{PI}}$ satisfying $Ly = g(x)$. The general solution to the inhomogeneous equation is then $y = y_{\mathrm{CF}} + y_{\mathrm{PI}}$.*

*To find the complementary function $y_{\mathrm{CF}}$, we use the commutativity of the $\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_i\right)$ to show that, since $y = y_i = A_i \mathrm{e}^{\alpha_i x}$ satisfies $\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_i\right) y = 0$, it is a solution to $Ly = 0$. If all $\alpha_i$ are distinct this gives $n$ independent solutions and $n$ constants and thus $y_{\mathrm{CF}} = \sum_i y_i$ is the general solution to $Ly = 0$. However, if there are $m > 1$ roots taking the same value $\alpha$ (repeated roots) then we will be $m - 1$ independent solutions/constants too few. In this case, the required $m$ independent solutions/constants are found by noting that $y = (\sum_{k=0}^{m-1} b_k x^k)\mathrm{e}^{\alpha x}$ satisfies $\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\right)^m y = 0$ and thus $y$ is a superposition of $m$ independent solutions to $Ly = 0$.*

*We find the particular integral $y_{\mathrm{PI}}$ by trial and (hopefully not) error. The first common case is $g(x) = \sum_{i=0}^{m-1} c_k x^k$, in which case a trial solution $y = \sum_{i=0}^{m-1} C_k x^k$ can be used. The $C_k$ corresponding to $y_{\mathrm{PI}}$ are found in terms of the $c_k$ by substituting the trial function $y$ into $Ly = g(x)$. We obtain $m$ linear equations for the $m$ variables $C_k$, one for each power of $x$, equations we know how to solve.*

*The second common case is $g(x) = g\mathrm{e}^{\alpha x}$, and we use the trial solution $y = G\mathrm{e}^{\alpha x}$. If $\alpha$ is not a root of the auxiliary equation, we find $G = g/\left(\sum_{i=1}^{n} a_n \alpha\right)$. If $\alpha$ is an $m$-times degenerate root then we use that $y = (\sum_{k=0}^{m} G_k x^k)\mathrm{e}^{\alpha x}$ satisfies $\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\right)^m y = G\mathrm{e}^{\alpha x}$ and thus is a suitable trial function.*

*It follows from this how to solve the cases $g(x) = g\sinh(\alpha x)$, $g(x) = g\cosh(\alpha x)$ or $g(x) = g_- \sinh(\alpha x) + g_+ \cosh(\alpha x)$ (the same reasoning applies to sin and cos, with a few factors of $\mathrm{i}$ inserted). We can write each as $g(x) = h_+ \mathrm{e}^{\alpha x} + h_- \mathrm{e}^{-\alpha x}$ and solve using the previous method. The trial function becomes $y = (\sum_{k=0}^{m} H_k x^k)(H_- \sinh(\alpha x) + H_+ \cosh(\alpha x))$ for all cases, where $m$ is the number of roots of the auxiliary equation equal to $\alpha$. Note that in this trial function, only one of the constants $H_-$ and $H_+$ is required, the other is redundant.*

[a] There are an infinite number since if we add the complementary function to a particular integral we obtain another valid particular integral.

*Last, consider the case $g(x) = (\sum_{k=0}^{l-1} g_k x^k)\mathrm{e}^{\alpha x}$, which again can be extended to trigonometric and hyperbolic functions. Note in this case that $y = (\sum_{k=0}^{m+l} G_k x^k)\mathrm{e}^{\alpha x}$ satisfies $\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\right)^m y = (\sum_{k=0}^{l-1} G_l' x^k)\mathrm{e}^{\alpha x}$ and thus is a suitable trial function if $\alpha$ is an $m$-times degenerate root of the auxiliary equation.*

*For complicated sums $g(x) = \sum_i g_i(x)$ of terms of the above types we can, due to linearity, divide solve for each term separately $y_{\mathrm{PI}} = \sum_i y_{\mathrm{PI},i}$.*

**6**. Find the general solution to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 10\cos(x)\,.$$

**Solution**: $y = \mathcal{C}_1\mathrm{e}^{3x} + \mathcal{C}_2\mathrm{e}^{x} + \cos(x) - 2\sin(x)$.

**7**. Show that the general solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2\mathrm{e}^{-x} + x^3\,,$$

is $y = (A + Bx + x^2)\mathrm{e}^{-x} + x^3 - 6x^2 + 18x - 24$, where $A$, $B$ are arbitrary constants.

