Tomi Johnson[1]

Keble College - Michaelmas 2014

CP3&4: Mathematical methods I&II

Tutorial 2 - Calculus I

Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.
Leave these at Keble lodge by 5pm on Friday of 1st week.
Look at the 'class problems' in preparation for the tutorial session.
Suggested reading: RHB 2 and 11.1, and the lecturer's problem sets.

Goals

- Become proficient in differentiating and integrating, including solving non-standard problems for which several basic techniques may be required.
- Learn to deduce properties of definite integrals from their integrand.
- Relate values of definite integrals to physical quantities like volumes or lengths of curves.

Problems

The derivative of a function $f(x)$ at x is defined as the limit

$$f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} = \left.\frac{\mathrm{d}f(y)}{\mathrm{d}y}\right|_{y=x} = \lim_{\delta x \to 0} \left\{ \frac{f(x+\delta x) - f(x)}{\delta x} \right\} .$$

Note, worry about, understand and accept the variety of notations. This includes the unfortunate use of $f(x)$ to mean both the function and the value it takes for argument x, and the use of x in $\mathrm{d}f(x)/\mathrm{d}x$ to denote both the quantity with respect to which the differentiation is taking place and the value at which the derivative is being evaluated. It also often includes the omission in $\mathrm{d}f/\mathrm{d}x$ of the value at which the derivative is being evaluated, even when this is not obvious. There are many such notational issues in calculus. It is up to you to carefully think about what is meant by each expression and to be as clear as possible when writing. Don't ignore these subtleties as this would prevent you from gaining a comprehensive understanding of the topic.

The derivative of the derivative of the function at x is denoted $f''(x) = \mathrm{d}^2 f(x)/\mathrm{d}x^2$ and similarly the n-th derivative is denoted $f^{(n)}(x) = \mathrm{d}^n f(x)/\mathrm{d}x^n$. From these fundamental definitions one obtains the derivatives of a large number of common functions, which form the basic tools of a differentiator. You are asked to work through two examples.

1. Starting from the above definition for the derivative, evaluate $f'(x)$ for (i) $f(x) = x^2$ and (ii) $f(x) = \sin x$.

Solution: (i) $2x$, (ii) $\cos x$.

[1]These problems were compiled by Prof. D. Jaksch based on problem sets by Prof. J. Yeomans and past Oxford Prelims exam questions.

Also from the above definition one can show (you may wish to satisfy yourself of this) the following rules for calculating derivatives:

- *Product rule:* $\mathrm{d}(f(x)g(x))/\mathrm{d}x = f(x)g'(x) + g(x)f'(x)$.
- *Chain rule:* $\mathrm{d}f(g(x))/\mathrm{d}x = f'(g(x))g'(x)$.

(From these two rules also follows the quotient rule.) These rules allow one to differentiate a complicated function quickly using prior knowledge of the derivatives of its constituent parts. Work through a single example that requires the use of both rules.

2. Using the solution to the previous question, calculate the derivative of $f(x) = x^2 \cos(x^2)$.

Solution: $2x[\cos(x^2) - x^2 \sin(x^2)]$.

The chain rule is more powerful than how it is used to answer the above question. It can be used to do more than differentiate a function that is explicitly represented (if you don't know what this means, note $f(x) = x^2$ is explicit but $x^2 + (f(x))^2 = 1$ is implicit). It allows one to find relationships between the derivatives of two functions given an implicit representation of one in terms of the other. This is helpful, as often the derivatives of one of the functions is known or considerably easier to find than the other. In the next three questions you will be asked to do this.

A first example is the use of the chain rule to find an expression for the derivative of the inverse of a function in terms of the derivative of the function, where the inverse $f^{-1}(x)$ of the function $f(x)$ is implicitly related to it by $f^{-1}(f(x)) = x$. In the next two questions, you are asked to derive and then use that expression.

3. Show (i) using the chain rule and (ii) graphically that the derivative of $f^{-1}(x)$ is given by

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))} ,$$

and (iii) explain when this can be re-written as

$$\frac{\mathrm{d}x(y)}{\mathrm{d}y} = \left(\frac{\mathrm{d}y(x)}{\mathrm{d}x}\right)^{-1} .$$

[Hint: All of the notational issues mentioned above arise in this question. Please think and write carefully. It is easy to write something incorrect whilst still arriving at the correct expression.]

Solution: (i) Apply the chain rule directly to the expression $f^{-1}(f(x)) = x$. (ii) Sketch the curves $y = f(x)$ and $y = f^{-1}(x)$ noting that they are reflections in the line $y = x$. Use this to argue that the product of the gradients of the lines at some x_0 and $f(x_0)$, respectively, must multiply to unity. (iii) Let $y(x) = f(x)$ so that $x(y) = f^{-1}(y)$, calculate the derivatives of x and y with respect, compare these to terms appearing in the solution to (i) and (ii).

