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Keble College - Michaelmas 2014

CP3&4: Mathematical methods I&II

Tutorial 4 - Calculus II

Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.
Leave these at Keble lodge by 5pm on Monday of 3rd week.
Look at the 'class problems' in preparation for the tutorial session.
Suggested reading: RHB 4 and 5, and the lecturer's problem sets.

Goals

- Develop an understanding of how series expansions and limits may be applied to mathematics and physics problems.
- Learn how to extend the concepts of differentiation and series expansions to higher dimensions.
- In particular, learn how to analyse properties of surfaces in three dimensional space using partial differentiation.

Problems

Taylor series are an extremely important tool for analysing functions, especially if one is concerned with the behaviour of/values taken by the function in the vicinity of a point. They can be used to e.g. aid differentiation and integration, deduce relationships between functions, approximate the value of the function in the vicinity of a point, deduce the type of behaviour the function exhibits in the vicinity of a point, extend the domain of function (as you saw for complex numbers and as you will see for matrices) and much much more.

The Taylor series of a function $f(x)$ about x_0 is defined by

$$\sum_{n=0}^{\infty} f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}\,.$$

The function is analytic if the Taylor series is convergent and equals $f(x)$ over some open interval. A Maclaurin series is a Taylor series with $x_0 = 0$ and is often simply called a power series expansion. Get used to Taylor series by answering the following two questions.

1. If possible, find the Maclaurin series of each of the following functions, writing out terms in up to and including those of order x^3, and comment on the continuity, differentiability and analyticity of the function: (i) e^x, (ii) $\sqrt{1+x}$, (iii) $\tan^{-1}(x)$, (iv) e^{-1/x^2}, (v) the step function $\theta(x) = 0, 1/2, 1$ for $x < 0, x = 0, x > 0$, and (vi) $|x|$.

Solution: (i) $\mathrm{e}^x = 1 + x + x^2/2 + x^3/6 + ...$, (ii) $\sqrt{1+x} = 1 + x/2 - x^2/8 + x^3/16 + ...$, (iii) $\tan^{-1}(x) = x - x^3/3 + ...$, (iv) 0, (v) and (vi) no Maclaurin series. All, except (v), are continuous. All, except (v) and (vi), are differentiable. All, except (iv), (v) and (vi), are analytic.

[1] These problems were compiled by Prof. D. Jaksch based on problem sets by Prof. J. Yeomans and past Oxford Prelims exam questions.

For $|x - x_0| \ll 1$ the terms in the Taylor series (see previous box) usually get smaller with increasing n so that the first few terms approximate the function well near x_0. Taylor's theorem (not on syllabus) gives quantitative estimates for the error in this approximation.

2. Obtain the value of $\sin(31°)$ by Taylor expanding $\sin(x)$ up to the fourth term about the point $x_0 = \pi/6$. How accurate is your answer?

Solution: $\sin(31°) \approx 0.515038$, accuracy better than 10^{-8}.

The very fact that we are able to exactly or approximately write a function as a Taylor series makes it simple to find the integral or derivative of a function exactly or approximately. This is another use of Taylor series.

3. Write down the power series expansion for $x^{-1}\sin(x)$. Hence evaluate, to four significant figures, the integral $I = \int_0^1 x^{-1}\sin(x)\mathrm{d}x$.

Solution: $I \approx 0.9461$.

Taylor series also offer a means of rewriting the limits of combinations of functions into a form from which its well-defined value can be deduced. For example, for $f(x)$ and $g(x)$ that are analytic around $x = c$, it can be shown from their Taylor series (try it) that, provided the limit is well defined,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}\,,$$

if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$. This is a special case of l'Hôpital's rule, which holds more generally in the case that $f(x)$ and $g(x)$ are merely differentiable and if $\lim_{x\to c}|f(x)| = \lim_{x\to c}|g(x)| = \infty$. This rule together with other clever manipulations and substitutions allow a wide range of limits to be evaluated. When evaluating such limits please try not to write down any ill-defined expressions, such as $0/0$, ∞/∞ or $\infty/0$.