**8**. Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + (\beta^2 + 1)y = \mathrm{e}^{x}\sin^2(x)\,,$$

for general values of the real parameter $\beta$, including $\beta = 0$ and $\beta = 2$, explaining why the most simple method fails for these two values of $\beta$.

*Probably the most important physical system is the harmonic oscillator. Any classical particle close enough to a global minimum/equilibrium point looks like it is a harmonic oscillator, e.g. a pendulum at small displacements, and many systems behave like one whatever the displacement from equilibrium, e.g. RLC electric circuits. The quantum analogue is also very important, being one of the few systems we can solve exactly and also describing a large number of phenomena, e.g. phonons (lattice vibrations) in solid state physics, and photons in quantum field theory. So a classical driven (potentially damped) harmonic oscillator is a pretty important system to study and gain intuition about. We'll do this next, starting with a driven but not damped oscillator. You'll get a chance to see the phenomenon of resonance.*

**9**. When a varying couple $I\sin(\omega t)$ is applied to a torsional pendulum with natural period $2\pi/\omega_0$ and moment of inertia $I$, the angle $\theta$ of the pendulum satisfies an equation of motion written $\ddot{\theta} + \omega_0^2\theta = \sin(\omega t)$. The couple is first applied at $t = 0$ when the pendulum is at rest in equilibrium, $\theta = 0$. Show that in the subsequent motion the root mean square $\theta_{\mathrm{rms}} = \sqrt{T^{-1}\int_0^T \mathrm{d}t\theta^2(t)}$ angular displacement is $\theta_{\mathrm{rms}} = 1/|\omega_0^2 - \omega^2|$ when the average is taken over a time $T$ large compared with $1/|\omega_0 - \omega|$. Discuss (i.e. use some words and physically interpret your solution) the motion as $|\omega_0 - \omega| \to 0$.

*This final question on the driven and damped harmonic oscillator reveals the usefulness of using complex exponentials rather than trigonometric functions.*

**10**. A mass $m$ is constrained to move in a straight line and is attached to a spring of strength $\lambda^2 m$ and a dashpot that produces a retarding force $-\alpha m v$, where $v$ is the velocity of the mass. Find the steady state displacement of the mass when an amplitude-modulated periodic force $Am\cos(pt)\sin(\omega t)$ with $p \ll \omega$ and $\alpha \ll \omega$ is applied to it, where all parameters given above are real.

Show that for $\omega = \lambda$ the displacement of the amplitude-modulated wave is approximately given by

$$-A\frac{\cos(\omega t)\sin(pt+\phi)}{\sqrt{4\omega^2p^2+\alpha^2\omega^2}}\,,$$

where

$$\cos(\phi) = \frac{2\omega p}{\sqrt{4\omega^2p^2+\alpha^2\omega^2}}\,,$$

where $A$ is a constant.

**Solution**: Your solution to this question may need some extra lines of work that do not fit onto the dots below, which, however, should still provide useful guidance. Using Newton's law for the position of the mass $x$ we find

$$m\ddot{x} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = Am\cos(pt)\sin(\omega t)\,.$$

We rearrange this equation by dividing through $m$. We then rewrite the right hand side as

$$A\cos(pt)\sin(\omega t) = \frac{A}{2}[\sin((\omega+p)t) + \ldots\ldots\ldots\ldots]\,.$$

We do not need to work out the solution to the homogeneous equation because we are only interested in the ............. and thus only consider the particular solution (the complimentary function will eventually decay in time, which can be verified — physically, any evolution due to the initial conditions will decay due to damping, only motion caused by the driving will remain at long times).