4. Use the solution to the previous question to calculate the derivatives of (i) $f^{-1}(x) = \sin^{-1}(x)$, (ii) $f^{-1}(x) = \tan^{-1}(x)$, and (iii) $f^{-1}(x) = \log_2(x)$.

Solution: (i) $\pm 1/\sqrt{1 - x^2}$, (ii) $1/(1 + x^2)$, (iii) $1/(x \ln 2)$.

Another application of the use of the chain rule to relate the derivatives of two functions is in solving differential equations. Often the solution to a seemingly difficult differential equation in $y(x)$ can be found by using the chain rule to first obtain from the original differential equation in $y(x)$ a more obviously tractable differential equation in a related variable $z(x)$, second solving this latter equation, and third obtaining the solution $y(x)$ from the solution $z(x)$. You will consider an example of this next.

5. Starting with the differential equation for $y(x)$ that is perhaps difficult to directly solve

$$x^2\frac{\mathrm{d}^2y(x)}{\mathrm{d}x^2} + (4x+3x^2)\frac{\mathrm{d}y(x)}{\mathrm{d}x} + (2+6x+2x^2)y(x) = x,$$

show that the variable $z(x)$, where $z(x) = y(x)x^2$, obeys the differential equation

$$\frac{\mathrm{d}^2z(x)}{\mathrm{d}x^2} + 3\frac{\mathrm{d}z(x)}{\mathrm{d}x} + 2z(x) = x\,.$$

This is a second order differential equation with constant coefficients that can be shown easily, using techniques you will learn this term, to have the solution

$$z(x) = A\mathrm{e}^{-x} + B\mathrm{e}^{-2x} + \frac{x}{2}\,,$$

with A and B constants. Find the solution to the original equation in $y(x)$.

Solution: We start by calculating the first and second derivatives of the new dependent variable $z(x)$

$$\frac{\mathrm{d}z(x)}{\mathrm{d}x} = 2xy(x) + \ldots\ldots\ldots\ldots\ldots\,,$$

and

$$\frac{\mathrm{d}^2z(x)}{\mathrm{d}x^2} = 2y(x) + \ldots\ldots\ldots\ldots\ldots + x^2\frac{\mathrm{d}^2y(x)}{\mathrm{d}x^2}\,,$$

where we have used the .... rule. We now substitute the first term in the differential equation, which gives

$$\frac{\mathrm{d}^2z(x)}{\mathrm{d}x^2} - 2y(x) + (4x+3x^2-4x)\frac{\mathrm{d}y(x)}{\mathrm{d}x} + (2+6x+2x^2\ldots\ldots..)y(x) = x\,.$$

Next, we substitute the first derivative term to get

$$\frac{\mathrm{d}^2z(x)}{\mathrm{d}x^2} + \ldots\ldots..\frac{\mathrm{d}z(x)}{\mathrm{d}x} + (6x+2x^2-6x)y(x) = x\,,$$

and finally use .... to get

$$\frac{\mathrm{d}^2z(x)}{\mathrm{d}x^2} + 3\frac{\mathrm{d}z(x)}{\mathrm{d}x} + 2z(x) = x\,.$$

The solution to this is

$$z(x) = A\mathrm{e}^{-x} + B\mathrm{e}^{-2x} + \frac{x}{2}\,,$$

and therefore the solution to the original equation must be

$$y(x) = A\frac{e^{-x}}{x^2} + B\ldots\ldots.. + \frac{1}{2x}\,.$$

A further use of the chain rule is to express the differential relationship between two variables x and y in terms of their derivatives with respect to another variable t. This can be essential to obtaining a expressions relating x and y given simple (but difficult to invert) relationships between each of x and y, and t. This is called parametric differentiation.

6. Use the chain rule to show that for $x(t)$ and $y(t)$ (and therefore implicitly also $y(x)$) that

$$\frac{\mathrm{d}y(x(t))}{\mathrm{d}x} = \frac{\mathrm{d}y(t)/\mathrm{d}t}{\mathrm{d}x(t)/\mathrm{d}t}\,,$$

and

$$\frac{\mathrm{d}^2 y(x(t))}{\mathrm{d}x^2} = \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)^{-2} \left[\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}y(t)}{\mathrm{d}t}\frac{\mathrm{d}^2 x(t)/\mathrm{d}t^2}{\mathrm{d}x(t)/\mathrm{d}t}\right].$$

