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Keble College - Hilary 2015

CP3&4: Mathematical methods I&II

Easter vacation problems

Prepare full solutions to the 'problems' with a self assessment of your progress on a cover page.
Leave these at Keble lodge by 5pm on Thursday of 0th week, Trinity.
Suggested reading: DJM, FR and RHB.

Goals

- Gain a deeper understanding of the topics covered during Michaelmas and Hilary terms.
- Build up confidence in solving prelims exam questions.
- Practise exam technique.

Problems

1. Two points $x_1 = 0$ and $x_2 = L$ on a string are observed as a traveling wave passes them. The transverse motions of the two points are given by

$$y_1(t) = A\sin(\omega t), \quad \text{and} \quad y_2(t) = A\sin(\omega t + \pi/8)\,.$$

(i) What is the frequency of the wave?

(ii) What is its wave length?

(iii) With what speed does the wave travel?

(iv) Which way is the wave travelling?

Discuss any ambiguities that might arise because of the limited amount of information given.

2. A symmetrical triangular pulse of maximum height H and total length L is moving in the positive x direction along a string for which the wave speed is v. At time $t = 0$ the pulse is entirely located between $x = 0$ and $x = L$. Draw a graph of the transverse velocity as a function of time t at $x = L$.

3. A closed loop of uniform string is rotated rapidly at some constant angular velocity ω. The mass of the string is M and the radius is R. A tension T is set up circumferentially in the string as a result of its rotation.

(i) By considering the instantaneous centripetal acceleration of a small segment of the string, show that the tension must be equal to $T = M\omega^2 R/2\pi$.

(ii) The string is suddenly deformed at some point, causing a kink to appear in it. Show that this could produce a distortion of the string that remains stationary with respect to the laboratory, regardless of the particular values of M, ω and R. Are there other solutions? How do these move?

[1] These problems were compiled by Prof. D. Jaksch based on problem sets by Prof. S. Rawlings, Prof. G.G. Ross and past Oxford Prelims exam questions.

4. Show that

$$d'f = y(1 + x - x^2)\mathrm{d}x + x(x + 1)\mathrm{d}y,$$

is not an exact differential.

Find the differential equation that a function $g(x)$ must satisfy if $\mathrm{d}\Phi = g(x)d'f$ is to be an exact differential. Verify that $g(x) = \mathrm{e}^{-x}$ is a solution of this equation and deduce the form of $\Phi(x, y)$.

5. Evaluate the following line integrals, showing your working. The path of integration in each case is anticlockwise around the four sides of the square OABC in the x–y plane whose edges are aligned with the coordinate axes. The length of each side of the square is a and one corner O is the origin.

(i) $\int \mathrm{d}l$, (ii) $\int \mathbf{dl}$, (iii) $\int \mathbf{b}\,\mathrm{d}l$, (iv) $\int \mathbf{b} \cdot \mathbf{dl}$, (v) $\int \mathbf{b} \times \mathbf{dl}$, (vi) $\int \mathbf{r}\,\mathrm{d}l$, (vii) $\int \mathbf{r} \cdot \mathbf{dl}$, (viii) $\int \mathbf{r} \times \mathbf{dl}$.

Here $\mathrm{d}l$ is the scalar infinitesimal path element, $\mathbf{dl}$ is the corresponding vector, $\mathbf{b}$ is a constant vector, and $\mathbf{r}$ is the radius vector of length r.

State Stokes' theorem. By applying it to (vii) above, verify that it yields the same result as direct integration.

6. An alternating voltage $V = V_0 \sin(\omega t)$ is applied to the circuit shown in figure 1. The following

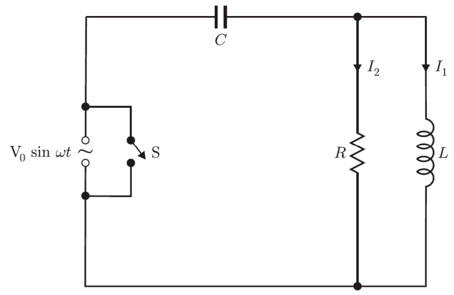

Figure 1: AC-circuit.

equations may be derived from Kirchhoff's law.

$$I_2 R + \frac{Q}{C} = V\,, \qquad L\frac{\mathrm{d}I_1}{\mathrm{d}t} = I_2 R, \quad \text{and} \quad \frac{\mathrm{d}Q}{\mathrm{d}t} = I_1 + I_2\,,$$

where Q is the charge on the capacitor.

Derive a second order differential equation for I_1, and hence obtain the steady state solution for I_1 after transients have decayed away.

Determine the angular frequency ω at which I_1 is in phase with V, and obtain expressions for the amplitudes of I_1 and I_2 at this frequency.

Suppose now that the switch S is closed and the voltage supply removed when I_1 is at its maximum value. Obtain the solution for the subsequent variation of I_1 with time for the case $L = 4CR^2$ and sketch the form of your solution.

7. Writing the n-th derivative of $f(x) = \sinh^{-1}(x)$ as

$$f^{(n)}(x) = \frac{P_n(x)}{(1 + x^2)^{n-1/2}}\,,$$

where $P_n(x)$ is a polynomial (of degree $n - 1$), show that the $P_n(x)$ satisfy the recurrence relation

$$P_{n+1}(x) = (1 + x^2)P_n'(x) - (2n - 1)xP_n(x)\,.$$

Hence generate the coefficients necessary to express $\sinh^{-1}(x)$ as a McLaurin series up to terms of order x^5.

8. A function may be expressed as $f(x, y)$ or alternatively as $g(u, v)$ where

$$u = y - 2x \quad \text{and} \quad v = 2y + x\,.$$

Find the value of the Jacobian $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ and hence show that

$$\iint f(x, y)\mathrm{d}x\mathrm{d}y = \frac{1}{5}\iint g(u, v)\mathrm{d}u\mathrm{d}v$$

A region S in the (x, y) plane is confined by the lines: $y = 1 + 2x$, $y = 2x$, $x = -2y$, and $2y + x = 2$. Sketch the area defined.

Using the Jacobian derived earlier, evaluate the following area integral over the region S.

$$\iint (2y^2 - 3xy - 2x^2)\mathrm{d}x\mathrm{d}y\,.$$

9. A double pendulum consists of two masses m suspended from a fixed point P by light strings of length ℓ and 2ℓ, as shown in figure 2.

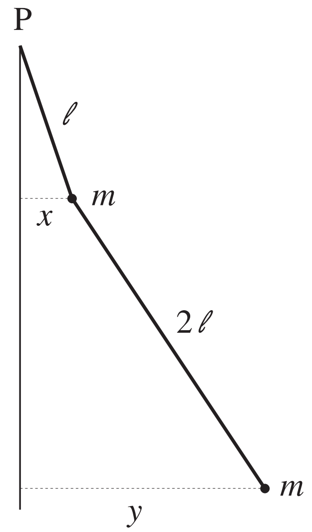

Figure 2: Double pendulum.

The (small) horizontal displacements x of the upper mass and y of the lower mass satisfy the differential equations

$$m\ddot{x} = \frac{mg(y - x)}{2\ell} - \frac{2mgx}{\ell}\,,$$

$$m\ddot{y} = -\frac{mg(y - x)}{2\ell}\,.$$

Explain what is meant by the *normal modes* of the double pendulum, and find their angular frequencies. Describe the relative motions of the two masses in each of the normal modes.

Find the motion of the pendulum, given that it is released from rest at $t = 0$ with $x = a$, $y = 0$.