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Keble College - Michaelmas 2014

CP3&4: Mathematical methods I&II

Tutorial 1 - Complex numbers

Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.
Leave these at Keble lodge by 5pm on Monday of 1st week.
Look at the 'class problems' in preparation for the tutorial session.
Suggested reading: RHB 1 and 3, and the lecturer's problem sets.

Goals

- Identify and close gaps in your A level knowledge.
- Gain proficiency in manipulating expressions containing complex numbers.
- Use complex numbers to solve otherwise difficult mathematics problems.
- Develop an understanding of how complex numbers may be used to simplify the solution of physics problems.

I am aware that complex numbers will not have been covered in lectures by the time of the first tutorial. However, in previous years students mostly agreed that this topic was reasonably accessible and did not require going much beyond A-level maths knowledge or equivalent (in particular for those having done Further Maths).

Problems

A complex number $z = x + \mathrm{i}y$ (with x, y real numbers) consists of a real part $x = \Re\{z\}$ and an imaginary part $y = \Im\{z\}$. The complex conjugate of z is $z^* = x - \mathrm{i}y$ and the modulus is given by $|z| = \sqrt{x^2 + y^2}$. Two complex numbers z_1 and z_2 with $z_j = x_j + \mathrm{i}y_j$ are added (subtracted) by adding (subtracting) their real and imaginary parts, $z_1 + z_2 = (x_1 + x_2) + \mathrm{i}(y_1 + y_2)$. They are multiplied by using the distributive law and $\mathrm{i}^2 = -1$, and multiplication is found to be associative and commutative.

1. For $z_1 = 1 + \mathrm{i}$ and $z_2 = -3 + 2\mathrm{i}$, find (i) $z_1 + z_2$, (ii) $z_1 - z_2$, (iii) $z_1 z_2$, (iv) $|z_1|$, (v) z_1^*.

Solution: (i) $-2 + 3\mathrm{i}$, (ii) $4 - \mathrm{i}$, (iii) $-5 - \mathrm{i}$, (iv) $\sqrt{2}$, (v) $1 - \mathrm{i}$.

Complex numbers can also be divided if we interpret this as the inverse of multiplication. The next question asks you to explain this.

2. Show that $z_1 z_2 = 1$ for $z_2 = z_1^*/|z_1|^2$ i.e. $x_2 = x_1/|z_1|^2$ and $y_2 = -y_1/|z_1|^2$. Discuss how this can be used to define division of complex numbers.

Solution: $z_1 z_2 = (x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) = (x_1 x_2 - y_1 y_2 + \mathrm{i}(x_1 y_2 + y_1 x_2))/|z_1|^2 = 1 + \mathrm{i}0 = 1$. We can thus define $z_1^{-1} = z_2$ and division of two complex numbers by $z_3/z_1 = z_3 z_1^{-1} = z_3 z_2$.

[1] These problems were compiled by Prof. D. Jaksch based on problem sets by Prof. G.G. Ross and past Oxford Prelims exam questions.

3. Given this definition of division, find z_3/z_1 for $z_3 = 1 + \mathrm{i}$ and $z_1 = -3 + 2\mathrm{i}$.

Solution: $-1/13 - 5\mathrm{i}/13$.

With the concepts of addition, subtraction, multiplication and division defined for complex numbers, we can perform a whole range of operations on them. Demonstrate this with the following examples.

4. For $z = x + \mathrm{i}y$ find the real and imaginary parts of (i) $2 + z$, (ii) z^2, (iii) z^* (iv) $1/z$, (v) $|z|$, (vi) i^5, (vii) $(1 + \mathrm{i})^2$, (viii) $(2 + 3i)/(1 + 6\mathrm{i})$.

Solution: (i) $2+x, y$ (ii) x^2-y^2, $2xy$, (iii) x, $-y$, (iv) $x/(x^2+y^2)$, $-y/(x^2+y^2)$, (v) $\sqrt{x^2+y^2}$, 0, (vi) 0, 1 (vii) 0, 2 (viii) $20/37$, $-9/37$.

We have covered addition, subtraction, multiplication and division. From our experience with real numbers, we know it is useful to go beyond this define complicated functions (of real numbers, or with the domain of real numbers), e.g. e^x. *How can we extend the domain of such functions to complex numbers, i.e., can we define* e^z*? It turns out that since we know how to add, subtract and multiply complex numbers, the domain of functions, such as the exponential function, equal to Taylor series can be easily extended to complex numbers, since Taylor series involve only addition, subtraction and multiplication (with division the same is true for Laurent series).*

For instance, we may extend the domain of the following functions to complex numbers, defined by their Maclaurin series as

$$\begin{aligned}
\mathrm{e}^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!}, \\
\cos(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \\
\sin(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \\
\cosh(z) &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \\
\sinh(z) &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \\
\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n,
\end{aligned}$$

which converge for all z *except for the last, which converges for* $|z| < 1$.