4. Use Taylor series or directly apply L'Hôpital's rule to evaluate the values taken by the functions

(i) $\sin(x)/x$,

(ii) $(1 - \cos^2(x))/x^2$,

(iii) $(\sin(x) - x)/(\mathrm{e}^{-x} - 1 + x)$,

in the limits (a) $x \to 0$, (b) $x \to \infty$:

Solution: (a) (i) 1, (ii) 1, (iii) 0; (b) (i) 0, (ii) 0, (iii) -1.

5. Expand $[\ln(1+x)]^2$ in powers of x as far as x^4. Hence determine the limit of $[\ln(1+x)]^2/[x(1-\cos(x)]$ as $x \to 0$.

Solution: ∞.

Consider a small change δx in x away from x_0. It is clear from the Taylor series that for an analytic function $f(x)$ the corresponding change in its value will be

$$\delta f = \sum_{n=1}^{\infty} f^{(n)}(x_0) \frac{\delta x^n}{n!} \,.$$

Usually, by analysing the first few terms of this expansion we can deduce whether x_0 is a stationary point of the function and what type of stationary point it is. It is a stationary point iff δf is zero to first order in δx i.e. $f'(x_0) = 0$. It is a maximum if to leading order δf is always positive, e.g. if $f''(x_0) < 0$, and a minimum if to leading order δf is always negative, e.g. if $f''(x_0) > 0$. Otherwise it is a saddle point (also known for single-variable functions as a point of inflection). Note that if $f''(x_0) = 0$ then we need to consider higher order terms to infer the type of stationary point.

6. Using the results of the previous question, or not, determine whether $\cos(2x) + [\ln(1+x)]^2$ has a maximum, minimum or point of inflection at $x = 0$.

Solution: Maximum.

It is clear that physics involves quantities that depend on more than one variable, which in turn necessitates our learning about functions of more than one variable. The variables could be independent, e.g. the temperature of the atmosphere as function of 3D coordinates, or they could be related, e.g. the temperature on a balloon's surface as a function of 3D coordinates. In the same way that a quantity $Q = f(x)$ dependent a single variable x could instead be represented by a function $Q = g(y)$ of a related variable $y = y(x)$, a quantity dependent on more than one variable may be described by many different functions of different combinations of the variables. For instance, one can express the energy E of an ideal gas in terms of its particle number and temperature (N, T) or in terms of the volume and pressure (V, p). These parameters are related by the ideal gas law $pV = NRT$ with R the ideal gas constant. Formally we write $E = f(N, T) = g(V, p)$ to represent the two functional dependencies. However, as we noted for functions of a single variable, often it will sloppily be written that $E = E(N, T)$ and $E = E(V, p)$ even though the functions E appearing in the two equations are different. As another example, consider expressing height $z = f(x, y)$ as a function of two variables x and y. If $x = x(t)$ and $y = y(t)$ follow a path parameterised by time t, then we could equally think of $z = g(t)$ as depending on a single variable t. Be aware of such notational issues and try to avoid confusion. Separate the concepts of a quantity and a function in your mind. Don't assign a quantity and function the same symbol unless you plan on using only a single functional representation or make it very clear what is going on through other means.

The concepts of gradients, Taylor series, limits and stationary points can all vitally be extended to functions of more than one variable. Their mathematical treatment rests on the concept of a partial derivative. The partial derivative with respect to x of a function in more than two (generalisation to more variables is trivial) variables $f(x, y)$ is defined as

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_y = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} \,.$$

The variables listed right of $|$ are treated as being constants. It is important to indicate which variables are being held constant in expressions of the above type. If it is absolutely clear what variables are being held constant one often writes e.g. $\partial f/\partial x = \partial_x f = f_x$, $\partial^2 f/\partial x \partial y = \partial_x \partial_y f = \partial_{xy} f = f_{xy}$. All of the notational ambiguities present in ordinary differentiation remain here, so work as clearly as you can, especially while you are learning.