Use this to then show for $y(t) = t^m + t^{-m}$ and $x(t) = t + t^{-1}$ that

$$(x^2-4)\left(\frac{\mathrm{d}y(x)}{\mathrm{d}x}\right)^2 = m^2(y^2(x)-4), \quad \text{and} \quad (x^2-4)\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + x\frac{\mathrm{d}y(x)}{\mathrm{d}x} - m^2 y(x) = 0.$$

These equations could be solved using advanced techniques to obtain an expression for $y(x)$. [Hint: Notice that $x^2(t) = (t-t^{-1})^2 + 4$ and $y^2(t) = (t^m - t^{-m})^2 + 4$.] [Apology: Notice also that, as is common, two different functions $y(x)$ and $y(t)$ are denoted by the same symbol y so that, for example, $y(4)$ is ambiguous and could mean the evaluation of either function for the argument 4. This is commonly done because it is usually clear which function is meant. To avoid this confusion one might start by writing $x = f(y)$, $y = g(t) = h(x)$. While explicitly clear this notation is perhaps more awkward to use. Think carefully.]

The definite integral of $f(x)$ between a and b can similarly be defined as a limit, this time

$$I[f](a,b) = \int_a^b f(x)\mathrm{d}x = \lim_{N\to\infty} \sum_{n=0}^{N-1} h\left(a + n\delta x_N\right)\delta x_N\,,$$

where $\delta x_N = (b-a)/N$. For some simple functions $f(x)$ it is possible to obtain the integral directly using this definition. You are asked to consider one example of this.

7. Starting from the above definition for the integral, evaluate $\int_a^b x\mathrm{d}x$.

Solution: $(b^2 - a^2)/2$.

Integrals are not normally obtained this way. More often, integrals are calculated by exploiting the relationship of integration to differentiation. You will discover this relationship by answering the next question.

8. Calculate the derivative of

$$F(x) = \int_a^x f(y)\mathrm{d}y\,.$$

Solution: $f(x)$.

In this way, integration can be viewed as the inverse of differentiation. The fact that the derivative of the definite integral $F(x) = \int_a^x f(s)\mathrm{d}s$ is given by $F'(x) = f(x)$ leads to the introduction of indefinite integrals as the inverse operation to taking derivatives, defined as $\int f(x)\mathrm{d}x = F(x) + \mathcal{C}$, where $F'(x) = f(x)$. The constant $\mathcal{C}$ is an arbitrarily chosen constant that reflects the ambiguity in the limits of the integral and the fact that the derivative of $F(x) + \mathcal{C}$ equals $f(x)$ for all choices of $\mathcal{C}$. Going back to the definite integral above we can then write $I[f](a, b) = F(b) - F(a)$. Most integration is done in this way, by seeking a function $F(x)$ whose derivative is known to be the integrand $f(x)$, and, for a definite integral, calculating the difference between the values of this function at the upper and lower limits.

As with differentiation, there are a few basic tools to relate one integral to others that may be easier to solve, for example integration by change of variables, parts, and partial fractions. A change of variable makes use of the identity

$$\int f(x)\mathrm{d}x = \int f(x(y)) \left(\frac{\mathrm{d}x(y)}{\mathrm{d}y}\right) \mathrm{d}y \,,$$

which follows from the chain rule of differentiation when considering x as a function of y. Integration by parts follows from the product rule of differentiation and uses the identity

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x \,.$$

You are now asked to use this knowledge to evaluate a number of definite and indefinite integrals.

9. Calculate the following integrals using the suggested method: (i) $\int (x+a)/(1+2ax+x^2)^{3/2}\mathrm{d}x$ by inspection, (ii) $\int_0^{\pi/2} \cos(x)\mathrm{e}^{\sin(x)}\mathrm{d}x$ by change of variable, (iii) $\int (x(1+x^2))^{-1}\mathrm{d}x$ using partial fractions, and (iv) $\int x\sin(x)\mathrm{d}x$ by parts.

Solution: (i) $-1/\sqrt{1+2ax+x^2} + \mathcal{C}$, (ii) $\mathrm{e} - 1$, (iii) $\ln(x) - \ln(x^2+1)/2 + \mathcal{C}$, (iv) $-x\cos(x) + \sin(x) + \mathcal{C}$.

10. Calculate the integrals (i) $\int(\cos^5(x) - \cos^3(x))\mathrm{d}x$, (ii) $\int \sin^2(x)\cos^4(x)\mathrm{d}x$, and (iii) $\int x^{-2}(16-x^2)^{-1/2}\mathrm{d}x$.

Solution: (i) $-\sin(x)/8 + \sin(3x)/48 + \sin(5x)/80 + \mathcal{C} = -\sin^3(x)/3 + \sin^5(x)/5 + \mathcal{C}$, (ii) $(12x + 3\sin(2x) - 3\sin(4x) - \sin(6x))/192 + \mathcal{C}$, (iii) $-\sqrt{16-x^2}/16x + \mathcal{C}$.