You should satisfy yourself that properties of these functions that follow from the power series representations (as well as associativity, distributivity and commutativity) must also hold for complex arguments, e.g. $\mathrm{e}^{z_1+z_2} = \mathrm{e}^{z_1}\mathrm{e}^{z_2}$, $\mathrm{e}^z = \cosh(z) + \sinh(z)$ *and* $1 = \cosh^2(z) - \sinh^2(z)$.

5. Use the above to express the following functions in the form $a + \mathrm{i}b$, where a and b are functions of x, y and real number φ: (i) $\mathrm{e}^{\mathrm{i}\varphi}$, (ii) e^z, (iii) $\cos(z)$, (iv) $\sin(z)$, (v) $\cosh(z)$, (vi) $\sinh(z)$.

Solution: (i) $\cos(\varphi) + \mathrm{i}\sin(\varphi)$, (ii) $\mathrm{e}^x\cos(y) + \mathrm{i}\mathrm{e}^x\sin(y))$, (iii) $\cos(x)\cosh(y) - \mathrm{i}\sin(x)\sinh(y)$, (iv) $\sin(x)\cosh(y) + \mathrm{i}\cos(x)\sinh(y)$, (v) $\cosh(x)\cos(y) + \mathrm{i}\sinh(x)\sin(y)$, (vi) $\sinh(x)\cos(y) + \mathrm{i}\cosh(x)\sin(y)$.

A complex number can be written in polar form $z = r\mathrm{e}^{\mathrm{i}\varphi}$ with modulus $r = |z| \geq 0$ and argument $\varphi = \arg(z)$. We can also interpret it as a two-dimensional coordinate (x, y) and thus represent it as a point on a plane, a so-called Argand diagram. The real part is the x-coordinate and the imaginary part is the y-coordinate. The next few questions ask you to explore this representation.

6. Show that a complex number $z = r\mathrm{e}^{\mathrm{i}\varphi}$ corresponds, as well as to a point (see above), to a vector in an Argand diagram with length $|z| = r$ and angle φ with the x-axis.

Solution: $z = r\cos(\varphi) + \mathrm{i}r\sin(\varphi) = x + \mathrm{i}y$. The length of a vector from the origin to the point (x, y) is thus $\sqrt{x^2 + y^2} = r = |z|$ and its angle with the x axis is φ.

7. Change the following complex numbers to polar form and draw them in an Argand diagram: (i) $-\mathrm{i}$, (ii) $(1 - \sqrt{3}\mathrm{i})/2$, (iii) $3 + 4\mathrm{i}$, (iv) $1 + \mathrm{i}$, (v) $1 - \mathrm{i}$, (vi) $(1 + \mathrm{i})/(1 - \mathrm{i})$.

Solution: (i) $\exp(-\mathrm{i}\pi/2)$, (ii) $\exp(-\mathrm{i}\pi/3)$, (iii) $5\exp(\mathrm{i}\arctan(4/3))$, (iv) $\sqrt{2}\exp(i\pi/4)$, (v) $\sqrt{2}\exp(-\mathrm{i}\pi/4)$, (vi) $\exp(\mathrm{i}\pi/2)$.

8. Figure 1 shows a Mathematica printout in which two different methods are used to convert z^4 into polar form, where $z = -4 - 3\mathrm{i}$. They yield different results. Explain what goes wrong and describe how you would correct for this problem when converting complex numbers to polar form (when not using Mathematica).

```
In[1]:= z = -4 - ⅈ 3;
```

(i) We work out the argument r and magnitude φ of z^4 using the built in *Mathematica* functions:

```
In[2]:= Print["r = ", Abs[z^4], ", φ = ", Arg[z^4]]

        r = 625, φ = π - ArcTan[336/527]
```

(ii) Now we compare these results with those obtained using the explicit formulas introduced above:

```
In[3]:= x = Re[z^4];
        y = Im[z^4];
        Print["r = ", Sqrt[x^2 + y^2], ", φ = ", ArcTan[y / x]]

        r = 625, φ = -ArcTan[336/527]
```

Figure 1: Converting a complex number into polar form in Mathematica.