7. (a) Find $\partial f/\partial x|_y$ for $f(x,y)$ equal to (i) $(x^2+y^2)^{1/2}$, (ii) $\tan^{-1}(y/x)$, (iii) y^x.

(b) Verify that $f_{xy} = f_{yx}$ for $f(x,y)$ equal to (i) $(x^2+y^2)\sin(x+y)$, (ii) $x^m y^n$.

(c) The function $f(x,y)$ is such that $f_{xy} = 0$. Find the most general forms for f_x and f_y and show that $f(x,y)$ has the form $f(x,y) = F(x) + G(y)$ with arbitrary functions F and G.

When dealing with quantities Q written as functions $Q = f(x)$ of a single variable, we introduced the idea of an ordinary derivative $\frac{\mathrm{d}Q}{\mathrm{d}x} = f'(x)$ with respect to this variable x. We showed that since $Q = g(y)$ can be written as a function of another variable $y = y(x)$ we can consider the derivative $\frac{\mathrm{d}Q}{\mathrm{d}y} = g'(y)$ of the same quantity with respect to that variable. The two derivatives are related by $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}g(y(x))}{\mathrm{d}y}\frac{\mathrm{d}y(x)}{\mathrm{d}x}$ or less precisely $\frac{\mathrm{d}Q}{\mathrm{d}x} = \frac{\mathrm{d}Q}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}$, i.e. the chain rule. With quantities that depend on more than one variable there are even more ways in which a quantity Q can be written as functions of different variable and therefore even more derivatives to consider and relate. The first thing to note is that it means we have to be very precise to express exactly which derivative we mean, as the next problem demonstrates.

8. Calculate $\partial Q/\partial x|_y$, $\partial Q/\partial x|_u$ and $\partial Q/\partial x|_v$ for $Q = x + y = f(x,y) = g(x,u) = h(x,v)$ and $u = x + y$, $v = x - y$, showing them to be different.

Solution: $\partial Q/\partial x|_y = 1$, $\partial Q/\partial x|_u = 0$, $\partial Q/\partial x|_v = 2$.

With functions of more than one variable, there is no one general formula for relating the different derivatives as was the case for the chain rule. However, the same careful thought through which one can derive the chain rule can be applied to derive the relations in each case. A good way to think about this is to imagine infinitesimal changes, called differentials, of the variables/quantity and how these must be related. Then to use these relations to find the relations between the derivatives. In the case of the chain rule, we know that for $Q = f(x) = g(y)$ and $y = y(x)$ that infinitesimal changes $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}Q$ around the point x must, from the definition of a derivative and an assumption of differentiability, be related by $\mathrm{d}Q = \frac{\mathrm{d}f(x)}{\mathrm{d}x}\mathrm{d}x = \frac{\mathrm{d}g(y(x))}{\mathrm{d}y}\mathrm{d}y$. Rearranging this, we get $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}g(y(x))}{\mathrm{d}y}\frac{\mathrm{d}y(x)}{\mathrm{d}x}$. The ratio $\frac{\mathrm{d}y(x)}{\mathrm{d}x}$ of two related changes in the limit that they are infinitesimal is correctly identified as the derivative of y with respect to x.

This approach generalises. Consider now $Q = f(x,y)$. The definition of a partial derivative and an assumption of differentiability, leads to differentials related by

$$\mathrm{d}Q = \left.\frac{\partial f(x,y)}{\partial x}\right|_y \mathrm{d}x + \left.\frac{\partial f(x,y)}{\partial y}\right|_x \mathrm{d}y\,.$$

This is the infinitesimal change in the value of Q if the variable x is changed from x by $\mathrm{d}x$ and y from y by $\mathrm{d}y$. It is easily generalised to more variables. Manipulating such relations between differentials, one can derive relations between derivatives, as we will now practice.