Integrals appear frequently in physics because so many quantities can be defined in terms of them: lengths, areas, and volumes are a few basic examples that you will consider in the following question.

11. (a) Find the arc length of the curve $y = \cosh(x)$ between $x = 0$ and $x = 1$.

(b) Find the arc length of the curve $x = \cos(t)$, $y = \sin(t)$ for $0 < t < \pi/2$.

(c) Find the surface area and volume of a sphere of radius R by treating it as obtained by rotating the curve $y = \sqrt{R^2 - x^2}$ about the x-axis.

For (b) and (c) do you get the answers that you expect?

Integrals also appear in the definitions of some important common mathematical functions and can be used to calculate their properties. Consider an example of this next.

12. If $\ln(x)$ is defined as $\int_1^x t^{-1}\mathrm{d}t$, show that $\ln(x) + \ln(y) = \ln(xy)$.

Finally, to confirm your understanding of both differentiation and integration, consider a generalisation of the earlier question regarding the derivative of an integral. To answer it you will need to understand, only in a very basic way, the derivative of a function of two arguments with respect to one of its arguments, called a partial derivative. The partial derivative of a function $f(x,y)$ *at* (x,y) *with respect to its first argument is defined as the limit*

$$\frac{\partial f(x,y)}{\partial x} = \left.\frac{\partial f(x,y)}{\partial x}\right|_y = \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x, y) - f(x,y)}{\delta x}\right\},$$

which is identical to the ordinary derivative of $f(x,y)$ *if* y *is a constant.*

13. The definite integral $I(x)$ is defined as

$$I(x) = \int_{f(x)}^{g(x)} h(x,s)\mathrm{d}s\,.$$

Calculate the derivative $I'(x)$.

Solution: $I'(x) = h(x,g(x))g'(x) - h(x,f(x))f'(x) + \int_{f(x)}^{g(x)} \frac{\partial h(x,s)}{\partial x}\mathrm{d}s$.

Class problems

14. Calculate the 8th derivative of $x^2\sin(x)$ using the Leibnitz theorem.

15. Definite integration maps a function f to a number $I[f](a,b) = \int_a^b f(s)\mathrm{d}s$ (such maps from functions f to numbers are often called functionals). Show that this mapping is linear.

16. Find $\int \sin^2(x)\mathrm{d}x$ using integration by parts.

17. Derive a recursion formula for evaluating integrals of the form $\int_0^\infty x^n \mathrm{e}^{-x^2}\mathrm{d}x$, with integer $n \geq 2$. Calculate the integral for $n = 1$ and use your results to evaluate it for $n = 3$ and $n = 4$.

18. Prove that

$$\int_{-a}^{a} f(x)\mathrm{d}x = 0\,,$$

if $f(x)$ is an odd function of x.

19. Which of the following integrals is zero?

$$\int_{-\infty}^{\infty} x\mathrm{e}^{-x^2}\mathrm{d}x\,, \qquad \int_{-\pi}^{\pi} x\sin(x)\mathrm{d}x\,, \qquad \int_{-\pi}^{\pi} x^2\sin(x)\mathrm{d}x.$$

20. Figure 1 shows a Mathematica calculation of an indefinite and a definite integral of a Gauss function $\mathrm{Erf}(x)$. Show that from these results it follows that $\mathrm{Erf}(0) = 0$ and calculate $\mathrm{Erf}(-\infty)$ and $\mathrm{Erf}(\infty)$.

21. From the definition of a normal single variable integral, infer the meaning of and evaluate the following line integrals in terms of many related variables:

(a) $\int_C (x^2 + 2y)\mathrm{d}x$ from $(0,1)$ to $(2,3)$ where C is the line $y = x + 1$.

(b) $\int_C xy\mathrm{d}x$ from $(0,4)$ to $(4,0)$ where C is the circle $x^2 + y^2 = 16$.

(c) $\int_C (y^2\mathrm{d}x + xy\mathrm{d}y + zx\mathrm{d}z)$ from $A = (0,0,0)$ to $B = (1,1,1)$ where (i) C is the straight line from A to B; (ii) C is the broken line from A to B connecting $(0,0,0)$, $(0,0,1)$, $(0,1,1)$ and $(1,1,1)$.

We work out the indefinite integral of a Gauss function

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Integrate[ Exp[-x^2], x]
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$\frac{1}{2} \sqrt{\pi}$ Erf[x]

The definite integral between -∞ and ∞ is

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Integrate[ Exp[-x^2], {x, -∞, ∞}]
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$\sqrt{\pi}$

This function is odd since

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Erf[x] == -Erf[-x]
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True
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Figure 1: Calculating indefinite and definite integrals in Mathematica.