9. The complex numbers z_1, z_2 and z_3 represent points in the complex plane as well as vectors from the origin to these points. Find geometrical interpretations of (i) $|z_1 - z_2|$ and (ii) $\arg[(z_2 - z_1)/(z_3 - z_1)]$.

Solution: (i) The distance between the points representing z_1 and z_2, or the length of the vector represented by $z_1 - z_2$, (ii) the angle between the vectors $z_2 - z_1$ and $z_3 - z_1$, as defined by the positive angled rotation of $z_3 - z_1$ needed to be parallel to $z_2 - z_1$.

We now have the tools needed to solve many equations with complex variables. The next few questions ask you to practice this. You will find it useful to use a combination of algebraic and geometric methods, the latter generally being quicker and more insightful.

10. Solve the equation $|2x - 1 + \mathrm{i}y| = x^2 + \mathrm{i}y$ for all possible values $z = x + \mathrm{i}y$, with real x and y.

Solution: The left hand side of the equation is a modulus and thus purely .... We immediately find that

$$y = \ldots\ldots .$$

The equation therefore simplifies to $|2x - 1| = x^2$. We now distinguish two cases

(a) For $x \geq$ ...... the term inside the modulus is larger or equal to zero. The equation therefore further simplifies to

$$\text{...............} = x^2 ,$$

which has the solution $x =$ ..... This does .... fulfil the above assumption for this case and so

$$(x, y) = (\text{......}, \text{.....}) ,$$

is .... a valid solution to the original equation.

(b) For $x <$ ...... the term inside the modulus is smaller than zero. The equation then reads

$$\text{...............} = x^2 ,$$

which has solutions $x = -1 +$ .... and $x = -1 -$ .... These do .... fulfil the above assumption for this case and therefore

$$(x, y) = (-1 + \text{......}, \text{.....}) ,$$

and

$$(x, y) = (-1 - \text{......}, \text{.....}) ,$$

are .... valid solutions to the original equation. We can check this answer using Mathematica by inputting `Solve[Abs[2 Re[z] - 1 + I Im[z]] == Re[z]∧2 + I Im[z], z]` but note that replacing `Re[z]` by `x` and `Im[z]` by `y` in this expressions will not work unless you explicitly set `Im[x] = Im[y] = 0`.

11. Solve (i) $2\mathrm{i}x + 3 = y - \mathrm{i}$, (ii) $(x + 2y + 3) + \mathrm{i}(3x - y - 1) = 0$, (iii) $z^2 = (z^*)^2$.

12. Solve (i) $|z - 3| = |z - 1|$, (ii) $\arg[(z - 3)/(z - 1)] = \pi/2$.

Class problems

13. Draw, in the complex plane, the solutions to (i) $z = 3 - 2\mathrm{i}$, (ii) $z = 4\mathrm{e}^{-\mathrm{i}\pi/6}$, (iii) $|z - 1| = 1$, (iv) $\Re\{z^2\} = 4$, (v) $z - z^* = 5\mathrm{i}$, (vi) $z = t\mathrm{e}^{\mathrm{i}\pi t}$ (for real valued parameter t).

14. Find (i) $(1 + 2\mathrm{i})^7$ (ii) $(1 - 2\mathrm{i})^7/(1 + 2\mathrm{i})^7$.

15. By noting that $\mathrm{e}^{\mathrm{i}n\theta} = (\cos\theta + \mathrm{i}\sin\theta)^n$, show that

- $\sin 6\theta = 6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta$, and
- $\cos^4\theta = (\cos 4\theta + 4\cos 2\theta + 3)/8$.

16. The series representations of the natural logarithm function is perhaps more complicated than some of the others used. However we can still define the natural logarithm function as the inverse of the exponential function, which we have already discussed, and use this to define what is meant by the logarithm of a complex number. Do so.

17. We consider the geometric series

$$S_N = \sum_{n=0}^{N} z^n,$$

where z is a complex number.

(i) Calculate the sum S_N. Consider the cases $z = 1$ and $z \neq 1$ separately.

(ii) For $|z| < 1$, find the limit $N \to \infty$ and show that

$$S_\infty = \frac{1}{1 - z} .$$

(iii) Discuss this limit when $|z| < 1$ is not fulfilled.

(iv) For $z = \mathrm{e}^{\mathrm{i}\pi/k}$ with $k \in \mathbb{N}$ find all values of $N \in \mathbb{N}$ for which $S_N = 0$.

(v) Calculate the sum

$$S_{100} = \sum_{n=0}^{100} \mathrm{e}^{\mathrm{i}n\pi/6},$$

first using the results from (i) and then repeating the calculation using the result from (iv).