There are many ways to proceed. We could use the trial function $x = B\sin((\omega+p)t) + C\cos((\omega+p)t) + D\sin((\omega-p)t) + E\cos((\omega-p)t)$ or the trial function $x = Be^{i(\omega+p)t} + Ce^{-i(\omega+p)t} + De^{i(\omega-p)t} + Ee^{-i(\omega-p)t}$. You are at risk of dying from boredom with these two approaches. A better idea is to consider the equation

$$m\ddot{z} + \alpha m\dot{z} + \lambda^2 mz = \frac{A}{2}[e^{i(\omega+p)t} + e^{-i(\omega+p)t}]\,.$$

By linearity, taking the real part, we see that if $z$ is a solution to the above equation then $x = \Im\{z\}$ is a solution to the original equation. Thus we focus on solving the equation for $z$ using the trial function $x = C_+e^{(\omega+p)t} + C_-e^{-(\omega+p)t}$, or using linearity to trial the two terms separately. This is half the number of terms we previously had and is just about bearable.

Doing this we find

$$C_\pm = \frac{A}{2(\lambda^2 - (\omega\pm p)^2 + i\ldots\ldots(\omega\pm p))}\,.$$

We can put this into exponential form

$$C_\pm = \frac{A}{2}\frac{e^{i\phi_\pm}}{\sqrt{(\lambda^2-(\omega\pm p)^2)^2+\alpha^2(\omega\pm p)^2}}\,,$$

where $\phi_\pm$ satisfies $\tan(\phi_\pm) = \ldots\ldots\ldots\ldots$ We therefore obtain the solution

$$z = \frac{A}{2}\sum_\pm \frac{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}{\sqrt{(\lambda^2-(\omega\pm p)^2)^2+\alpha^2(\omega\pm p)^2}},$$

to the complex equation and thus

$$x = \frac{A}{2} \sum_{\pm} \frac{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}{\sqrt{(\lambda^2 - (\omega \pm p)^2)^2 + \alpha^2(\omega \pm p)^2}},$$

for the original real equation.

We now put $\lambda = \omega$ and only keep the lowest order terms in $p/\omega$ and $\alpha/\omega$, which gives

$$x = \frac{A}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

and can be re-written in the form given in the question by using ........................................................ ........................................................ . You won't get a much more complicated question than that!

## Class problems

**11**. State the order of the following differential equations and whether they are linear or non-linear: (i) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + k^2y = f(x)$ (ii) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(x)$ (iii) $\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = yx$.

**12**. $L_1$ is the differential operator

$$L_1 = \left(\frac{\mathrm{d}}{\mathrm{d}x} + 2\right).$$

Evaluate (i) $L_1x^2$, (ii) $L_1(x\mathrm{e}^{2x})$, (iii) $L_1(x\mathrm{e}^{-2x})$.

**13**. $L_2$ is the differential operator

$$L_2 = \left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right).$$

Express the operator $L_3 = L_2L_1$ in terms of $\frac{\mathrm{d}}{\mathrm{d}x}$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, etc. Show that $L_1L_2 = L_2L_1$. What is different if instead

$$L_2 = \left(\frac{\mathrm{d}}{\mathrm{d}x} - x\right).$$

**14**. Solve the following first order differential equations:

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = (x - y\cos(x))/\sin(x)$.

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} + 2x/y = 3$.

(c) $\frac{\mathrm{d}y}{\mathrm{d}x} + y/x = 2x^{3/2}y^{1/2}$.

(d) $2\frac{\mathrm{d}y}{\mathrm{d}x} = y/x + y^3/x^3$.

(e) $2x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^2$.

(f) $\frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^y/(1 + x^2)$, where $y = 0$ at $x = 0$.

**Solution**: (a) $y = (x^2/2 + \mathcal{C})/\sin(x)$, (b) $(2x - y)^2 = \mathcal{C}(y - x)$, (c) $\sqrt{y} = x^{5/2}/3 + \mathcal{C}x^{-1/2}$, (d) $y^2 = x^2/(1 + \mathcal{C}x)$, (e) $y = x^2/3 + \mathcal{C}\sqrt{x}$, (f) $\mathrm{e}^{-y} = 1 - \ln[1 + x^2]/2$.