9. If $f(x,t) = g(x - ct) + h(x + ct)$ where c is a constant, prove that $f_{xx} - f_{tt}/c^2 = 0$.

10. (a) Let $u = f(x,y) = g(t)$, $x = x(t)$, $y = y(t)$. Show that $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = \left.\frac{\partial f(x(t),y(t))}{\partial x}\right|_y \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \left.\frac{\partial f(x(t),y(t))}{\partial y}\right|_x \frac{\mathrm{d}y(t)}{\mathrm{d}t}$, often abbreviated as $\frac{\mathrm{d}u}{\mathrm{d}t} = \left.\frac{\partial u}{\partial x}\right|_y \frac{\mathrm{d}x}{\mathrm{d}t} + \left.\frac{\partial u}{\partial y}\right|_x \frac{\mathrm{d}y}{\mathrm{d}t}$.

(b) Use this to find $\mathrm{d}u/\mathrm{d}t = g'(t)$ when (i) $u = x^n y^n$ and $x = \cos(at)$, $y = \sin(at)$, where a, n are constants, and (ii) $u = x^2 y + y^{-1}$ and $y = \ln(x)$.

(c) Verify your solutions for both (i) and (ii) by applying the normal rules (product and chain) of ordinary differentiation directly to the expression of u.

Solution: (i) $na\cos^{n-1}(at)\sin^{n-1}(at)(\cos^2(at) - \sin^2(at))$, (ii) $x + 2x\ln(x) - 1/[x(\ln(x))^2]$.

11. (a) The perfect gas law $pV = RT$ may be regarded as defining any one of the quantities pressure p, volume V, or temperature T, for a fixed particle number N of perfect gas, as a function of the other two. Verify explicitly that

$$\left.\frac{\partial p}{\partial V}\right|_T \left.\frac{\partial V}{\partial T}\right|_p \left.\frac{\partial T}{\partial p}\right|_V = -1\,, \quad \text{and} \quad \left.\frac{\partial p}{\partial V}\right|_T = 1/\left.\frac{\partial V}{\partial p}\right|_T .$$

Note the notational simplifications in this question and many that follow. You may choose to work with more precise notation.

(b) Show that this is true whenever there is a relationship of the form $f(p, V, T) = 0$ between p, V and T.

12. Change of variable [from Prelims 1997]. A variable z may be expressed either as a function of (u, v) or of (x, y), where $u = x^2 + y^2$, $v = 2xy$.

(a) Find

$$\left.\frac{\partial z}{\partial x}\right|_y \quad \text{in terms of} \quad \left.\frac{\partial z}{\partial u}\right|_v \quad \text{and} \quad \left.\frac{\partial z}{\partial v}\right|_u .$$

(b) Find

$$\left.\frac{\partial z}{\partial u}\right|_v \quad \text{in terms of} \quad \left.\frac{\partial z}{\partial x}\right|_y \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_x .$$

(c) Express

$$\left.\frac{\partial z}{\partial u}\right|_v - \left.\frac{\partial z}{\partial v}\right|_u \quad \text{in terms of} \quad \left.\frac{\partial z}{\partial x}\right|_y \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_x .$$

(d) Check your expressions by seeing if they hold for the specific case $z = u + v$.

We can generalise the concept of a Taylor series to functions of more than one variable. Consider a function $f(x, y)$ of two variables. The Taylor series about (x_0, y_0) is defined by

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left[\left((x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y} \right)^n f \right]_{x_0, y_0} ,$$

where the subscript of the square bracket indicates the points at which the derivatives are evaluated. Again, the function is analytic if the Taylor series is convergent and equals $f(x, y)$ over some open interval. Assuming this, the Taylor series representation of $f(x, y)$ may be derived by assuming $f(x, y)$ can be expanded as a Taylor series in x and y separately (try it), as well as by other means. Also, one can show (try it) that continuity of the partial derivatives implied by analyticity means that the order of multiple partial differentiations is irrelevant, in particular $f_{xy} = f_{yx}$, which simplifies the expression when the Taylor series is expanded.

Taylor series of functions of more than one variable have all the same uses as for functions of a single variable. For the rest of this problem set, however, we'll focus on the use of Taylor series in identifying and classifying stationary points of functions of two variables. It may be useful to compare with what we previously said for a function of one variable. Consider a small change δx in x and δy in y away from (x_0, y_0). It is clear from the Taylor series that for an analytic function $f(x,y)$ the corresponding change in its value will be

$$\delta f = \sum_{n=1}^{\infty} \frac{1}{n!} \left[\left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y} \right)^n f \right]_{x_0, y_0} .$$

Usually, by analysing the first few terms of this expansion we can deduce whether (x_0, y_0) is a stationary point of the function, and what type of stationary point it is. It is a stationary point iff δf is zero to first order in δx and δy. Looking at the first order terms $\delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y}$ reveals that (x_0, y_0) is a stationary point iff $f_x = f_y = 0$ at that point. It is a maximum if to leading order δf is always positive and a minimum if to leading order δf is always negative. Otherwise it is a saddle point. We can usually tell this from the second order terms $\frac{1}{2}\left[\delta x^2 f_{xx} + \delta y^2 f_{yy} + 2\delta x \delta y f_{xy}\right]$, which for $f_{xx} \neq 0$ and $f_{yy} \neq 0$ respectively rearrange to $\frac{1}{2}\left[f_{xx}\left(\delta x + f_{xy}\delta y/f_{xx}\right)^2 + \delta y^2\left(f_{yy} - f_{xy}^2/f_{xx}\right)\right]$ and $\frac{1}{2}\left[f_{yy}\left(\delta y + f_{xy}\delta x/f_{yy}\right)^2 + \delta x^2\left(f_{xx} - f_{xy}^2/f_{yy}\right)\right]$. It is then clear that if f_{xx} and f_{yy} take finite values with different signs then the second order contribution to δf can be both positive and negative and thus (x_0, y_0) is a saddle point. If $f_{xx}, f_{yy} < 0$ and $f_{xy}^2 < f_{xx} f_{yy}$ then the second order contribution to δf is strictly negative and thus (x_0, y_0) is a maximum. Similarly, if $f_{xx}, f_{yy} > 0$ and $f_{xy}^2 < f_{xx} f_{yy}$, it is strictly positive and thus (x_0, y_0) is a minimum. If $f_{xy}^2 > f_{xx} f_{yy}$ then it can be both positive and negative and thus (x_0, y_0) is a saddle point. The unexplored cases are when $f_{xy}^2 = f_{xx} f_{yy}$, and f_{xx} and f_{yy} are not both finite and of opposite sign. For these cases we need to consider higher order terms to infer the sign of the leading order terms to δf and the type of stationary point.

13. Find the position and nature of the stationary points of the following functions and sketch rough contour graphs in each case. (i) $f(x,y) = x^2 + y^2$, (ii) $f(x,y) = x^3 + y^3 - 2(x^2 + y^2) + 3xy$, (iii) $f(x,y) = \sin(x)\sin(y)\sin(x+y)$, the latter for $0 < x < \pi/2$ and $0 < y < \pi/2$.
[Note: The symmetry of the value taken by the functions when swapping x and y suggests that the mathematics may be simpler/or sketching easier when using rotated coordinates $u = (x+y)/\sqrt{2}$ and $v = (x-y)/\sqrt{2}$. One could work with another function $g(u,v) = f(x,y) = Q$, whose form is easily found. Since they describe the same quantity Q, they must have corresponding stationary points of the same type. This is perhaps obvious for a rotation but holds more generally. Even if you don't use these alternative variables, the symmetry should help you in your sketching.]

14. Figure 1 shows a Mathematica file producing contour and surface plots of the function $f(x,y) = x^3 + y^3 - 2(x^2 + y^2) + 3xy$. Use the results from question **13** to mark the stationary points in these plots.

Class Problems

15. Sketch the following functions and state whether they are (i) continuous and/or (ii) differentiable throughout the domain $-1 \leq x \leq 1$?

(a) $f(x) = 0$ for $x \leq 0$, $f(x) = x$ for $x > 0$,
(b) $f(x) = 0$ for $x \leq 0$, $f(x) = x^2$ for $x > 0$,
(c) $f(x) = 0$ for $x \leq 0$, $f(x) = \cos(x)$ for $x > 0$,
(d) $f(x) = |x|$.

```
In[90]:= f[x_, y_] = x^3 + y^3 - 2 (x^2 + y^2) + 3 x y;
GraphicsRow[
 {ContourPlot[{f[x, y] == f[0.1, 0.1], f[x, y] == f[1.75, 1.75], f[x, y] == f[1 / 3, 1 / 3],
    f[x, y] == -0.5, f[x, y] == -2, f[x, y] == 0.5, f[x, y] == 3, f[x, y] == -5}, {x, -1, 2},
   {y, -1, 2}, FrameTicks → {{-1, 0, 1, 2}, {-1, 0, 1, 2}}, FrameLabel → {x, y}],
  Plot3D[f[x, y], {x, -1, 2}, {y, -1, 2}, Ticks → {{-1, 0, 1, 2}, {-1, 0, 1, 2}, {0, 10}},
   AxesLabel → {"x", "y", "f(x,y)"}]}]
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Out[91]=

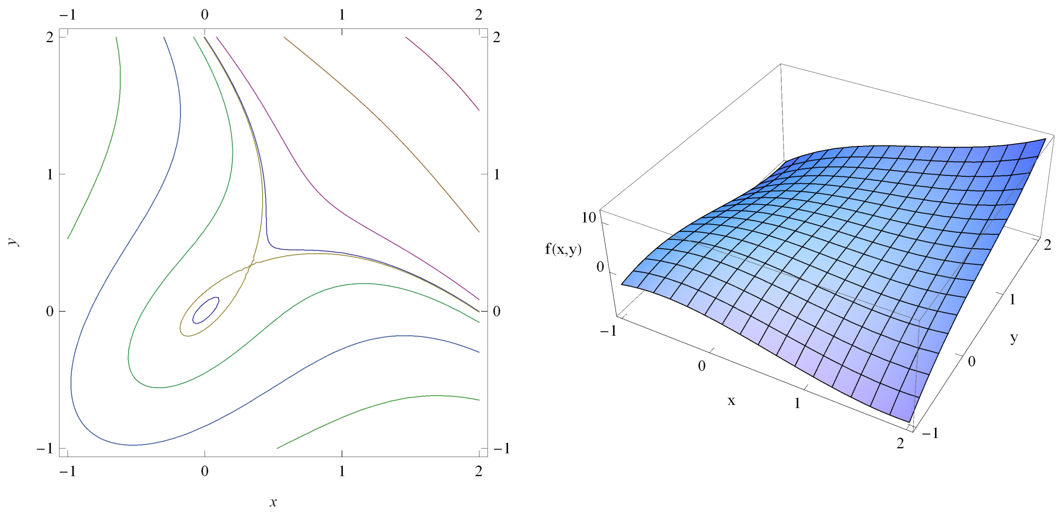

Figure 1: Surface and contour plots of $f(x,y) = x^3 + y^3 - 2(x^2+y^2) + 3xy$.

16. Chain rule. If $w = \mathrm{e}^{-x^2-y^2}$, $x = r\cos(\theta)$, $y = r\sin(\theta)$, find $\frac{\partial w}{\partial r}\big|_\theta$ and $\frac{\partial w}{\partial \theta}\big|_r$ in two ways.

17. Taylor series in two variables. Expand $f(x,y) = e^{xy}$ to three terms around the point $x = 2$, $y = 3$.

18. Exact differentials.

(a) Which of the following differentials are exact? For those that are exact, find f. (i) $\mathrm{d}f = x\mathrm{d}y + y\mathrm{d}x$, (ii) $\mathrm{d}f = x\mathrm{d}y - y\mathrm{d}x$, (iii) $\mathrm{d}f = x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z$.

(b) What is the value of $\oint(x\mathrm{d}y + y\mathrm{d}x)$ around the closed curve $x^4 + y^4 = 1$?